(1) # Geometry in $\mathbb{R}^n$

Def Dot product / standard inner product of vectors $\bar{u}, \bar{v}$ in $\mathbb{R}^n$ is

$$\bar{u} \cdot \bar{v} = u_1 v_1 + \dots + u_n v_n \quad , \text{ where}$$

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Matrix multiplication interpretation

$$\underset{\substack{\uparrow \\ \text{dot product}}}{\bar{u} \cdot \bar{v}} = \underset{\substack{\uparrow \\ \text{matrix multiplication}}}{\bar{u}^{T} \cdot \bar{v}} \;\leftarrow \text{transpose} = \underbrace{[u_1 \dots u_n]}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{n \times 1} = \underbrace{\overset{\text{number} \downarrow}{}}_{1 \times 1}$$

Example

$$\bar{u} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$\bar{u} \cdot \bar{v} = 3 \cdot 6 + (-1) \cdot (-2) + (-5) \cdot 3 = 18 + 2 - 15 = 5$$

$$\bar{v} \cdot \bar{u} = [6 \;\; -2 \;\; 3] \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 6 \cdot 3 + (-2) \cdot (-1) + 3 \cdot (-5) = 5$$

Properties:

$$\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$$

$$\bar{u} \cdot \bar{u} = 0 \iff \bar{u} = \bar{0}$$

$$\bar{u} \cdot \bar{u} \geq 0 \text{ for any } \bar{u}.$$

$$\bar{u} \cdot \bar{u} = 3 \cdot 3 + (-1) \cdot (-1) + (-5)(-5) = 9 + 1 + 25 = 35 > 0$$

(2) Def. Length of a vector $\bar{u}$:

$$\|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

notation for length

dot product

where $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Example find length of vector

$$\bar{u} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$$

Solution: $\bar{u} \cdot \bar{u} = 35$ ← computed in previous example.

$$\|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}} = \sqrt{35}$$

Def. $\bar{u}$ is Unit vector if $\|\bar{u}\| = 1$

Example Is vector $\bar{v} = \begin{bmatrix} \frac{3}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ -\frac{5}{\sqrt{35}} \end{bmatrix}$ unit?

Solution: $\bar{v} \cdot \bar{v} = \frac{3 \cdot 3}{\sqrt{35} \cdot \sqrt{35}} + \frac{(-1) \cdot (-1)}{\sqrt{35} \cdot \sqrt{35}} + \frac{(-5)(-5)}{\sqrt{35} \cdot \sqrt{35}} =$

$= \frac{9 + 1 + 25}{35} = 1 \rightsquigarrow \|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}} = 1.$

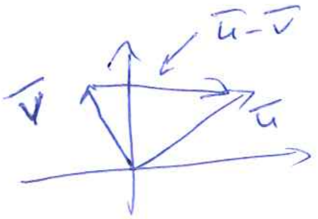
Notice that $\bar{v} = \frac{1}{\sqrt{35}} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = \frac{1}{\left\| \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \right\|} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

Definition Normalization of $\bar{u} \neq \bar{0}$ (non zero vector) is obtaining a unit vector in the same direction as $\bar{u}$,

i.e. $\bar{v} = \frac{1}{\|\bar{u}\|} \cdot \bar{u}$ ← normalization of $\bar{u}$

③ **Def** Distance between two vectors $\bar{u}$ and $\bar{v}$:

$$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$$



Example compute the distance between

$$\bar{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad \bar{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$$

Solution:

1. Compute $\bar{u} - \bar{z}$

$$\bar{u} - \bar{z} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$$

2. $d(\bar{u}, \bar{z}) = \|\bar{u} - \bar{z}\| = \sqrt{(\bar{u}-\bar{z})\cdot(\bar{u}-\bar{z})} =$

$$= \sqrt{4\cdot 4 + (-4)\cdot(-4) + (-6)\cdot(-6)} =$$

$$= \sqrt{16 + 16 + 36} = \sqrt{68} = \sqrt{4\cdot 17} = 2\sqrt{17}$$

Def. $\bar{u}$ is orthogonal to $\bar{v}$ iff $\bar{u}\cdot\bar{v} = 0$. Notation: $\bar{u} \perp \bar{v}$ orthogonal.

