Example:

(5) Is $\overline{u}$ in Col(A), where $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$

and $\overline{u} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$?

Solution:

1. ~~To answer the question~~

The question can be rephrased in the following way.

Is the system $A\overline{x} = \overline{u}$ consistent or not? → yes, $\overline{u} \in$ Col(A) (in)

→ no, $\overline{u} \notin$ Col(A) (not in)

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -7 \\ 2 & 0 & 7 & 3 \\ -3 & -5 & -3 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & -1 & 5 & -7 \\ 0 & \boxed{2} & -3 & 17 \\ 0 & -8 & 12 & -19 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} \boxed{1} & -1 & 5 & -7 \\ 0 & \boxed{2} & -3 & 17 \\ 0 & 0 & 0 & \boxed{49} \end{array}\right]$$ ⇝ pivot in last column of augmented matrix

⇝ inconsistent system

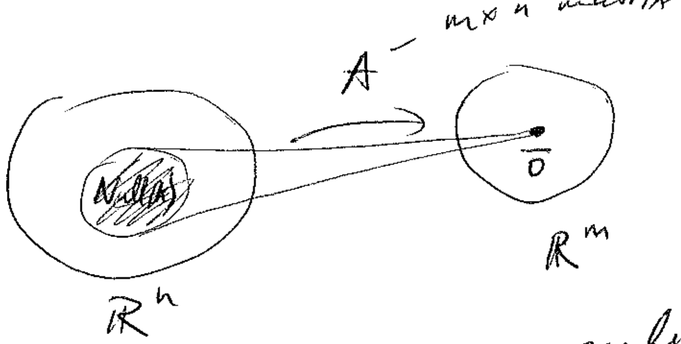
⇝ $\overline{u}$ is not in Col(A)

① **Def**

$Null(A)$ = null space of a matrix A =

= all $\bar{x}$ such that $A\bar{x} = \bar{0}$ =

= all solutions of equation $A\bar{x} = \bar{0}$



Let us see that it is a subspace of R^n:

1) $A\bar{0} = \bar{0} \rightarrow \bar{0} \in Null(A)$

2) $\bar{u}, \bar{v} \in Null(A) \Rightarrow A\bar{u} = \bar{0}$ and $A\bar{v} = \bar{0} \Rightarrow$

$\Rightarrow A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v} = \bar{0} + \bar{0} = \bar{0} \Rightarrow \bar{u} + \bar{v} \in Null(A)$

3) $\bar{u} \in Null(A) \Rightarrow A\bar{u} = \bar{0} \Rightarrow$

$\Rightarrow A(c \cdot \bar{u}) = c \cdot A\bar{u} = c \cdot \bar{0} = \bar{0} \Rightarrow c\bar{u} \in Null(A)$ for any constant c.

Lemma:

$\dim(Null(A))$ = number of free variables in the system $(A|\bar{0})$

Example Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 5 & 11 \end{bmatrix}$.

Find a basis and the dimension for the null space of A.

② Example

Given $A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$.

Find a basis and the dimension for the null space of A if A is row equivalent to $\begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution:

1. Find the solution of $A\bar{x} = \bar{0}$:

$$[A \mid \bar{0}] \longrightarrow \left[\begin{array}{cccc|c} -3 & 9 & -2 & -7 & 0 \\ 2 & -6 & 4 & 8 & 0 \\ 3 & -9 & -2 & 2 & 0 \end{array}\right] \longrightarrow \cdots$$

$$\cdots \longrightarrow \left[\begin{array}{cccc|c} \boxed{1} & -3 & 6 & 9 & 0 \\ 0 & 0 & \boxed{4} & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

to solve we need RREF

row operations do not change last column

REF

$$\xrightarrow{R_2 \to \frac{R_2}{4}} \left[\begin{array}{cccc|c} \boxed{1} & -3 & 6 & 9 & 0 \\ 0 & 0 & \boxed{1} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$$\xrightarrow{R_1 \to R_1 - 6R_2} \left[\begin{array}{cccc|c} \boxed{1} & -3 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & \boxed{1} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x_1 - 3x_2 + \frac{3}{2}x_4 = 0 \\ x_3 + \frac{5}{4}x_4 = 0 \end{cases}$$

RREF

$$\longrightarrow \begin{cases} x_1 = 3x_2 - \frac{3}{2}x_4 \\ x_3 = -\frac{5}{4}x_4 \end{cases}$$

(3) 2. How many free variables? x_2, x_4 – free variables
(no pivot in a column)
2

$\Rightarrow \dim(\mathrm{Null}(A)) = 2.$

3. Write the solution in a parametric vector form.

$$\bar{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_2 - \frac{3}{2}x_4 \\ x_2 \\ -\frac{5}{4}x_4 \\ x_4 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -\frac{3}{2} \\ 0 \\ -\frac{5}{4} \\ 1 \end{pmatrix}$$

basis vectors (they are automatically linearly independent)

$$\boxed{\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ -\frac{5}{4} \\ 1 \end{bmatrix} \right\}}$$

a basis of $\mathrm{Null}(A) = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ -5/4 \\ 1 \end{bmatrix} \right\}$

What is the dimension and a basis of the column space of A?

$\dim(\mathrm{Col}(A)) = 2$

a basis of $\mathrm{Col}(A) = \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}$
