# Homework 12 - Solutions

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"(*)" means that the problem is optional.

1. Check that
$$\left\{ u_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$
is an orthogonal basis for the plane $x + y + z = 0$ in $\mathbb{R}^3$. Find the orthogonal projection of $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto this plane. Find the distance from the plane to $v$.

   *Solution.* First, it is easy to see that both vectors lie in the plane $x+y+z = 0$. Since this plane is 2-dimensional, it is enough to check that these vectors are linearly independent to check that this is a basis, but that will follow from checking that they are orthogonal. We check
$$u_1 \cdot u_2 = 1 \cdot 1 + 0 \cdot (-2) + (-1) \cdot 1 = 0$$
so we indeed have orthogonality.

   The projection of $v$ onto the plane, which we denote by $\hat{v}$, is given by
$$\hat{v} = \frac{u_1 \cdot v}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot v}{u_2 \cdot u_2} u_2 = \frac{-2}{2} u_1 + \frac{0}{6} u_2 = -u_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$
The distance from $v$ to the plane is
$$\|v - \hat{v}\| = \left\| \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\| = \sqrt{12} = 2\sqrt{3}.$$
□

2. Apply the Gram-Schmidt algorithm to produce an orthogonal basis of $\mathbb{R}^3$ from the basis.
$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

*Solution.* We set

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then,

$$u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

and

$$u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/6 \\ -1/6 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

Therefore, the resulting orthogonal basis is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix} \right\}$$

□

3. Is the matrix $M = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ orthogonal?

*Solution.* $M$ is orthogonal if and only if $M^T M = I$. We compute:

$$\begin{aligned} M^T M &= \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} \end{aligned}$$

which is not the identity matrix so $M_1$ is not orthogonal.

Alternatively, if $M$ is orthogonal, then $M$ has orthonormal columns, i.e., column vectors are orthogonal to each other and each column vector has length 1. The first column is a vector $\bar{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The length of $\bar{v}_1$ is equal to $\|\bar{v}_1\| = \sqrt{10} \neq 1$. Therefore, $M$ is not orthogonal. □

4. Orthogonally diagonalize the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. That is, find an orthogonal matrix $P$ and diagonal matrix $D$ such that $A = PDP^T$.

*Solution.* We compute that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) - (-\lambda) + \lambda = -\lambda^3 + 3\lambda^2 = \lambda^2(3-\lambda)$$

so the eigenvalues of $A$ are 0 and 3. Thus, we know that

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We now need orthonormal bases of the eigenspaces. We first look at $E_3$.

$$A - 3I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, an orthonormal basis for $E_3$ is $\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Now, we look at $E_0$.

$$A - 0I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, a basis for $E_0$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$. Applying Gram-Schmidt, we replace the second vector with

$$v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = v_2 - \frac{1}{2} v_1 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

to get an orthogonal basis. Thus, $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$. Therefore, we can take

$$P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}.$$

□

5. For any $n \times n$ matrix $A$, show that there is an orthogonal matrix $P$ and diagonal matrix $D$ such that $A^T A = PDP^T$.

   *Solution.* We have that $(A^T A)^T = A^T (A^T)^T = A^T A$. Thus, this matrix is symmetric so the spectral theorem implies that we can find $P$ and $D$ as desired. □

6. Is **True** or **False** that the product of symmetric matrices is symmetric. Justify your answer. How does this relate orthogonal diagonalizability?

*Solution.* False. Take

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then we compute

$$AB = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \qquad (AB)^T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

which are not equal so $AB$ is not symmetric.

This shows that the product of orthogonally diagonalizable matrices might not be orthogonally diagonalizable. □

7. (*) Suppose $W$ is a subspace of $\mathbb{R}^n$. Show that the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$v \mapsto \mathrm{Proj}_W(v)$$

is a linear transformation, where $\mathrm{Proj}_W(v)$ is the orthogonal projection of $v$ onto $W$.

*Solution.* Every vector $v \in \mathbb{R}^n$ can be written uniquely as

$$v = \hat{v} + v^\perp$$

where $\hat{v} = \mathrm{Proj}_W(v) \in W$ and $v^\perp \in W^\perp$ by the orthogonal decomposition theorem.

Let $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Then as $cv = c\hat{v} + cv^\perp$ and $W^\perp$ is also a subspace of $\mathbb{R}^n$, we must have $c\mathrm{Proj}_W(v) = c\hat{v} = \mathrm{Proj}_W(cv)$ by uniqueness of the orthogonal decomposition.

Similarly, $v + w = \hat{v} + v^\perp + \hat{w} + w^\perp = \hat{v} + \hat{w} + v^\perp + w^\perp$. Again, as $W$ and $W^\perp$ are both subspaces of $\mathbb{R}^n$, uniqueness of the orthogonal decomposition implies that $\mathrm{Proj}_W(v + w) = \hat{v} + \hat{w} = \mathrm{Proj}_W(v) + \mathrm{Proj}_W(w)$. Thus, we have shown that $T$ is a linear transformation.

Alternatively, choose an orthogonal basis $\{u_1, \ldots, u_r\}$ of $W$. Then $\mathrm{Proj}_W(v) = \sum_{i=1}^r \frac{u_i \cdot v}{u_i \cdot u_i} u_i$. If $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have $u_i \cdot (cv) = cu_i \cdot v$ and $u_i \cdot (v + w) = u_i \cdot v + u_i \cdot w$ by the properties of the dot product so that

$$\mathrm{Proj}_W(cv) = \sum_{i=1}^r \frac{u_i \cdot (cv)}{u_i \cdot u_i} u_i = \sum_{i=1}^r c \frac{u_i \cdot v}{u_i \cdot u_i} u_i = c \sum_{i=1}^r \frac{u_i \cdot v}{u_i \cdot u_i} u_i = c\,\mathrm{Proj}_W(v)$$

and

$$\begin{aligned} \mathrm{Proj}_W(v + w) &= \sum_{i=1}^r \frac{u_i \cdot (v + w)}{u_i \cdot u_i} u_i \\ &= \sum_{i=1}^r \frac{u_i \cdot v + u_i \cdot w}{u_i \cdot u_i} u_i \\ &= \sum_{i=1}^r \frac{u_i \cdot v}{u_i \cdot u_i} u_i + \sum_{i=1}^r \frac{u_i \cdot w}{u_i \cdot u_i} u_i = \mathrm{Proj}_W(v) + \mathrm{Proj}_W(w). \end{aligned}$$

This implies that $T$ is a linear transformation.

□