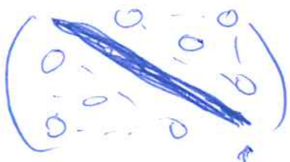
①

Diagonalization

Def An $n\times n$ matrix A is said to be diagonal matrix if all entries except the main diagonal are zeros.

Examples



main diagonal

$$A=\begin{pmatrix}4&0\\0&2\end{pmatrix},\quad B=\begin{pmatrix}7&0&0\\0&0&0\\0&0&1\end{pmatrix}\ \Bigg\}\ \text{diagonal}$$

$$C=\begin{pmatrix}4&1\\0&2\end{pmatrix} \leftarrow \text{not diagonal}$$

← non-zero off diagonal entry

Example Let $A=\begin{bmatrix}5&0\\0&3\end{bmatrix}$.

Find A^5.

Solution: $A^2=A\cdot A=\begin{bmatrix}5&0\\0&3\end{bmatrix}\begin{bmatrix}5&0\\0&3\end{bmatrix}=\begin{bmatrix}5^2&0\\0&3^2\end{bmatrix}$

$A^3=A\cdot A\cdot A=A\cdot A^2=\begin{bmatrix}5&0\\0&3\end{bmatrix}\begin{bmatrix}5^2&0\\0&3^2\end{bmatrix}=\begin{bmatrix}5^3&0\\0&3^3\end{bmatrix}$

$$A^k=\underbrace{A\cdot\ldots\cdot A}_{k\text{ times}}=\begin{bmatrix}5^k&0\\0&3^k\end{bmatrix}$$

In particular, $A^5=\begin{bmatrix}5^5&0\\0&3^5\end{bmatrix}=\begin{bmatrix}3125&\\&243\end{bmatrix}$

Example (Warning!) Let $B=\begin{bmatrix}5&1\\0&3\end{bmatrix}$

Then, $B^2\neq\begin{bmatrix}5^2&1^2\\0&3^2\end{bmatrix}=\begin{bmatrix}25&1\\0&9\end{bmatrix}$ different

Compute $B^2=\begin{bmatrix}5&1\\0&3\end{bmatrix}\begin{bmatrix}5&1\\0&3\end{bmatrix}=\begin{bmatrix}25&8\\0&9\end{bmatrix}$

(2) Example Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

a) Check that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

b) Find A^5.

Solution: a) $$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$= \frac{1}{-1} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 10 & 5 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = A \checkmark$$

b) $A^5 = ?$

$$A^2 = A \cdot A = (PDP^{-1}) \cdot (PDP^{-1}) =$$

$$= PD\underbrace{P^{-1}P}_{= I_2 \text{ (by definition of the inverse of matrix)}}DP^{-1} = PDI_2DP^{-1} = PDDP^{-1} =$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$= PD^2P^{-1}$$

$$A^3 = A \cdot A \cdot A = A \cdot A^2 = (PDP^{-1})(PD^2P^{-1}) = PD\overbrace{P^{-1}P}^{= I_2}D^2P^{-1} =$$

$$= PD^3P^{-1}$$

$\vdots$

$$A^k = PD^kP^{-1}$$

(3) In particular,

$$A^5 = P D^5 P^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}^5 \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^5 & 0 \\ 0 & 3^5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 3125 & 0 \\ 0 & 243 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 6250 & 3125 \\ -243 & -243 \end{bmatrix} = \begin{bmatrix} 6007 & 2882 \\ -5764 & -2639 \end{bmatrix}$$

Def. A, B – $n \times n$ square matrix.

A is similar to B if there exists an $n \times n$ invertible matrix P such that $A = PBP^{-1}$.

Example $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Fact: A is similar to B $\longrightarrow$

$\longrightarrow$ A and B have the same characteristic polynomial

$\longrightarrow$ A and B have the same eigenvalues

Example

$$\chi_{\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}}(t) = \det \begin{bmatrix} 7-t & 2 \\ -4 & 1-t \end{bmatrix} = (7-t)(1-t) - (-4) \cdot 2 = 7 + t^2 - 8t + 8 = t^2 - 8t + 15$$

$$\chi_{\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}}(t) = \det \begin{bmatrix} 5-t & 0 \\ 0 & 3-t \end{bmatrix} = (5-t)(3-t) = t^2 - 8t + 15$$

(4) Eigenvalues of $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ are 5 and 3.

<u>Warning!</u>

A and B have the same eigenvalues/ same characteristic polynomial $\not\Rightarrow$ A is similar to B

A and B can be brought to the same reduced row echelon form $\not\Rightarrow$ A is similar to B.

