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Eigenvalues and Eigenvectors

We know how to solve

$$A \cdot \bar{x} = \bar{b}$$

$m \times n$ matrix; n-vector of variables; m-vector of values,

where $A, \bar{b}$ are given and we want to find x.

Today we work with another equation

$$A \, \bar{x} = \lambda \bar{x}$$

$n \times n$ matrix (number of rows = number of columns); number; n-vector of variables

For this equation: A is given

We want to find $\bar{x}$ and λ.

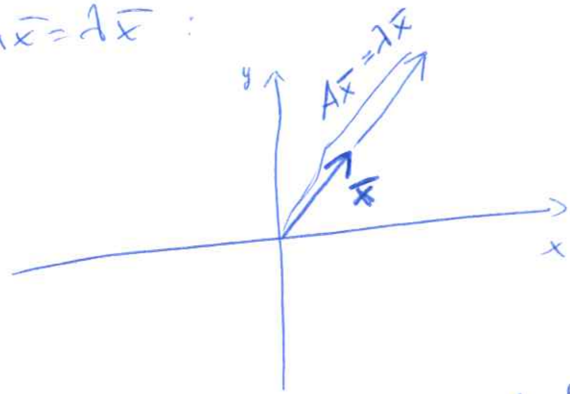
<u>Note!</u> $\bar{x} = 0$ always satisfies $A \cdot \bar{x} = \lambda \bar{x}$ as $A \cdot \bar{0} = \bar{0} = \lambda \cdot \bar{0}$ for any λ.

From now, $\bar{x} = \bar{0}$ is <u>not</u> an interesting solution.

② <u>Def</u> Let λ be a number
and $\bar{x} \neq \bar{0}$ - an n-vector
such that $A\bar{x} = \lambda \bar{x}$.

Then we say:
λ is an <u>eigenvalue</u> for A
$\bar{x}$ is an <u>eigenvector</u> for A with eigenvalue λ.

Cartoon for $A\bar{x} = \lambda\bar{x}$:



<u>Note</u>! 1) If $\bar{x}$ is an eigenvector such that $A\bar{x} = \lambda\bar{x}$, then λ is <u>uniquely defined</u>

2) If λ is an eigenvalue, then can be a lot of vectors $\bar{x}$ such that $A\bar{x} = \lambda\bar{x}$

<u>Example</u> $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Any $\bar{x} \neq \bar{0} \in \mathbb{R}^2$ is an eigenvector with eigenvalue 2.

Let $\bar{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $A\bar{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix} = 2 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 2\bar{x}$

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General strategy for finding eigenvalues and eigenvectors

(solving $A\bar{x} = \lambda\bar{x}$ if A is given)

Special case $\lambda = 0 \longrightarrow A\bar{x} = \bar{0}$
(homogeneous linear system - we know how to solve!)

$\{\text{all eigenvectors for } \lambda=0\} = \underbrace{Null(A) \setminus \{\bar{0}\}}$ excluding

empty (nothing in this set)
$\lambda=0$ is <u>not</u> an eigenvalue

there is at least one vector in this set
$\lambda=0$ is an eigenvalue

General approach

$$A\bar{x} = \lambda\bar{x} \Longleftrightarrow A\bar{x} - \lambda\bar{x} = \bar{0}$$
$$\Longleftrightarrow (A - \lambda I)\bar{x} = \bar{0}$$

where $\lambda I = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}$ (λI scales vectors by λ, i.e. $(\lambda I)\bar{v} = \lambda\bar{v}$ for any $\bar{v}$)

$(A-\lambda I)\bar{x} = \bar{0}$ has non-trivial solution $(\bar{x} \neq \bar{0})$
if and only if $A - \lambda I$ is not invertible
if an only if $\det(A - \lambda I) = 0$.

⑤ ~~Nf~~ Notation / Definition

If λ is an eigenvalue of A,

then $\boxed{E_\lambda = Null(A-\lambda I)}$

is the **eigenspace** of λ

$$Null(A-\lambda I) \setminus \{\bar{0}\} = \{\text{all eigenvectors for } \lambda\}$$

exclude

λ is an eigenvalue of A $\Longleftrightarrow$ $\dim(E_\lambda) = \dim(Null(A-\lambda I)) > 0$

$\Longleftrightarrow$ $E_\lambda = Null(A-\lambda I) \neq \{\bar{0}\}$

$\Longleftrightarrow$ $\det(A-\lambda I) = 0$

$\overset{\text{def.}}{\Longleftrightarrow}$ λ is a root of characteristic polynomial of A

$$\chi_A(t) = \det(A - tI)$$

i.e., λ satisfies characteristic equation

$$\chi_A(t) = 0,$$

i.e., $\chi_A(\lambda) = 0.$

6 Example

Find eigenvalues and eigenspaces for the following A.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

Solution: 1. Find characteristic polynomial $\chi_A(t)$

$$\chi_A(t) = \det(\underbrace{A - tI}_{\substack{\text{subtract} \\ t \text{ from} \\ \text{the diagonal} \\ \text{entries of } A}}) = \det\begin{pmatrix} 2-t & 3 \\ 0 & -1-t \end{pmatrix} =$$

$$= (2-t)(-1-t) - 0 \cdot 3 =$$

$$= (2-t)(-1-t)$$

2. Find roots of characteristic polynomial $\chi_A(t)$

(roots of $\chi_A(t)$ ~~are~~ eigenvalues)

$$\chi_A(t) = 0$$

$$(2-t)(-1-t) = 0$$

$$t = 2 \text{ or } t = -1$$

Eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$.

3. How to find eigenspaces?

$$E_\lambda = Null(A - \lambda I)$$

$\lambda_1 = 2$

Find $Null(A - \lambda_1 I) = Null(A - 2I)$

$$\left(\begin{array}{cc|c} 2-2 & 3 & 0 \\ 0 & -1-2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{array}\right).$$

$$\boxed{Null(A-2I) = Span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}$$

$$\boxed{E_2 = Span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}$$

$\lambda_2 = -1$

Find $Null(A - \lambda_2 I) = Null(A + I)$

$$\left(\begin{array}{cc|c} 2+1 & 3 & 0 \\ 0 & -1+1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} \boxed{3} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\boxed{Null(A+I) = Span\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}$$

$$\boxed{E_{-1} = Span\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}$$