⑤ Def Let $\bar{v}_1, \dots, \bar{v}_k$ be vectors in $\mathbb{R}^n$.

1) $\{\bar{v}_1, \dots, \bar{v}_k\}$ is orthogonal set if $\bar{v}_i \perp \bar{v}_j$ for all $i \neq j$

2) $\{\bar{v}_1, \dots, \bar{v}_k\}$ is orthonormal set if it is orthogonal set and $\|\bar{v}_i\| = 1$ for all $i = 1, \dots, k$

3) $\{\bar{v}_1, \dots, \bar{v}_k\}$ is orthogonal basis if it is orthogonal set and basis

4) $\{\bar{v}_1, \dots, \bar{v}_k\}$ is orthonormal basis if it is orthonormal set and basis.

Thm $\bar{v}_1, \dots, \bar{v}_k$ orthogonal and $\bar{v}_i \neq \bar{0}$ for all i. Then, $\bar{v}_1, \dots, \bar{v}_k$ - linearly independent.

Example Is $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ orthogonal basis? orthonormal basis?

$$\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \bar{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

solution:

1. Check that $\bar{v}_1, \bar{v}_2, \bar{v}_3$ if orthogonal set?

$\bar{v}_2 \cdot \bar{v}_1 = \bar{v}_1 \cdot \bar{v}_2 = 2 \cdot 1 + (-1) \cdot 2 + 0 \cdot 1 = 0$

$\bar{v}_3 \cdot \bar{v}_1 = \bar{v}_1 \cdot \bar{v}_3 = 2 \cdot (-1) + (-1) \cdot (-2) + 0 \cdot 5 = 0$

$\bar{v}_3 \cdot \bar{v}_2 = \bar{v}_2 \cdot \bar{v}_3 = 1 \cdot (-1) + 2 \cdot (-2) + 1 \cdot 5 = 0$

$\bar{v}_i \cdot \bar{v}_j = 0 \quad \forall\, i \neq j \Rightarrow \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ - orthogonal set.

2. $\bar{v}_1, \bar{v}_2, \bar{v}_3$ - orthogonal set + $\bar{v}_i \neq \bar{0}$ for all i $\xrightarrow{\text{By Thm}}$ $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are linearly independent

(6) $(1)+(2) \Longrightarrow \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ – orthogonal basis.

3. $\|\bar{v}_1\| = \sqrt{\bar{v}_1 \cdot \bar{v}_1} = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5} \neq 1$

$\rightarrow$ basis is not orthonormal

Why are orthogonal bases useful?

Example Find coordinates of $\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ with respect to basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

$$\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

(i.e. find x_1, x_2, x_3 such that
$$\bar{v} = x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + x_3 \bar{v}_3 \text{)}$$

Solution: Traditional way: Solve system
$A\bar{x} = \bar{v}$, where $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

and $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ← coordinates you look for.

Easier way if $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ – orthogonal.

Notice: $\bar{v} \cdot \bar{v}_1 = x_1 \cdot \bar{v}_1 \cdot \bar{v}_1 + x_2 \cdot \underbrace{\bar{v}_2 \cdot \bar{v}_1}_{=0} + x_3 \underbrace{\bar{v}_3 \cdot \bar{v}_1}_{=0} \Rightarrow$

from orthogonality

$$\Rightarrow x_1 = \frac{\bar{v} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}$$

$$x_1 = \frac{1 \cdot 2 + 0 \cdot (-1) + 0 \cdot 1}{2^2 + (-1)^2 + 0^2} = \frac{2}{5}$$

$$x_2 = \frac{\bar{v} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} = \frac{1 \cdot 1 + 0 \cdot 2 + 1 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{\bar{v} \cdot \bar{v}_3}{} = \frac{1 \cdot (-1) + 0 \cdot (-2) + 1 \cdot 5}{(-1)^2 + (-2)^2 + 5^2} = \frac{4}{30} = \frac{2}{15}$$

(7)

$$\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{5}\bar{v}_1 + \frac{1}{3}\bar{v}_2 + \frac{2}{15}\bar{v}_3 .$$

