Unit Step Function

The unit step function is

$$u_c(t) = \begin{cases} 0, & \text{if } t < c \\ 1, & \text{if } t \geqslant c \end{cases}$$

Example

$$u_2(1) = 0, \ \textbf{as } 1 < 2$$
$$u_2(4) = 1, \ \textbf{as } 4 > 2$$

Let

$$f(t) = \begin{cases} f_1(t), & \text{if } t < c_1 \\ f_2(t), & \text{if } c_1 \leqslant t < c_2 \\ f_3(t), & \text{if } c_2 \leqslant t \end{cases}$$

Then, we can **rewrite** $f(t)$ **using step functions** in the following way

$$f(t) = f_1(t)(1 - u_{c_1}(t)) + f_2(t)(u_{c_1}(t) - u_{c_2}(t)) + f_3(t)u_{c_2}(t).$$

The Laplace transform of a unit step function is

$$\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}.$$

A useful property for computing the Laplace transform of a product of a step function and some function is

$$\mathcal{L}[u_c(t)g(t)] = e^{-cs}\mathcal{L}[g(t + c)].$$

A useful property for computing the inverse Laplace transform of the expression

$$e^{-cs}\frac{P(s)}{Q(s)},$$

where $P(s)$, $Q(s)$ are polynomials in s, is

$$\mathcal{L}[u_c(t)f(t - c)] = e^{-cs}\mathcal{L}[f(t)].$$

To find the inverse Laplace transform of the expression $e^{-cs}\frac{P(s)}{Q(s)}$, where $P(s)$, $Q(s)$ are polynomials in s:

1. Find the inverse Laplace transform of $\frac{P(s)}{Q(s)}$. Let $f(t) = \mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right]$.
2. Using the property above and $f(t)$ we found in the first item, we get

$$\mathcal{L}^{-1}\left[e^{-cs}\frac{P(s)}{Q(s)}\right] = u_c(t)f(t - c).$$

Impulse/Delta Function

$$\delta(t-c) = 0, \text{ if } t \neq c$$
$$\int_{-\infty}^{+\infty} \delta(t-c)dt = 1$$

The Laplace transform of an impulse function

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\delta(t-c)] = e^{-cs} \text{ for } c \geqslant 0$$

The Laplace transform of a product of an impulse function and some function

$$\mathcal{L}(\delta(t-c)f(t)) = e^{-cs}f(c) \text{ for } c \geqslant 0$$