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PROCEEDINGS OF THE INTERNATIONAL SYMPOSIUM
MONTREAL, 1963

ON MULTIVARIATE GENERALIZED POWER SERIES DISTRIBUTION
AND ITS APPLICATION TO THE MULTINOMIAL AND
NEGATIVE MULTINOMIAL

G. P. Patil

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### TABLE OF CONTENTS

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Edited by G. P. Patil

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Table of Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>... v</td>
</tr>
<tr>
<td>Symposium Programme</td>
<td>... vii</td>
</tr>
<tr>
<td><strong>Chapter</strong></td>
<td><strong>Page</strong></td>
</tr>
<tr>
<td>1. Inaugural Session</td>
<td></td>
</tr>
<tr>
<td>1.1. Welcome to the delegates</td>
<td>... H. Rooks Robertson</td>
</tr>
<tr>
<td>1.2. Opening remarks</td>
<td>... G. P. Patil</td>
</tr>
<tr>
<td>1.3. Inaugural Address: Certain chance mechanisms involving discrete distributions</td>
<td>... Jerzy Neyman</td>
</tr>
<tr>
<td>2. Certain Stochastic Systems</td>
<td></td>
</tr>
<tr>
<td>2.1. A type-resisting distribution generated from considerations of an inventory decision model</td>
<td>... Barnard H. Bissinger</td>
</tr>
<tr>
<td>2.2. The weakly contagious discrete stochastic process</td>
<td>... Tosio Kitagawa</td>
</tr>
<tr>
<td>2.3. Subclustering</td>
<td>... E. L. Scott</td>
</tr>
<tr>
<td>3. Structural Properties</td>
<td></td>
</tr>
<tr>
<td>3.1. Incomplete and absolute moments of some discrete distributions</td>
<td>... A. E. Kamat</td>
</tr>
<tr>
<td>3.2. Characterization problems for discrete distributions</td>
<td>... Eugene Lukacs</td>
</tr>
<tr>
<td>3.3. A moment generating function of the hyper-geometric distributions</td>
<td>... Chia Kwei Tsao</td>
</tr>
<tr>
<td>4. Limit Distributions</td>
<td></td>
</tr>
<tr>
<td>4.1. Normal approximations to the classical discrete distributions</td>
<td>... Zakkula Govindarajulu</td>
</tr>
<tr>
<td>4.2. Some discrete distribution limit theorems, using a new derivative</td>
<td>... Vivian Pessin</td>
</tr>
<tr>
<td>4.3. Conditional limit-distributions for the entries in a $2 \times k$ contingency table</td>
<td>... Constance Van Esden</td>
</tr>
</tbody>
</table>
ON MULTIVARIATE GENERALIZED POWER SERIES DISTRIBUTION* AND ITS APPLICATION TO THE MULTINOMIAL AND NEGATIVE MULTINOMIAL

By G. P. PATIL
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INTRODUCTION AND SUMMARY

Let $T$ be a countable subset of the set of real numbers without any limit point. Define $f(\theta) = \sum a(x)\theta^x$ where the summation extends over $T$ and $a(x) > 0$, $\theta > 0$ with $\theta \in H$, the parameter space, such that $f(\theta)$ is finite and differentiable. Then a random variable $X$, with probability function

$$
\text{Prob} \{X = x\} = p(x, \theta) = a(x)\theta^x f(\theta) \quad x \in T
$$

is said to have the generalized power series distribution (GPSD) with range $T$ and the series function $f(\theta)$. It is clear that when $T$ is restricted to non-negative integers, one has $H = \{\theta : 0 < \theta < R\}$ where $R$ is the radius of convergence of the power series of $f(\theta)$. The author has discussed some problems of statistical inference associated with the GPSD which may be looked upon as a unifying form for the classical discrete distributions like the binomial, Poisson, negative binomial, logarithmic series and their truncations.

