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Symmetric matrices & their diagonalization

Def. A is a symmetric matrix if $A^T = A$.

Example:

$$A = \begin{bmatrix} 0 & -1 & 5 \\ -1 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix} \to A^T = \begin{bmatrix} 0 & -1 & 5 \\ -1 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix}$$

Notice $A = A^T \to \begin{bmatrix} 0 & -1 & 5 \\ -1 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix}$ - symmetric

different

$$B = \begin{bmatrix} 0 & -1 & 3 \\ -1 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix} \to B^T = \begin{bmatrix} 0 & -1 & 5 \\ -1 & 7 & 3 \\ 3 & 3 & 5 \end{bmatrix} \to$$

different

$\to B \neq B^T \to \begin{bmatrix} 0 & -1 & 3 \\ -1 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix}$ - not symmetric

$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ - symmetric

$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ - not symmetric.

(18) <u>Thm</u> If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

<u>Example</u>

Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. Find eigenvectors of A and show that two eigenvectors from different eigenspaces are orthogonal.

<u>Solution:</u> 1. Characteristic polynomial

$$\chi_A(t) = \det\begin{pmatrix} 1-t & 3 \\ 3 & 1-t \end{pmatrix} = (1-t)^2 - 3^2 =$$
$$= (1-t-3)(1-t+3) =$$
$$= (-t-2)(-t+4) =$$
$$= (t+2)(t-4)$$

2. Roots of $\chi_A(t)$ = eigenvalues of A

$\chi_A(t) = 0 \qquad (t+2)(t-4) = 0$

$t = -2 \qquad t = 4$

$\lambda_1 = -2$, $\lambda_2 = 4$ are eigenvalues of A.

3. Find eigenvectors

$\lambda_1 = -2$

$$E_{-2} = \text{Null}(A - (-2)I) =$$
$$= \text{Null}\begin{pmatrix} 1-(-2) & 3 \\ 3 & 1-(-2) \end{pmatrix} = \text{Null}\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} \boxed{1} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{cases} x_1 = -x_2 \\ x_2 - \text{free} \end{cases}$$

$\bar{x} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where $x_2 \neq 0$ is eigenvector for $\lambda_1 = -2$

$\lambda_2 = 4$

$$E_4 = \text{Null}(A - 4I) =$$
$$\text{Null}\begin{pmatrix} 1-4 & 3 \\ 3 & 1-4 \end{pmatrix} = \text{Null}\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\bar{x} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where $x_2 \neq 0$ is an eigenvector for $\lambda_2 = 4$

(19) Let $\bar{u} = \begin{bmatrix} -a \\ a \end{bmatrix}$, where $a \neq 0$, be an eigenvector for $\lambda_1 = -2$

and $\bar{v} = \begin{bmatrix} b \\ b \end{bmatrix}$, where $b \neq 0$, be an eigenvector for $\lambda_2 = 4$.

$\bar{u} \cdot \bar{v} = (-a) \cdot b + a \cdot b = -ab + ab = 0 \Rightarrow \bar{u} \perp \bar{v}$,

i.e. any vector in E_{-2} is orthogonal to any vector in E_4.

Spectral Theorem

Let A be $n \times n$ matrix.

A - symmetric $\overset{\text{definition}}{\Longleftrightarrow} A^T = A \Longleftrightarrow$ ~~such that~~

~~there exists P~~

$\cancel{A = P}$

$\Longleftrightarrow$ there exists an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors of A

$\Longleftrightarrow$ A is orthogonally diagonalizable,

i.e. there exists an orthogonal matrix P such that $A = PDP^{-1}$, where D is diagonal. (Notice that $P^{-1} = P^T$ in this case)

Warning: A - diagonalizable $\not\Rightarrow A = A^T$

Example $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(20) How to get orthogonal matrix P if A is symmetric?

Example Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.
Find an orthogonal matrix P such that $A = PDP^T$, where D = diagonal.

Solution: 1. We found that
$\lambda_1 = -2$ and $\lambda_2 = 4$ are eigenvalues of A.

2. $E_{-2} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

$E_4 = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Recall P is orthogonal if columns of P form orthonormal basis (i.e, orthogonal + lengths of vectors equal to 1)

We have Every vector in E_{-2} orthogonal to vector in E_4.

Take vector in E_{-2}:
$\bar{u}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is in E_{-2}, then $\bar{v}_1 = \frac{\bar{u}_1}{\|\bar{u}_1\|}$ = the unit vector in E_{-2}

$\bar{u}_1 \cdot \bar{u}_1 = (-1)^2 + 1^2 = 2 \Rightarrow \|\bar{u}_1\| = \sqrt{2}$

$\bar{v}_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ - the unit vector in E_{-2}

$\bar{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in E_4, then $\bar{v}_2 = \frac{\bar{u}_2}{\|\bar{u}_2\|}$ - the unit vector in E_4

$\bar{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ - the unit vector in E_4

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Therefore,

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$

and $P^{-1} = P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

General algorithm for finding P if A = $n \times n$ symmetric matrix

Step 1. Find eigenvalues of A. Let $\lambda_1, \dots, \lambda_k$ - all distinct eigenvalues

Step 2. Find eigenspaces $E_{\lambda_1}, \dots, E_{\lambda_k}$

Step 3. For each λ_i: a) find basis of E_{λ_i}

b) make orthogonal bases out of bases you found in (a) (Gram-Schmidt process)

c) divide each vector in the bases from (b) by its length $\to$ orthonormal bases of E_{λ_i}

Step 4. ~~To~~ build P use vectors from orthonormal bases of $E_{\lambda_1}, \dots, E_{\lambda_k}$.

Notice that if A is symmetric, then

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \dots + \dim(E_{\lambda_k}) = \underset{\text{size of } A}{\underset{\uparrow}{n}}.$$