<u>Example</u>

(See homework) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is <u>not</u> similar to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

though both matrices have the same characteristic polynomial $(1-t)^2$ and eigenvalue 1.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Difference between $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ that

$$E_1\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \neq E_1\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

eigenspaces for eigenvalue 1.

$$E_1\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

$$E_1\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

(5) Def. An $n \times n$ matrix A is said to be diagonalizable if it is similar to diagonal matrix.

Example $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is diagonalizable as it is similar to matrix $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1}$$

diagonal

Warning! Not every matrix is diagonalizable.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Thm If $\bar{v}_1, \dots, \bar{v}_k$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of a matrix A, then $\{\bar{v}_1, \dots, \bar{v}_k\}$ are linearly independent.

(6) <u>Example</u>

we showed that for $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$

we have eigenvectors for $\lambda_1 = 2$ are

$\overline{x} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where $x_1 \neq 0$

and for $\lambda_2 = -1$ are $\overline{x} = x_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

where $x_2 \neq 0$.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are linearly independent;

<u>Theorem</u> (The Diagonalization Thm)

Let A be $n \times n$ matrix.

A is diagonalizable $\Longleftrightarrow$

$\Longleftrightarrow$ A has n linearly independent eigenvectors.

A is diagonalizable $\Longrightarrow$ $A = PDP^{-1}$ where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ - eigenvalues with multiplicity.

and $$P = \begin{pmatrix} | & | & & | \\ \overline{v_1} & \overline{v_2} & \cdots & \overline{v_n} \\ | & | & & | \end{pmatrix}$$ where $\overline{v_1}, \ldots, \overline{v_n}$ - linearly independent

$\overline{v_1}$ is an eigenvector corresponding to eigenvalue λ_1

$\overline{v_2}$ is an eigenvector corresponding to eigenvalue λ_2

$\vdots$

$\overline{v_n}$ is an eigenvector corresponding to eigenvalue λ_n

7 Example

Dиagonalize the matrix
$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

Solution

1. Find characteristic polynomial
$$\chi_A(t) = \det\left(\begin{bmatrix} 7-t & 2 \\ -4 & 1-t \end{bmatrix}\right) = (7-t)(1-t)-(-4)\cdot 2 = t^2-8t+15$$

2. Find roots of $\chi_A(t)$ = eigenvalues of A
$$t^2-8t+15=0$$

Notice
$$(t-5)(t-3)=0$$
$t_1=5$ $t_2=3$

or

$$t^2-8t+15=0$$
$$\xrightarrow{\text{descriminant}} D = (-8)^2-4\cdot 1\cdot 15 = 64-60 = 2^2$$
$$t_{1,2} = \frac{8\pm\sqrt{D}}{2\cdot 1} = \frac{8\pm 2}{2} \rightarrow t_1=5,\ t_2=3$$

$\lambda_1 = 5$ and $\lambda_2 = 3$ - eigenvalues of A

Remark: A is $\underline{2\times 2}$ matrix and has $\underline{2}$ distinct eigenvalues $\rightarrow$ A is diagonalizable
$\downarrow$
2 linearly independent eigenvectors.

3. Find eigenvectors for $\lambda_1=5$ and $\lambda_2=3$

$\lambda_1 = 5$

$E_{\lambda_1} = E_5 = Null(A-5I)$

$$(A-5I|\bar{0}) \rightarrow \left(\begin{array}{cc|c} 7-5 & 2 & 0 \\ -4 & 1-5 & 0 \end{array}\right) \Rightarrow$$
$$\Rightarrow \left(\begin{array}{cc|c} \boxed{2} & 2 & 0 \\ -4 & -4 & 0 \end{array}\right) \xrightarrow{R_2\to R_2+2R_1} \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$$
$$\xrightarrow{R_1\to\frac{R_1}{2}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} x_1 = -x_2 \\ x_2 - \text{free} \end{cases}$$

$\bar{x} = x_2\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where $x_2\neq 0$ - eigenvectors for $\lambda_1=5$

In particular $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector ($x_2=-1$)

$\lambda_2 = 3$

$E_{\lambda_2} = E_3 = Null(A-3I)$.

$$(A-3I|\bar{0}) \rightarrow \left(\begin{array}{cc|c} 7-3 & 2 & 0 \\ -4 & 1-3 & 0 \end{array}\right) \rightarrow$$
$$\rightarrow \left(\begin{array}{cc|c} 4 & 2 & 0 \\ -4 & -2 & 0 \end{array}\right) \xrightarrow{R_2\to R_2+R_1} \left(\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{R_1\to\frac{R_1}{4}} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} x_1 = -\frac{1}{2}x_2 \\ x_2 - \text{free} \end{cases}$$

$\bar{x} = x_2\cdot\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$, where $x_2\neq 0$ - eigenvectors for $\lambda_2=5$

In particular, $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector for $\lambda_2=5$ ($x_2=-2$)

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4. Write down D and P.