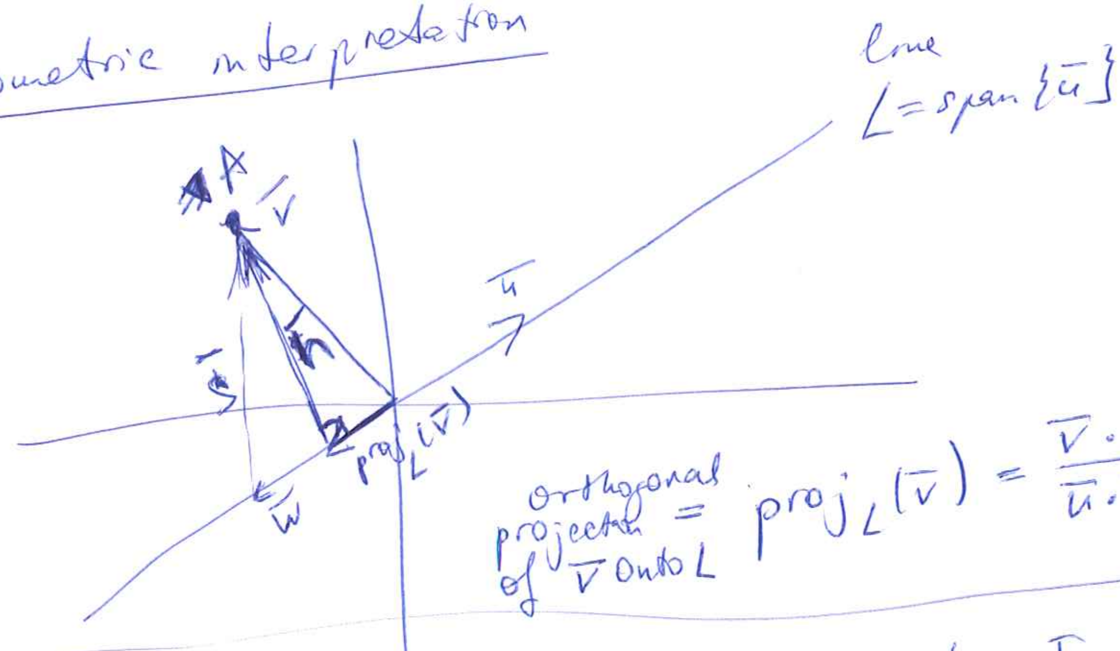
<u>Thm</u> Let $\{\bar{v}_1, \dots, \bar{v}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then for each $\bar{v}$ in W, we have

$$\bar{v} = x_1\bar{v}_1 + x_2\bar{v}_2 + \dots + x_k\bar{v}_k,$$

where $$x_i = \frac{\bar{v}\cdot\bar{v}_i}{\bar{v}_i\cdot\bar{v}_i} \qquad (i = 1, 2, \dots, k)$$

<u>Geometric interpretation</u>

$\mathbb{R}^2$



orthogonal projection of $\bar{v}$ onto L $= \text{proj}_L(\bar{v}) = \dfrac{\bar{v}\cdot\bar{u}}{\bar{u}\cdot\bar{u}}\,\bar{u}$

<u>Claim</u>: 1) $\bar{h} = (\bar{v} - \text{proj}_L(\bar{v})) \perp \bar{u}$ (i.e., $\bar{h}$ is orthogonal to $\bar{u}$, $\bar{h}\cdot\bar{u} = 0$)

Notice that $\bar{v} = \text{proj}_L(\bar{v}) + \bar{h}$

2) $\text{proj}_L(\bar{v})$ is the closest vector in L to $\bar{v}$.

(see picture: $\|\bar{s}\| > \|\bar{h}\|$.

and $\|\bar{h}\| = \|\bar{v} - \text{proj}_L(\bar{v})\|$ - the distance from the point identified with $\bar{v}$

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<u>Example</u> Find the orthogonal projection of $\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ onto line spanned by $\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

<u>Solution:</u> $L = \text{Span}\{\bar{v}_1\}$

$$\text{proj}_L(\bar{v}) = \frac{\bar{v} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 = \frac{2}{5} \bar{v}_1$$

distance from the point identified with $\bar{v}$ and L?

