Solve differential equation using Laplace transform

Assume we are given a linear differential equation

$$ay'' + by' + cy = f(t), \tag{1}$$

where a, b, c-constants, $f(t)$ - some function of t.
And we need to find a solution of it such that

$$y(0) = \alpha, \qquad y'(0) = \beta. \tag{2}$$

The algorithm to solve the differential equation is the following:

1. Take the Laplace transform from both sides of the equation (1). Use the linearity property of Laplace transform to simplify the expression. Also, compute the Laplace transform of $f(t)$. Therefore, you get

$$\mathcal{L}(ay'' + by' + cy) = \mathcal{L}(f(t))$$
$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = \mathcal{L}(f(t))$$

2. Then, use the properties that

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) \qquad \text{and} \qquad \mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0).$$

So, you get

$$a(s^2\mathcal{L}(y) - sy(0) - y'(0)) + b(s\mathcal{L}(y) - y(0)) + c\mathcal{L}(y) = \mathcal{L}(f(t))$$

3. Distribute the numbers a, b over the terms in the parentheses in the expression above. And plug in the values for $y(0)$ and $y'(0)$ from the given initial conditions (2).

$$as^2\mathcal{L}(y) - asy(0) - ay'(0) + bs\mathcal{L}(y) - by(0) + c\mathcal{L}(y) = \mathcal{L}(f(t))$$
$$as^2\mathcal{L}(y) - as\alpha - a\beta + bs\mathcal{L}(y) - b\alpha + c\mathcal{L}(y) = \mathcal{L}(f(t))$$

4. Express $\mathcal{L}(y)$ as a function of s

$$(as^2 + bs + c)\mathcal{L}(y) = \mathcal{L}(f(t)) + as\alpha + a\beta + b\alpha$$
$$\mathcal{L}(y) = \frac{\mathcal{L}(f(t)) + as\alpha + a\beta + b\alpha}{as^2 + bs + c}$$

5. The solution of the initial value problem is equal to

$$y(t) = \mathcal{L}^{-1}\Big(\frac{\mathcal{L}(f(t)) + as\alpha + a\beta + b\alpha}{as^2 + bs + c}\Big)$$

To compute the inverse Laplace transform you need to simplify the function by factoring the denominator if possible and using partial fraction decomposition.