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

Remarks. 1. We can pick any eigenvectors as long as we pick linearly independent

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix} \quad \text{work as well}$$

2. Order of eigenvalues in D doesn't matter as long as correct order in P.

If $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, then $P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

Thm A – $n \times n$ matrix.
let $\lambda_1, \dots, \lambda_r$ – all distinct eigenvalues of A.

$\boxed{A \text{ is diagonalizable}} \longleftrightarrow \boxed{\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \dots + \dim(E_{\lambda_r}) = n}$

Example Let $A = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$.

Is A diagonalizable?

Solution

1. Characteristic polynomial $\chi_A(t) = \det(A-t) =$

$$= \det\begin{pmatrix} 5-t & 1 \\ 0 & 5-t \end{pmatrix} = (5-t)^2$$

2. Roots of $\chi_A(t)$ = eigenvalues of A.

$$(5-t)^2 = 0 \longrightarrow \lambda_1 = \lambda_2 = 5 \leftarrow \text{twice the same eigenvalue}$$

3. Find eigenspace for eigenvalue 5.

$$E_5 = Null\,(A - 5I)$$

$$(A-5I\,|\,\bar{0}) \to \left(\begin{array}{cc|c} 5-5 & 1 & 0 \\ 0 & 5-5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{cases} x_2 = 0 \\ x_1 - \text{free} \end{cases}$$

$$\bar{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow \text{all solutions}$$

$$E_5 = Span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

$$\dim(E_5) = 1 \neq 2 \leftarrow \text{size of } A \implies$$

$\longrightarrow$ A is not diagonalizable

(10) Example

Let $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

a) Is A diagonalizable?

b) If yes, then diagonalize it.

Solution:

a) 1. Find characteristic polynomial ~~and eigen~~

$\chi_A(t) = \det(A - tI) = \det\begin{bmatrix} 4-t & 0 & -2 \\ 2 & 5-t & 4 \\ 0 & 0 & 5-t \end{bmatrix} =$

$\overset{\text{use 3rd row}}{\underset{\downarrow}{=}} (5-t) \cdot \det\begin{bmatrix} 4-t & 0 \\ 2 & 5-t \end{bmatrix} = (5-t)(4-t)(5-t) = (5-t)^2(4-t)$

$\chi_A(t) = 0 \Rightarrow (5-t)^2(4-t) = 0 \to t_1 = t_2 = 5$, $t_3 = 4$

2. Find eigenvalues of A:

$\lambda_1 = \lambda_2 = 5$ and $\lambda_3 = 4$ ← eigenvalues

3. Find eigenspaces

$\lambda_1 = \lambda_2 = 5$

$E_5 = Null(A - 5I)$

$(A - 5I \mid \bar{0}) \to \left(\begin{array}{ccc|c} 4-5 & 0 & -2 & 0 \\ 2 & 5-5 & 4 & 0 \\ 0 & 0 & 5-5 & 0 \end{array}\right) \to$

$\to \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$

$\xrightarrow[R_1 \to -R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{cases} x_1 = -2x_3 \\ x_2, x_3 \text{ - free} \end{cases}$

$\bar{x} = \begin{pmatrix} -2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$E_5 = Span\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\dim(E_5) = 2$

$\lambda_3 = 4$

$E_4 = Null(A - 4I)$

$(A - 4I \mid \bar{0}) \to \left(\begin{array}{ccc|c} 4-4 & 0 & -2 & 0 \\ 2 & 5-4 & 4 & 0 \\ 0 & 0 & 5-4 & 0 \end{array}\right) \to$

$\to \left(\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to$

$\to \left(\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \to$

$\to \left(\begin{array}{ccc|c} \boxed{1} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\to \begin{cases} x_1 = -\frac{1}{2}x_2 \\ x_3 = 0 \\ x_2 \text{ - free} \end{cases} \to \bar{x} = \begin{pmatrix} -\frac{1}{2}x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

$E_4 = Span\left\{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}\right\} \to \dim(E_4) = 1$

(11) $\dim(E_5) + \dim(E_4) = 2 + 1 = 3 \leftarrow$ size of A

$\rightarrow$ A is diagonalizable

b) $$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

linearly independent eigenvectors for eigenvalue 5

eigenvector for eigenvalue 4 (took $x_2 = -2$ in $\bar{x} = x_2 \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$)

$\underline{A = P D P^{-1}}$.