(1) Continue about Determinants

Recall Let A be $n \times n$ matrix

A_{ij} - matrix $(n-1)\times(n-1)$ we obtain from A by removing i-th row and j-th column

$C_{ij} = (-1)^{i+j}\det(A_{ij})$. Let $C = (C_{ij})$ - matrix of cofactors with (i,j)-entry being C_{ij}

Then,

$$\det(A) \overset{\text{cofactor expansion along the } i\text{-th row}}{=} a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

Example Compute determinant of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

Solution

$$\det(A) \overset{\text{use 2nd row}}{=} a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} =$$

$$= 0\cdot(-1)^{2+1}\det\left(\begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix}\right) + 3\cdot(-1)^{2+2}\cdot\det\left(\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}\right) +$$

$$+ (-4)\cdot(-1)^{2+3}\cdot\det\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) =$$

$$= 3\cdot(1\cdot(-2) - 0\cdot 2) + (-4)\cdot(-1)(1\cdot 1 - 0\cdot 2) =$$

$$= -6 + 4 = \boxed{-2}$$

Can we use column for cofactor expansion?

Yes. $\det(A) \underset{\text{cofactor expansion along } j\text{th column}}{=} a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$

(2)

$$\det(A) \overset{\text{along 1st column}}{=} a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} =$$

$$= 1 \cdot (-1)^{1+1} \cdot \det\left(\begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix}\right) + \overbrace{0 \cdot (-1)^{2+1} \det\left(\begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix}\right) +}^{=0}$$

$$\underbrace{+\, 0 \cdot (-1)^{3+1} \cdot \det\left(\begin{bmatrix} 2 & 2 \\ 3 & -4 \end{bmatrix}\right)}_{=0} =$$

$$= 1 \cdot (3 \cdot (-2) - 1 \cdot (-4)) = -6 + 4 = \boxed{-2}$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow C = \begin{bmatrix} -2 & 0 & 0 \\ -(-6) & -2 & -1 \\ -14 & -(-4) & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 6 & -2 & -1 \\ -14 & 4 & 3 \end{bmatrix}$$

! Formula for the inverse !

Suppose A is invertible $n \times n$ matrix.

Then $A^{-1} = \dfrac{1}{\det(A)} \cdot C^T$

(Recall: $\det(A) \neq 0 \iff A$ is invertible)

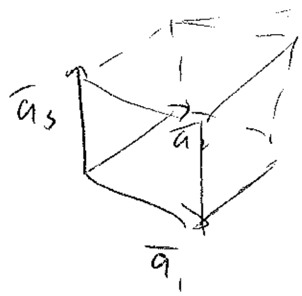
For our example: $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{bmatrix}$

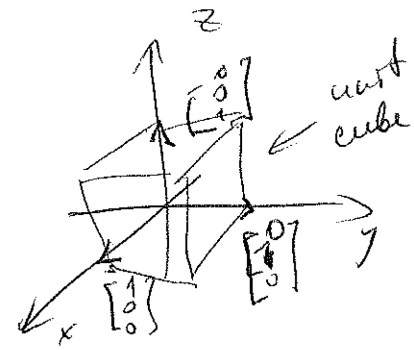
We obtain $A^{-1} = \dfrac{1}{-2} \cdot \begin{bmatrix} -2 & 0 & 0 \\ 6 & -2 & -1 \\ -14 & 4 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 1/2 \\ 7 & -2 & -3/2 \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 1 & -2 \\ 0 & 1/2 & -3/2 \end{bmatrix}$

③ Geometric interpretation

$|\det(A)| =$ Volume of parallelepiped constructed on n-vectors $\bar{a}_1, \ldots, \bar{a}_n$ if

$$A = \begin{bmatrix} - & \bar{a}_1 & - \\ - & \bar{a}_2 & - \\ & \vdots & \\ - & \bar{a}_n & - \end{bmatrix}$$





$$I_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(I_n) = 1 \cdot \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot (1-0) = 1$$

Alternative notation for det of matrix A:

$\det(A) = |A|$ ← means determinant, too

matrix

④ **Behavior under row operations of det(A)**

Row operation	det(A)
RO1. Add a multiple of a row to another row	unchanged
RO2. Switch rows	$\det \rightsquigarrow -\det$ (change sign)
RO3. Scale a row by a non-zero number $\lambda \neq 0$	$\det \rightsquigarrow \lambda \cdot \det$ (determinant is multiplied by λ)