$$\text{dist}(\bar{v}, L) = \| \bar{v} - \text{proj}_L(\bar{v}) \| =$$

$$= \left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 - \frac{4}{5} \\ 0 + \frac{2}{5} \\ 1 - 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} \right\| =$$

$$= \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + 1} = \sqrt{\frac{1}{25} + \frac{4}{25} + 1} = \sqrt{\frac{1}{5} + 1} = \boxed{\sqrt{\frac{6}{5}}}$$

<u>What is projection on the subspace?</u>
(not necessarily line)

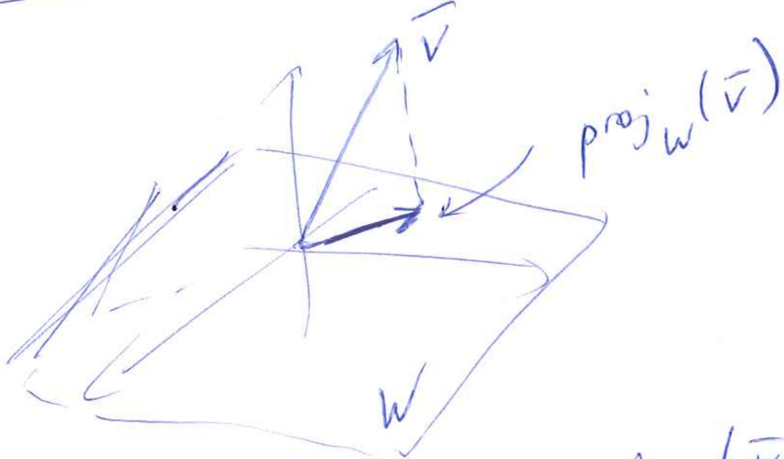
<u>Thm</u> Let W be any subspace in R^n.
Any vector $\bar{v}$ in R^n has a unique decomposition

$\bar{v} = \bar{w} + \bar{z}$, where $\bar{w}$ is in W
$\bar{z}$ is in $W^{\perp}$, i.e. $\bar{z}$ is orthogonal to every vector in W.

In particular,
$\bar{w} = \text{proj}_W(\bar{v})$. – projection of $\bar{v}$ onto W, – the closest vector in W to $\bar{v}$.

$(\bar{v} - \bar{w})$ … $(\bar{v} - \text{proj}_W(\bar{v})) \perp \bar{w}$, $\|\bar{z}\|$ – distance from the end of $\bar{v}$ to W

(9) Cartoon picture:



How do find $proj_W(\bar{v})$?

If $\{\bar{v}_1, \dots, \bar{v}_k\}$ is any orthogonal basis of W, then

$$proj_W(\bar{v}) = \frac{\bar{v}\cdot\bar{v}_1}{\bar{v}_1\cdot\bar{v}_1}\cdot\bar{v}_1 + \dots + \frac{\bar{v}\cdot\bar{v}_k}{\bar{v}_k\cdot\bar{v}_k}\cdot\bar{v}_k$$

Example We showed that

$$\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

is an orthogonal basis

$$\bar{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\frac{2}{5}\bar{v}_1 + \frac{1}{3}\bar{v}_2}_{proj_{Span\{\bar{v}_1, \bar{v}_2\}}(\bar{v})} + \underbrace{\frac{2}{15}\bar{v}_3}_{\text{ortogonal to } Span\{\bar{v}_1, \bar{v}_2\}}$$

(10) Example

Let $\overline{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\overline{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\overline{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\overline{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

$\{\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

Let $W = Span\{\overline{u}_2, \overline{u}_3\}$.

Find $proj_W(\overline{v})$, where $\overline{v} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$

Solution:

$$proj_W(\overline{v}) = \frac{\overline{v} \cdot \overline{u}_2}{\overline{u}_2 \cdot \overline{u}_2} \cdot \overline{u}_2 + \frac{\overline{v} \cdot \overline{u}_3}{\overline{u}_3 \cdot \overline{u}_3} \cdot \overline{u}_3$$

as $\{\overline{u}_2, \overline{u}_3\}$ - orthogonal basis of W.

