Eigenvalues and eigenfunctions

Let us consider a certain two-point boundary value problem

$$X'' + \lambda X = 0$$

together with two boundary conditions, where λ is a constant real number.

Definition. The values of λ for which nontrivial solutions $X(x)$ (i.e., $X(x) \not\equiv 0$) of the above problem exist are called eigenvalues. The corresponding nontrivial solutions are called eigenfunctions.

To find eigenvalues, you need to find values of λ such that **NOT BOTH** constants c_1, c_2 in the general solution $X(x)$ of the given equation are 0.
Notice that the general solution has different forms depending on $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$.

The characteristic equation for the equation $X'' + \lambda X = 0$ is $r^2 + \lambda = 0$. So, $r^2 = -\lambda$.

The general solution $X(x)$ for different cases:

1. If $\lambda < 0$, then $\lambda = -a^2$, where $a > 0$ real number.
   The characteristic equation is $r^2 = a^2$. The roots are $r_{1,2} = \pm a$.
   The general solution is
   $$X(x) = c_1 e^{ax} + c_2 e^{-ax},$$
   where c_1, c_2 are any constants.
2. If $\lambda > 0$, then $\lambda = a^2$, where $a > 0$ real number.
   The characteristic equation is $r^2 = -a^2$. The roots are $r_{1,2} = \pm ai$.
   The general solution is
   $$X(x) = c_1 \cos(ax) + c_2 \sin(ax),$$
   where c_1, c_2 are any constants.
3. If $\lambda = 0$, then the characteristic equation is $r^2 = 0$. The roots are $r_1 = r_2 = 0$.
   The general solution is
   $$X(x) = c_1 + c_2 x,$$
   where c_1, c_2 are any constants.

For eigenvalue problems, the following trigonometric identities are helpful:

$$\sin(x) = 0 \Rightarrow x = \pi n \quad \text{for all integer } n$$

$$\cos(x) = 0 \Rightarrow x = -\frac{\pi}{2} + \pi n \quad \text{for all integer } n$$