(4) Def

The rank of a matrix A

$$\boxed{\operatorname{rank}(A) = \dim(\operatorname{Col}(A))}$$

In previous example rank$(A) = 2$

$$\boxed{\operatorname{rank}(A) = \text{number of pivots in } A}$$

Thm

If A has n columns, then

$$\operatorname{rank}(A) + \dim(\operatorname{Null}(A)) = n.$$

In previous example

rank$(A) = 2$

dim$(\operatorname{Null}(A)) = 2$

number of columns in A $= 4$

$$2 + 2 = 4.$$

Thm (The Basis Theorem)

$\dim(H) = p$ and H is a subspace of $\mathbb{R}^4$.

Any set of exactly p linearly independent vectors is automatically a basis of H.

Any set of p vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p$ such that $H = \operatorname{Span}\{\bar{v}_1, \ldots, \bar{v}_p\}$ is automatically a basis of H.

⑤ Example

We know $\dim(\mathbb{R}^2) = 2$

$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are linearly independent

$$a_1 \cdot \begin{bmatrix} 5 \\ 0 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5a_1 + 3a_2 \\ 2a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a_2 = 0 = a_1$$

$\left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ - a basis of $\mathbb{R}^2$.

Example

Let A be an 7×5 matrix whose column space is 5-dimensional. Are the columns of A linearly independent?

$$A = \begin{bmatrix} | & | & & | \\ \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_5 \\ | & | & & | \end{bmatrix}$$

$$Col(A) = Span\{\underbrace{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_5}_{\text{exactly 5 vectors}}\}$$

$$\dim(Col(A)) = 5$$

$\longrightarrow$ $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_5$ - a basis of $Col(A)$,

i.e. $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_5$ are linearly independent.

(6) Thm Let A be $n \times n$ matrix

A - invertible $\Longleftrightarrow$ the columns of A form a basis of $\mathbb{R}^n$

$\Longleftrightarrow$ $Col(A) = \mathbb{R}^n$

$\Longleftrightarrow$ $\dim(Col(A)) = n$

$\Longleftrightarrow$ $rank(A) = n$

$\Longleftrightarrow$ $Null(A) = \{\bar{0}\}$

$\Longleftrightarrow$ $\dim(Null(A)) = 0.$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ a, b, c - \text{any numbers} \right\}$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent

$Col(A) = \mathbb{R}^3$

$\dim(Col(A)) = 3$

$rank(A) = \dim(Col(A)) = 3$

$Null(A) = \{\bar{0}\}$ $\quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ has a unique solution $\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\dim(Null(A)) = 0.$