Mechanical Vibrations

A mass-spring system undergoing free vibration (i.e. without a forcing function) is described by the equation:

$$mu'' + \gamma u' + ku = 0,$$

where $m > 0, \gamma \geqslant 0, k > 0$.
The behavior of the system is determined by the magnitude of the damping coefficient γ relative to m and k.

1. **Undamped system** (when $\gamma = 0$)

   Displacement: $u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

   Oscillation: Yes, periodic (at natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$)/ simple harmonic motion

   Notes: Steady oscillation with constant amplitude $R = \sqrt{C_1^2 + C_2^2}$.

2. **Underdamped system** (when $D = \gamma^2 - 4mk < 0$)

   Displacement: $u(t) = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$

   Oscillation: Yes, quasi-periodic (at quasi-frequency μ)

   Notes: Exponentially-decaying oscillation

3. **Critically Damped system** (when $D = \gamma^2 - 4mk = 0$)

   Displacement: $u(t) = C_1 e^{rt} + C_2 t e^{rt}$

   Oscillation: No

   Notes: The mass could cross its equilibrium position at most once

4. **Overdamped system** (when $D = \gamma^2 - 4mk > 0$)

   Displacement: $u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

   Oscillation: No

   Notes: The mass could cross its equilibrium position at most once