Nonhomogeneous Linear Equations. Method of Undetermined coefficients

The general form

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t). \tag{1}$$

The solution has form

$$y(t) = y_h(t) + Y(t),$$

where $Y(t)$ is *any* specific function that satisfies the nonhomogeneous equation (1) and $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$ is a general solution of the corresponding homogeneous equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{2}$$

(C_1, C_2 are arbitrary constants and y_1, y_2-linear independent solutions of the homogeneous equation (2), i.e. the Wronskian $W(y_1, y_2) \neq 0$)

If $g(t)$ is a sum of several functions, i.e.

$$g(t) = g_1(t) + g_2(t) + \cdots + g_n(t),$$

then any solution of the equation 1 has form

$$Y(t) = Y_1(t) + Y_2(t) + \cdots + Y_n(t),$$

where

$$\begin{aligned} Y_1(t) &\text{ is a solution of } y'' + p(t)y' + q(t)y = g_1(t), \\ Y_2(t) &\text{ is a solution of } y'' + p(t)y' + q(t)y = g_2(t), \\ &\vdots \\ Y_n(t) &\text{ is a solution of } y'' + p(t)y' + q(t)y = g_n(t). \end{aligned}$$

To solve the equation of the form

$$ay''(t) + by'(t) + cy(t) = g(t), \tag{3}$$

where a, b, c-constants and $g(t)$ is a sum of functions of a special form.

Step 1 Find the general solution $y_h(t)$ of the homogeneous equation with constant coefficients

$$ay''(t) + by'(t) + cy(t) = 0.$$

Step 2

Find a specific solution $Y(t)$ of the equation 3.
First, break down $g(t)$ into a sum of functions $g_i(t)$ such that they have one of the forms listed in the table.
Find a specific solution $Y_i(t)$ of the equation

$$ay''(t) + by'(t) + cy(t) = g_i(t) \tag{4}$$

by using the method of undetermined coefficients, i.e. try to find a solution in the form listed in the table.
Let $Y(t)$ be equal to the sum of all $Y_i(t)$.

Step 3

The solution of the equation (3) is $y(t) = y_h(t) + Y(t)$.

How to find a solution for the equation (4)

Let r_1, r_2 be the solutions of the quadratic equation

$$ar^2 + br + c = 0. \tag{5}$$

The form of $g_i(t)$	**The form of the solution $Y_i(t)$**
Polynomial of order n: $$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$$	$$Y_i(t) = t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n),$$ where $$s = \begin{cases} 0, & \text{if} \quad r_1 \neq 0 \text{ and } r_2 \neq 0; \\ 1, & \text{if} \quad r_1 = 0 \text{ and } r_2 \neq 0; \\ 1, & \text{if} \quad r_1 \neq 0 \text{ and } r_2 = 0; \\ 2, & \text{if} \quad r_1 = 0 \text{ and } r_2 = 0. \end{cases}$$
Polynomial of order n multiplied by $e^{\alpha t}$: $$P_n(t)e^{\alpha t} = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n)e^{\alpha t}$$	$$Y_i(t) = t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t},$$ where $$s = \begin{cases} 0, & \text{if} \quad r_1 \neq \alpha \text{ and } r_2 \neq \alpha; \\ 1, & \text{if} \quad r_1 = \alpha \text{ and } r_2 \neq \alpha; \\ 1, & \text{if} \quad r_1 \neq \alpha \text{ and } r_2 = \alpha; \\ 2, & \text{if} \quad r_1 = \alpha \text{ and } r_2 = \alpha. \end{cases}$$
Polynomial of order n multiplied by $e^{\alpha t}\sin(\beta t)$ or $e^{\alpha t}\cos(\beta t)$: $$P_n(t)e^{\alpha t}\sin(\beta t) = \\ = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n)e^{\alpha t}\sin(\beta t)$$ or $$P_n(t)e^{\alpha t}\cos(\beta t) = \\ = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n)e^{\alpha t}\cos(\beta t)$$	$$Y_i(t) = t^s e^{\alpha t}[(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)\cos(\beta t) + \\ +(B_0 t^n + B_1 t^{n-1} + \cdots + B_n)\sin(\beta t)],$$ where $$s = \begin{cases} 0, & \text{if} \quad r_{1,2} \neq \alpha \pm i\beta; \\ 1, & \text{if} \quad r_{1,2} = \alpha \pm i\beta. \end{cases}$$