REDUCTION OF ORDER

We apply the reduction of order method for the following type of problem:

The second order linear homogeneous differential equation

$$A(t)y'' + B(t)y' + D(t)y = 0,$$

where $A(t), B(t), D(t)$ are some functions, has a known solution $y_1(t)$. Use the **reduction of order** to find the general solution of this differential equation.

Solution.

1. Make coefficient 1 in front of y'' in the given equation, i.e. divide both sides of the equation by $A(t)$.
$$y'' + \frac{B(t)}{A(t)}y' + \frac{D(t)}{A(t)}y = 0. \tag{1}$$
Therefore, we get an equation in standard form
$$y'' + p(t)y' + q(t)y = 0. \tag{2}$$

2. Let us try to find another solution $y_2(t)$ of the given equation, which will be linearly independent from $y_1(t)$, in the following form
$$y_2(t) = y_1(t)v(t), \text{ where } v(t) \text{ is an unknown function.}$$

3. Let us compute Wronskian of $y_1(t)$ and $y_2(t)$ by definition
$$W(y_1, y_2)(t) = det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = det\begin{pmatrix} y_1(t) & y_1(t)v(t) \\ y_1'(t) & y_1'(t)v(t) + y_1(t)v'(t) \end{pmatrix} =$$
$$= y_1(t)(y_1'(t)v(t) + y_1(t)v'(t)) - y_1'(t)(y_1(t)v(t)) =$$
$$= y_1(t)y_1'(t)v(t) + (y_1(t))^2v'(t) - y_1'(t)y_1(t)v(t) = (y_1(t))^2v'(t)$$
Therefore, we get that
$$W(y_1, y_2)(t) = (y_1(t))^2v'(t) \tag{3}$$

4. Let us compute Wronskian by Abel's theorem
$$W(y_1, y_2)(t) = Ce^{-\int p(t)dt}$$
As we need just one solution linearly independent from $y_1(t)$, we can put $C = 1$. Therefore, we get
$$W(y_1, y_2)(t) = e^{-\int p(t)dt} \tag{4}$$

5. We got two expressions (3) and (4) for Wronskian of y_1, y_2, therefore they should coincide. We set the two expressions equal to each other to get a differential equation on $v(t)$.

$$(y_1(t))^2 v'(t) = e^{-\int p(t)dt} \tag{5}$$

6. Solve the equation (5) to find $v(t)$:

$$v'(t) = \frac{e^{-\int p(t)dt}}{(y_1(t))^2}$$

$$v(t) = \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2} dt + K, \text{ where } K \text{ is some constant}$$

We can set K equal to 0 as we need just one $v(t)$, which works. Therefore, we can get

$$v(t) = \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2} dt$$

7. After we found $v(t)$, we can write the expression for $y_2(t)$ using $v(t)$ we found:

$$y_2(t) = y_1(t)v(t)$$

8. The general solution of the equation has form

$$y(t) = C_1 y_1(t) + C_2 y_2(t), \text{ where } C_1, C_2 \text{ are any constants}$$