<u>Thus</u> let

$$A = \begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & * & * \\ 0 & 0 & \ddots & * \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

(upper triangular), $*$ – any number,

or

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ * & a_{22} & 0 & \vdots \\ * & * & \ddots & 0 \\ * & * & * & a_{nn} \end{bmatrix}$$

(lower triangular)

Then $\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$

<u>Example</u>

$$\det\left(\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}\right) = 1 \cdot \det\left(\begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix}\right) =$$

$$= 1 \cdot 2 \cdot (-3) = -6$$

<u>Corollary:</u> If A is $n \times n$ matrix and has less than n pivots, then $\det(A) = 0$.

<u>Example</u>

$$\det\left(\begin{bmatrix} \boxed{1} & 5 & 3 \\ 0 & \boxed{2} & [illegible] \\ 0 & 0 & \boxed{0} \end{bmatrix}\right) = 1 \cdot 2 \cdot (0) = 0.$$

(8) Find the determinant using row operations

$$A = \begin{pmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{pmatrix}$$

<u>Solution:</u>

1. Bring A to REF by keeping track of operations

$$A = \underset{\det(A)}{\begin{pmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{pmatrix}} \xrightarrow[\text{RO1}]{R_2 \to R_2 - R_1} \underset{\det(A)}{\begin{pmatrix} 3 & 3 & -3 \\ 0 & 1 & -1 \\ 2 & -3 & -5 \end{pmatrix}} \xrightarrow[\text{RO3}]{R_1 \to \frac{R_1}{3}} \underset{\frac{\det(A)}{3}}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 2 & -3 & -5 \end{pmatrix}}$$

$$\xrightarrow[\text{RO1}]{R_3 \to R_3 - 2R_1} \underset{\frac{\det(A)}{3}}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{pmatrix}} \xrightarrow[\text{RO1}]{R_3 \to R_3 + 5R_2} \underset{\frac{\det(A)}{3}}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{pmatrix}}$$

$$\cancel{\det\left(\begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix}\right) \overset{\text{RO1}}{=} \det\left(\begin{bmatrix} 3 & 3 & 3 \\ 0 & 1 & -1 \\ 2 & -3 & -5 \end{bmatrix}\right)}$$

$$\det\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{pmatrix} = 1 \cdot 1 \cdot (-8) = -8$$

by row operations $\to \; \| \; \frac{\det(A)}{3}$

$$\rightsquigarrow \frac{\det(A)}{3} = -8 \rightsquigarrow \det(A) = -8 \cdot 3 = -24$$

Answer: $-24 = \det\left(\begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix}\right)$

(6) **Thm** Let A and B be $n \times n$ matrices, then $\boxed{\det(A \cdot B) = \det(A) \cdot \det(B)}$

Example Verify that $\det(AB) = \det(A) \cdot \det(B)$ for $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}$

Solution:

$\det(A) = 3 \cdot (-2) - (-1) \cdot 6 = -6 + 6 = 0$

$\det(B) = 4 \cdot (-3) - (-1) \cdot 3 = -12 + 3 = -9$

$\det(A) \cdot \det(B) = 0 \cdot (-9) = 0$

$$A \cdot B = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -2 & 3 \end{bmatrix}$$

$\det(A \cdot B) = 6 \cdot 3 - (-2) \cdot (-9) = 18 - 18 = 0$

$\det(A \cdot B) = \det(A) \cdot \det(B)$.

Notice! $\boxed{\det(A+B) \neq \det(A) + \det(B)}$ $\det(A) + \det(B) = 0 + (-9) = -9$

see for the same A, B !

$$A + B = \begin{bmatrix} 7 & 9 \\ -2 & -5 \end{bmatrix}$$

$\det(A+B) = 7 \cdot (-5) - (-2) \cdot 9 = -35 + 18 = -17 \neq -9$

$\det(A+B) \neq \det(A) + \det(B)$.

Notice! $\boxed{\det(A^T) = \det(A)}$: $A^T = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$

$\det(A^T) = 3 \cdot (-2) - 6 \cdot (-1) = -6 + 6 = 0 = \det(A)$