Classification of Critical points

Critical point $(0, 0)$ of linear homogeneous system

$$\bar{x}' = A\bar{x}, \tag{1}$$

where A- 2×2 matrix of numbers such that $det(A) \neq 0$.

Eigenvalues r_1, r_2 of A	Solution of the system 1	Cases for eigenvalues	Type of the critical point	Behavior of solutions
Two distinct	$\bar{x}(t) = C_1\bar{k}_1 e^{r_1 t} + C_2\bar{k}_2 e^{r_2 t}$,	$r_1 > 0, r_2 > 0$	unstable node	all nonzero solutions diverge away from $(0, 0)$ to infinite-distance away (the limit does not exist as $t \to +\infty$)
real eigenvalues	where C_1, C_2 are	$r_1 < 0, r_2 < 0$	asymptotically stable node	all solutions converge to $(0, 0)$ as $t \to +\infty$
r_1, r_2 with eigenvectors $\bar{k}_1, \bar{k}_2$, respectively	some constants	$r_1 \cdot r_2 < 0$	saddle point (unstable)	some trajectories converge to $(0, 0)$, others moves to infinite distance away from $(0, 0)$ (i.e. for some trajectories the limit does not exist as $t \to \pm\infty$)

<table>
<tr><th>Eigenvalues r_1, r_2 of A</th><th>Solution of the system 1</th><th>Cases for eigenvalues</th><th>Type of the critical point</th><th>Behavior of solutions</th></tr>
<tr><td rowspan="3">Complex conjugate
eigenvalues $\lambda \pm i\mu$
with eigenvectors $\bar{a} \pm i\bar{b}$,
respectively</td><td rowspan="3">$\bar{x}(t) = C_1 e^{\lambda t}(\bar{a}\cos(\mu t) - \bar{b}\sin(\mu t)) +$
$+ C_2 e^{\lambda t}(\bar{a}\sin(\mu t) + \bar{b}\cos(\mu t))$
where C_1, C_2 are
some constants</td><td>$\lambda = 0$</td><td>center
(neutrally stable;
NOT asymptotically
stable)</td><td>all nonzero solutions
neither converge to $(0,0)$
nor move to
infinite-distant away
as $t \to \pm\infty$</td></tr>
<tr><td>$\lambda > 0$</td><td>spiral point
(unstable)</td><td>all non-zero solutions
spiral away from $(0,0)$
to infinite-distance away
as $t \to +\infty$</td></tr>
<tr><td>$\lambda < 0$</td><td>spiral point
(asymptotically
stable)</td><td>all solutions
converge to $(0,0)$
as $t \to +\infty$</td></tr>
<tr><td rowspan="2">Repeated real eigenvalue
$r_1 = r_2 = r$
with two linearly
independent eigenvectors
$\bar{k}_1, \bar{k}_2$
i.e. $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$,
where $\alpha \neq 0$
(all nonzero vectors
are eigenvectors)</td><td rowspan="2">$\bar{x}(t) = C_1 \bar{k}_1 e^{rt} + C_2 \bar{k}_2 e^{rt}$
where C_1, C_2 are
some constants</td><td>$r < 0$</td><td>asymptotically
stable
proper node
(star point)</td><td>all solutions
converge to $(0,0)$
as $t \to +\infty$</td></tr>
<tr><td>$r > 0$</td><td>unstable
proper node
(star point)</td><td>all nonzero solutions
diverge away from $(0,0)$
as $t \to +\infty$</td></tr>
<tr><td rowspan="2">Repeated real eigenvalue
$r_1 = r_2 = r$
with only one linearly
independent eigenvector $\bar{k}$</td><td rowspan="2">$\bar{x}(t) = C_1 \bar{k} e^{rt} + C_2(\bar{k} t e^{rt} + \bar{\eta} e^{rt})$,
where $(A - rI)\bar{\eta} = \bar{k}$,
C_1, C_2 are
some constants</td><td>$r < 0$</td><td>asymptotically
stable
improper node</td><td>all solutions
converge to $(0,0)$
as $t \to +\infty$</td></tr>
<tr><td>$r > 0$</td><td>unstable
improper node</td><td>all nonzero solutions
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</table>

Nonlinear Systems

$$\begin{cases} x' = F(x,y), \\ y' = G(x,y), \end{cases} \tag{2}$$

where F, g are functions of two variables $x = x(t)$ and $y = y(t)$.

Critical point is a solution of the system

$$\begin{cases} x' = 0, \\ y' = 0, \end{cases} \tag{3}$$

i.e. such point (x, y) that

$$\begin{cases} F(x,y) = 0, \\ G(x,y) = 0, \end{cases} \tag{4}$$

Every solution of the system 4 above is a critical point of the given nonlinear system 2

Let (α, β) be a critical point, then, to define a type of that point, it is needed to use the above table with

$$A = \begin{pmatrix} \frac{\partial F}{\partial x}(\alpha,\beta) & \frac{\partial F}{\partial y}(\alpha,\beta) \\ \\ \frac{\partial G}{\partial x}(\alpha,\beta) & \frac{\partial G}{\partial y}(\alpha,\beta) \end{pmatrix}.$$