$$\overline{v} \cdot \overline{u}_2 = 10 \cdot (-2) + (-8) \cdot 1 + 2 \cdot (-1) + 0 \cdot 1 = -20 - 8 - 2 + 0 = -30$$

$$\overline{v} \cdot \overline{u}_3 = 10 \cdot (1) + (-8) \cdot 1 + 2 \cdot (-2) + 0 \cdot (-1) = 10 - 8 - 4 + 0 = -2$$

$$\overline{u}_2 \cdot \overline{u}_2 = (-2)^2 + 1^2 + (-1)^2 + 1^2 = 7$$

$$\overline{u}_3 \cdot \overline{u}_3 = 1^2 + 1^2 + (-2)^2 + (-1)^2 = 7$$

$$proj_W(\overline{v}) = -\frac{30}{7}\overline{u}_2 - \frac{2}{7}\overline{u}_3 = -\frac{30}{7} \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{7}\begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{58}{7} \\ -\frac{32}{7} \\ \frac{34}{7} \\ -4 \end{bmatrix}$$

(11) To compute lengths and distances, we used dot product

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n, \text{ where}$$

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

<u>Question:</u> what $A : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation preserves these quantities?

i.e. For which matrix A, we have $(A\bar{u}) \cdot (A\bar{v}) = \bar{u} \cdot \bar{v}$?

<u>Example</u> $n=2$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Check that for any $\bar{u}, \bar{v}$ in $\mathbb{R}^2$

$$(A\bar{u}) \cdot (A\bar{v}) = \bar{u} \cdot \bar{v}$$

<u>Solution:</u>

$$A\bar{u} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

$$A\bar{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

$$A\bar{u} \cdot A\bar{v} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} =$$

$$= u_1 \cdot v_1 + (-u_2) \cdot (-v_2) =$$

$$= u_1 \cdot v_1 + u_2 \cdot v_2 = \bar{u} \cdot \bar{v}$$

(12) ~~Remark There are a lot of matrices that~~

Def An $n \times n$ matrix M is orthogonal if $(M\bar{u}) \cdot (M\bar{v}) = \bar{u} \cdot \bar{v}$.

Thm The following are equivalent:

M - orthogonal matrix $\Longleftrightarrow (M\bar{u}) \cdot (M\bar{v}) = \bar{u} \cdot \bar{v}$

$\Longleftrightarrow$ columns of M form orthonormal bases

$\Longleftrightarrow M^T \cdot M = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \Longleftrightarrow M^{-1} = M^T$

Example Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

We saw that $(A\bar{u}) \cdot (A\bar{v}) = \bar{u} \cdot \bar{v}$ for any $\bar{u}, \bar{v}$ in $\mathbb{R}^2$

Columns of A are $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$\left.\begin{array}{l} \bar{v}_1 \cdot \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 1 \cdot 0 + 0 \cdot (-1) = 0 \\ \bar{v}_1 \neq 0, \bar{v}_2 \neq 0 \end{array}\right\} \Rightarrow \{\bar{v}_1, \bar{v}_2\} \text{ - orthogonal basis}$$

$$\left.\begin{array}{l} \|\bar{v}_1\| = \sqrt{1^2 + 0^2} = 1 \\ \|\bar{v}_2\| = \sqrt{0^2 + (-1)^2} = 1 \\ \{\bar{v}_1, \bar{v}_2\} \text{ - orthogonal bases} \end{array}\right\} \Rightarrow \{\bar{v}_1, \bar{v}_2\} \text{ - orthonormal bases.}$$

(13)

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

<u>Does orthogonal basis exist?</u>

<u>Yes, Gram-Schmidt method!</u>

<u>Example</u> Let $W = \text{Span}\{\bar{v}_1, \bar{v}_2\}$, where

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Construct an orthogonal basis $\{\bar{u}_1, \bar{u}_2\}$ of W.

<u>Solution:</u> Set $\bar{u}_1 = \bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Recall $\bar{v}_2 - \text{proj}_{\text{Span}\{\bar{u}_1\}}(\bar{v}_2)$ is orthogonal to $\bar{u}_1$

Set $\bar{u}_2 = \bar{v}_2 - \text{proj}_{\text{Span}\{\bar{u}_1\}}(\bar{v}_2) =$

$$= \bar{v}_2 - \frac{\bar{v}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \cdot \bar{u}_1 =$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{0 \cdot 1 + 1 \cdot (-1) + (-1) \cdot 0}{1^2 + (-1)^2 + 0^2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

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Thm (Gram-Schmidt Process)

Let $\{\bar{v}_1, \dots, \bar{v}_k\}$ be a basis for a nonzero ~~[illegible]~~ subspace W of $\mathbb{R}^n$.