Remark $\bar{0}$ vector orthogonal to any vector as

$$\bar{0}\cdot\bar{u} = 0.$$

Example: Is $\bar{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$ orthogonal to $\bar{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$?

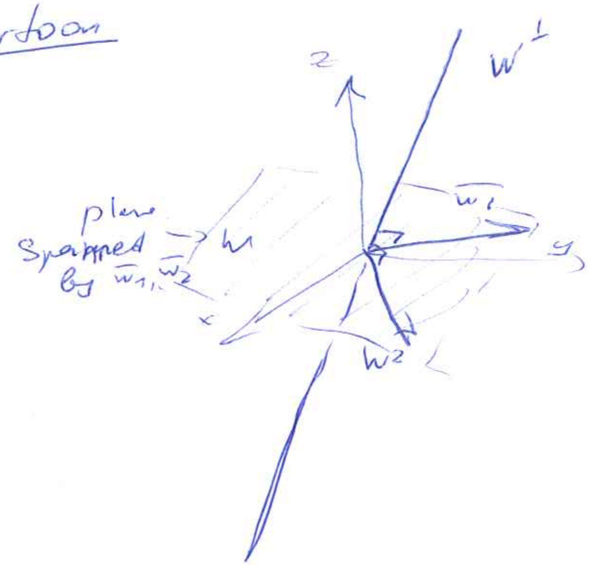
$\bar{a}\cdot\bar{b} = 8\cdot(-2) + (-5)\cdot(-3) = -16 + 15 = -1 \neq 0 \Rightarrow$ not orthogonal.

(4) <u>Def.</u> W be a subset in $\mathbb{R}^n$ (not necessarily subspace)

Then, its <u>orthogonal complement</u> is

$$W^{\perp} = \{ \overline{v} \in \mathbb{R}^n \mid \overline{v} \perp \overline{w} \text{ for all } \overline{w} \text{ in } W \}$$

<u>Cartoon</u>



<u>Theorem</u> A - $m \times n$ matrix.

$$(\mathrm{Col}(A))^{\perp} = \mathrm{Null}(A^T)$$

<u>Example</u>

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{then}} A^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle| \ a\text{ - any number} \right\}$$

$$\mathrm{Null}(A^T) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \middle| \ b\text{ - any number} \right\}$$

$$\underbrace{\begin{pmatrix} a \\ 0 \end{pmatrix}}_{\text{vector in Col}(A)} \cdot \underbrace{\begin{pmatrix} 0 \\ b \end{pmatrix}}_{\text{vector in Null}(A^T)} = 0 \cdot a + 0 \cdot b = 0. \rightarrow$$

Any vector in $\mathrm{Null}(A^T)$ is orthogonal to any vector in $\mathrm{Col}(A)$.

⑤ <u>Def</u> Let $\overline{v_1}, \dots, \overline{v_k}$ be vectors in $\mathbb{R}^n$.

1) $\{\overline{v_1}, \dots, \overline{v_k}\}$ is <u>orthogonal set</u> if $\overline{v_i} \perp \overline{v_j}$ for all $i \neq j$

2) $\{\overline{v_1}, \dots, \overline{v_k}\}$ is <u>orthonormal set</u> if
it is orthogonal set <u>and</u>
$\|\overline{v_i}\| = 1$ for all $i = 1, \dots, k$

3) $\{\overline{v_1}, \dots, \overline{v_k}\}$ is <u>orthogonal basis</u> if
it is orthogonal set <u>and</u> basis

4) $\{\overline{v_1}, \dots, \overline{v_k}\}$ is <u>orthonormal basis</u> if
it is orthonormal set <u>and</u> basis.

<u>Thm</u> $\overline{v_1}, \dots, \overline{v_k}$ orthogonal and $\overline{v_i} \neq \overline{0}$ for all i.
Then, $\overline{v_1}, \dots, \overline{v_k}$ - linearly independent.

