How to solve an initial value problem for a linear homogeneous system

Assume we need to solve the initial value problem

$$\bar{x}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bar{x}(t), \qquad x(h) = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1}$$

where a, b, c, d, h, f, g are given numbers.

Step 1. Make change of variable

$$T = t - h.$$

Step 2. Solve the initial value problem

$$\bar{y}'(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bar{y}(T), \qquad y(0) = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2}$$

1. Find eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
2. Find eigenvectors corresponding to the eigenvalues from the previous item.
3. Find the general solution (with some constants C_1, C_2) for the system

   $$\bar{y}'(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bar{y}(T).$$

   !!! Notice that the vector-function $\bar{y}(T)$ depends on T.

4. Plug the initial condition

   $$\bar{y}(0) = \begin{pmatrix} f \\ g \end{pmatrix}$$

   into the solution from the previous item, i.e.

   plug $T = 0$ in the solution from the previous item and make it equal to $\begin{pmatrix} f \\ g \end{pmatrix}$.

   You will get a system of two equations (USE BOTH EQUATIONS!) with C_1, C_2. Find constants C_1, C_2 which satisfy the system (there exist unique constants).

5. Write the particular solution $\bar{y}(T)$ with the constants C_1, C_2 replaced by their values, which you found in the previous item.

Step 3. Plug $t - h$ instead of T in the solution $\bar{y}(T)$ of the initial value problem 2. The expression you get is equal to the solution $\bar{x}(t)$ of the initial problem 1.