Define

$$\bar{u}_1 = \bar{v}_1$$

$$\bar{u}_2 = \bar{v}_2 - \operatorname{proj}_{\operatorname{Span}\{\bar{u}_1\}}(\bar{v}_2) = \bar{v}_2 - \frac{\bar{v}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1$$

$$\bar{u}_3 = \bar{v}_3 - \operatorname{proj}_{\operatorname{Span}\{\bar{u}_1, \bar{u}_2\}}(\bar{v}_3) = \bar{v}_3 - \frac{\bar{v}_3 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}u_1 - \frac{\bar{v}_3 \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2}\bar{u}_2$$

$$\vdots$$

$$\bar{u}_k = \bar{v}_k - \operatorname{proj}_{\operatorname{Span}\{\bar{u}_1, \dots, \bar{u}_{k-1}\}}(\bar{v}_k) =$$

$$= \bar{v}_k - \frac{\bar{v}_k \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 - \frac{\bar{v}_k \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2}\bar{u}_2 - \dots - \frac{\bar{v}_k \cdot \bar{u}_{k-1}}{\bar{u}_{k-1} \cdot \bar{u}_{k-1}}\bar{u}_k$$

Then, $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k\}$ is an orthogonal basis for W.

In particular, 1) $\left\{\frac{\bar{u}_1}{\|\bar{u}_1\|}, \frac{\bar{u}_2}{\|\bar{u}_2\|}, \dots, \frac{\bar{u}_k}{\|\bar{u}_k\|}\right\}$ is an orthonormal basis for W.

2) $\operatorname{Span}\{\bar{v}_1, \dots, \bar{v}_k\} = \operatorname{Span}\{\bar{u}_1, \dots, \bar{u}_k\}$

Example

Let $W = \operatorname{Span}\{\bar v_1, \bar v_2, \bar v_3\}$, where

$$\bar v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \quad \bar v_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \quad \bar v_3 = \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix}.$$

Find an orthogonal basis of W.

Solution: ~~One way~~

Remark: ! Nobody said that $\bar v_1, \bar v_2, \bar v_3$ are linearly independent.

One way: Step 1: Find ~~the~~ basis ~~orthogonal~~ of W

Step 2: Apply Gram-Schmidt

The other way: Let's skip step 1 and see what happens.

$$\bar u_1 = \bar v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$\bar v_2 - \operatorname{proj}_{\operatorname{Span}\{\bar u_1\}}(\bar v_2) = \bar v_2 - \frac{\bar v_2 \cdot \bar u_1}{\bar u_1 \cdot \bar u_1} \bar u_1 =$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{\overbrace{8+5+2}^{15}}{\underbrace{4+25+1}_{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \bar u_2$$

$$\bar v_3 - \operatorname{proj}_{\operatorname{Span}\{\bar u_1, \bar u_2\}}(\bar v_3) = \bar v_3 - \frac{\bar v_3 \cdot \bar u_1}{\bar u_1 \cdot \bar u_1} \bar u_1 - \frac{\bar v_3 \cdot \bar u_2}{\bar u_2 \cdot \bar u_2} \bar u_2 =$$

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$$= \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} - \frac{-45}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} - \frac{\frac{27}{2}}{\underbrace{9+\frac{9}{4}+\frac{9}{4}}} \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftarrow$$ cannot be in basis!

What we got $\bar{v}_3 = \text{proj}_{\text{Span}\{\bar{u}_1, \bar{u}_2\}}(\bar{v}_3) \Longrightarrow$

$\Longrightarrow \bar{v}_3$ is in $\text{Span}\{\bar{u}_1, \bar{u}_2\}$

Therefore, $\text{Span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \text{Span}\{\bar{u}_1, \bar{u}_2\}$ and $\{\bar{u}_1, \bar{u}_2\}$ – orthogonal basis of W.

<u>Remark</u> We could take

$$\bar{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$\bar{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ as $$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{3} \cdot \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

i.e. $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$ point the same direction.