<u>Example</u> Is $\{\overline{v_1}, \overline{v_2}, \overline{v_3}\}$ orthogonal basis?
orthonormal basis?

$$\overline{v_1} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \overline{v_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \overline{v_3} = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

<u>solution:</u> 1. Check that $\overline{v_1}, \overline{v_2}, \overline{v_3}$ if orthogonal set?

$\overline{v_2} \cdot \overline{v_1} = \overline{v_1} \cdot \overline{v_2} = 2 \cdot 1 + (-1) \cdot 2 + 0 \cdot 1 = 0$

$\overline{v_3} \cdot \overline{v_1} = \overline{v_1} \cdot \overline{v_3} = 2 \cdot (-1) + (-1) \cdot (-2) + 0 \cdot 5 = 0$

$\overline{v_3} \cdot \overline{v_2} = \overline{v_2} \cdot \overline{v_3} = 1 \cdot (-1) + 2 \cdot (-2) + 1 \cdot 5 = 0$

$\overline{v_i} \cdot \overline{v_j} = 0 \quad \forall\, i \neq j \Rightarrow \{\overline{v_1}, \overline{v_2}, \overline{v_3}\}$ - orthogonal set.

2. $\overline{v_1}, \overline{v_2}, \overline{v_3}$ - orthogonal set + $\overline{v_i} \neq \overline{0}$ for all i $\xRightarrow{\text{by Thm}}$ $\overline{v_1}, \overline{v_2}, \overline{v_3}$ are linearly independent

(6) (1)+(2) $\Rightarrow$ $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ – orthogonal basis.

3. $\|\bar{v}_1\| = \sqrt{\bar{v}_1 \cdot \bar{v}_1} = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5} \neq 1$

$\rightarrow$ basis is not orthonormal

Why are orthogonal bases useful?

Example: Find coordinates of $\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ with respect to basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

$$\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

(i.e. find x_1, x_2, x_3 such that $\bar{v} = x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + x_3 \bar{v}_3$)

Solution: Traditional way: Solve system $A\bar{x} = \bar{v}$, where $A = \begin{bmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{bmatrix}$

and $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\leftarrow$ coordinates you look for.

Easier way if $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ - orthogonal.

Notice: $\bar{v} \cdot \bar{v}_1 = x_1 \cdot \bar{v}_1 \cdot \bar{v}_1 + x_2 \cdot \underbrace{\bar{v}_2 \cdot \bar{v}_1}_{=0} + x_3 \underbrace{\bar{v}_3 \cdot \bar{v}_1}_{=0}$ $\Rightarrow$

from orthogonality

$$\Rightarrow x_1 = \frac{\bar{v} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}$$

$$x_1 = \frac{1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 1}{2^2 + (-1)^2 + 0^2} = \frac{2}{5}$$

$$x_2 = \frac{\bar{v} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} = \frac{1 \cdot 1 + 0 \cdot 2 + 1 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$x_3 = \frac{\bar{v} \cdot \bar{v}_3}{\bar{v}_3 \cdot \bar{v}_3} = \frac{1 \cdot (-1) + 0 \cdot (-2) + 1 \cdot 5}{(-1)^2 + (-2)^2 + 5^2} = \frac{4}{30} = \frac{2}{15}$$

(7)

$$\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{5}\bar{v}_1 + \frac{1}{3}\bar{v}_2 + \frac{2}{15}\bar{v}_3 .$$

<u>Thm</u> Let $\{\bar{v}_1, \dots, \bar{v}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then for each $\bar{v}$ in W, we have

