Fourier series

The Fourier Convergence Theorem

Suppose that f and f' are piecewise continuous on the interval $-L < x < L$. Further assume that f is defined outside the interval $-L < x < L$ so that it is a periodic function with a period $T = 2L$, i.e., $f(x + 2L) = f(x)$ for every real number x. Then, f has the Fourier series, $F(x)$, which is given by formula

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, \cdots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, \cdots$$

The Fourier series converges to the function given by

$$F(x_0) = \begin{cases} f(x_0), & \text{if } f \text{ is continuous at } x_0 \\ \frac{\lim\limits_{x \to x_0+} f(x) + \lim\limits_{x \to x_0-} f(x)}{2}, & \text{if } f \text{ is discontinuous at } x_0 \end{cases}$$

The Fourier series F is also a periodic function with a period $T = 2L$.

Remark.

Just because a Fourier series could have infinitely many terms does not mean that it will always have that many terms. If a periodic function f can be expressed by finitely many terms normally found in the Fourier series, then f must be the Fourier series of itself.

The following trigonometric identities are very helpful in this topic:

$$\sin(-n\pi) = -\sin(n\pi) = 0 \qquad \text{for all integer } n$$

$$\cos(-n\pi) = \cos(n\pi) = (-1)^n \qquad \text{for all integer } n$$