# Stability of an equilibrium solution of autonomous equations

Consider an autonomous differential equation with initial condition

$$y' = f(y), \qquad y(t_0) = y_0, \tag{1}$$

where $f$ is some function of $y$ and $t_0, y_0$ are some constants.

How to find equilibrium solutions:

1. Find **all** constants $a$ such that $f(a) = 0$, i.e. solve algebraic equation $f(y) = 0$.
2. If $a$ is a constant from previous item, then $y(t) = a$ is an equilibrium solution.

How to classify equilibrium solutions:

1. First, draw a vertical line, $y$-axis, where we have direction from bottom to the top, i.e. $-\infty$ on the bottom and $+\infty$ on the top. (Example: $a < b < c$). And mark on it all constants which correspond to equilibrium solutions. In our example they are $a, b, c$. See Figure 1.

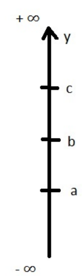



Figure 1

2. For each interval formed by equilibrium solutions on the $y$-axis define the sign of $f(y)$. To define a sign it is needed to pick any number $k$ between any two numbers corresponding to equilibrium solutions and define if $f(k)$ positive or negative. If $f(k) > 0$, then draw an upward arrow on that interval. If $f(k) < 0$, then draw a downward arrow on that interval. Make this procedure with every interval on $y$-axis.

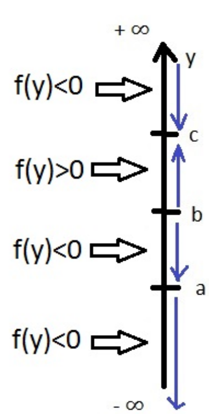
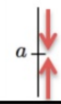
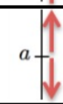
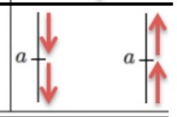



Figure 2

3. Depending on what direction of arrows you see around the point, you call the equilibrium solution either **stable**, or **unstable**, or **semi-stable**.

| Stability of an equilibrium solution y(t) = a | Picture around |
|---|---|
| $y = a$ is (asymptotically) stable |  |
| $y = a$ is unstable |  |
| $y = a$ is semistable |  |

Example: For Figure 2 we have

(a) $y(t) = c$ is stable;

(b) $y(t) = b$ is unstable;

(c) $y(t) = a$ is semi-stable;

How to define the behavior of the solution with initial condition $y(t_0) = y_0$

1. If $y_0$ coincide with a number corresponding to some equilibrium solution, then the solution of the initial equation is $y(t) = y_0$.

2. If you need to define the behavior as $t \to +\infty$. Then, mark a place of $y_0$ on $y$-axis on Figure 2. The limit is equal to the number to which the arrow on the interval points to. If the arrow goes to $+\infty$ or $-\infty$, then the limit is $+\infty$ or $-\infty$, respectively.

   Example: If we have the picture below, then $\lim_{t\to\infty} y(t) = a$.