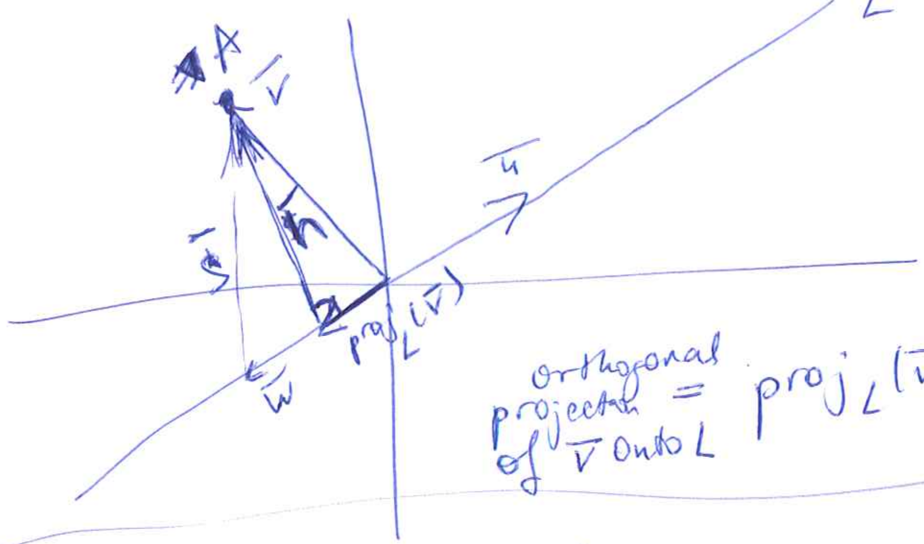
$$\bar{v} = x_1\bar{v}_1 + x_2\bar{v}_2 + \dots + x_k\bar{v}_k,$$

where $x_i = \dfrac{\bar{v} \cdot \bar{v}_i}{\bar{v}_i \cdot \bar{v}_i}$ $\quad (i = 1, 2, \dots, k)$

<u>Geometric interpretation</u>

line $L = \text{span}\{\bar{u}\}$

$\mathbb{R}^2$



orthogonal projection of $\bar{v}$ onto L $= \text{proj}_L(\bar{v}) = \dfrac{\bar{v} \cdot \bar{u}}{\bar{u} \cdot \bar{u}}\,\bar{u}$

<u>Claim:</u> 1) $\bar{h} = (\bar{v} - \text{proj}_L(\bar{v})) \perp \bar{u}$ (i.e. $\bar{h}$ is orthogonal to $\bar{u}$, $\bar{h} \cdot \bar{u} = 0$)

Notice that $\bar{v} = \text{proj}_L(\bar{v}) + \bar{h}$

2) $\text{proj}_L(\bar{v})$ is the closest vector in L to $\bar{v}$

(see picture: $\|\bar{s}\| > \|\bar{h}\|$.

and $\|\bar{h}\| = \|\bar{v} - \text{proj}_L(\bar{v})\|$ - the distance from the point A (end of $\bar{v}$) point identified with $\bar{v}$ and line L

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Example Find the orthogonal projection of $\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ onto line spanned by $\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Solution: $L = \text{Span}\{\bar{v}_1\}$

$$\text{proj}_L(\bar{v}) = \frac{\bar{v} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 = \frac{2}{5} \bar{v}_1$$

distance from the point identified with $\bar{v}$ and L?

$$\text{dist}(\bar{v}, L) = \|\bar{v} - \text{proj}_L(\bar{v})\| =$$

$$= \left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 - \frac{4}{5} \\ 0 + \frac{2}{5} \\ 1 - 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} \right\| =$$

$$= \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + 1} = \sqrt{\frac{1}{25} + \frac{4}{25} + 1} = \sqrt{\frac{1}{5} + 1} = \boxed{\sqrt{\frac{6}{5}}}$$

What is projection on the subspace?
(not necessarily line)

Thm Let W be any subspace in R^n.
Any vector $\bar{v}$ in R^n has a unique decomposition

$\bar{v} = \bar{w} + \bar{z}$, where $\bar{w}$ is in W
$\bar{z}$ is in $W^{\perp}$, i.e. $\bar{z}$ is orthogonal to every vector in W.

In particular,

$\bar{w} = \text{proj}_W(\bar{v})$. — projection of $\bar{v}$ onto W, — the closest vector in W to $\bar{v}$.

$\bar{z} = (\bar{v} - \bar{w}) = (\bar{v} - \text{proj}_W(\bar{v})) \perp \bar{w}$, $\|\bar{z}\|$ — distance from the end point of $\bar{v}$ to W.