Exact Equation

Consider an equation

$$M(x, y) + N(x, y)y' = 0.$$

To check if the equation exact:

1. Write the equation in the differential form

$$M(x, y)dx + N(x, y)dy = 0.$$

2. Check if the following equality is true:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If the equality is true then the equation is exact. Otherwise, the equation is not exact.

How to solve exact equation:

1. We want to find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x, y).$$

2. First, compute the following integral with respect to x:

$$f(x, y) = \int M(x, y)dx + C_1(y),$$

where $C_1(y)$ is an unknown function of y.

3. Second, compute the following integral with respect to y:

$$f(x, y) = \int N(x, y)dy + C_2(x),$$

where $C_2(x)$ is an unknown function of x.

4. We have two expressions of the same $f(x, y)$ but with the unknown functions $C_1(y)$ and $C_2(x)$. Find the expressions of $C_1(y)$, $C_2(x)$ from the following equality, which you get from two expressions of $f(x, y)$:

$$\int M(x, y)dx + C_1(y) = \int N(x, y)dy + C_2(x).$$

Keep in mind that $C_1(y)$ depends only on y, so there is no x in its expression. Similarly, $C_2(x)$ depends only on x, so there is no y in its expression.

5. Now, you know $C_1(y)$. So, you find $f(x, y)$ by plugging it in the formula for $f(x, y)$:

$$f(x, y) = \int M(x, y)dx + C_1(y).$$

6. The general implicit solution of the initial differential equation is

$$f(x, y) = C,$$

where C is a constant.

7. If you need to find a solution of the initial value problem, i.e. such that $y(x_0) = y_0$, then plug in x_0 instead of x and y_0 instead of y in the equation

$$f(x, y) = C.$$

So, you can find the corresponding C, i.e. the solution of the initial value problem is

$$f(x, y) = f(x_0, y_0).$$