To consider the multivariate analogue of the GPSD, let $T$ be a countable subset without any limit point of a $k$-fold cartesian product of the set $R$ of real numbers. Thus writing $R_i = R, i = 1, 2, \ldots, k$,

$$
T = \{(x_1, x_2, \ldots, x_k) \subset R_1 \times R_2 \times \cdots \times R_k
$$

Define $f(\theta_1, \theta_2, \ldots, \theta_k) = \sum a(x_1, x_2, \ldots, x_k)\theta_1^{x_1}\theta_2^{x_2}\cdots\theta_k^{x_k}$ where the summation extends over $T$ and $a(x_1, x_2, \ldots, x_k) > 0$, $\theta_i > 0$ with $(\theta_1, \theta_2, \ldots, \theta_k) \in H$, the $k$-dimensional parameter space, so that $f(\theta_1, \theta_2, \ldots, \theta_k)$ is finite and differentiable. Then a $k$-dimensional random variable $(x_1, x_2, \ldots, x_k)$ with probability function

$$
p(x_1, x_2, \ldots, x_k; \theta_1, \ldots, \theta_k) = a(x_1, x_2, \ldots, x_k)\theta_1^{x_1}\cdots\theta_k^{x_k} f(\theta_1, \ldots, \theta_k) \quad (x_1, \ldots, x_k) \in T
$$

may be defined to have the multivariate GPSD with range $T$ and the series function $f(\theta_1, \theta_2, \ldots, \theta_k)$. The multivariate GPSD is thus a discrete multivariate exponential-type distribution, whereas the multivariate power series distribution as defined by Khatri [2] is a special case of the multivariate GPSD. For the purpose of this paper, we take $R_i = I$, the set of non-negative integers so that $\sum a(x_1, x_2, \ldots, x_k)\theta_1^{x_1}\cdots\theta_k^{x_k}$ is the power series expansion of $f(\theta_1, \ldots, \theta_k)$ in $\theta_1, \theta_2, \ldots, \theta_k$. Clearly, the parameter space $H$ is then the region of convergence of the power series of $f(\theta_1, \ldots, \theta_k)$.

In this paper, we study some structural properties of the multivariate GPSD and provide certain characterization statements. Further, we study the problem of estimation with reference to the methods of maximum likelihood and minimum variance. In the last section, the minimum variance unbiased estimators for the probability and distribution functions of the multivariate GPSD are obtained and two applications of these results are cited.

*Research carried out while the author was a fellow of the 1963 Summer Research Institute of the Canadian Mathematical Congress.
1. Certain Properties of the Multivariate GPSD

(i) The probability function (2) satisfies the recursion relation

\[ p(x_1 + r_1, \ldots, x_k + r_k; \theta_1, \ldots, \theta_k) = \frac{a(x_1 + r_1, \ldots, x_k + r_k)}{a(x_1, \ldots, x_k)} \cdot \prod_{i=1}^{k} \theta_i^{x_i} \cdot \prod_{k=1}^{k} \theta_k^{r_k} \quad \ldots \quad (3) \]

which can be fruitfully used for the tabulation of the distribution.

(ii) Let \((x_1, \ldots, x_k)\) be a discrete random variable taking values in \(T\) according to some probability law \(p(x_1, \ldots, x_k)\). We define \((x_1, \ldots, x_k)\) to enjoy the "property of proportions" if

\[ p(x_1 + r_1, \ldots, x_k + r_k) = b(x_1, \ldots, x_k; r_1, \ldots, r_k) \cdot \prod_{i=1}^{k} \theta_i^{x_i} \cdot \prod_{k=1}^{k} \theta_k^{r_k} \]

where \(b(x_1, \ldots, x_k; r_1, \ldots, r_k) > 0\); \((x_1, \ldots, x_k)\) and \((x_1 + r_1, \ldots, x_k + r_k)\) are in \(T\).

Then we have a characterization statement saying that a discrete distribution of \((x_1, \ldots, x_k)\) is a multivariate GPSD if and only if \((x_1, \ldots, x_k)\) enjoys the property of proportions.

(iii) The marginal probability function for \(x_i\) is given by

\[ p_i(x_i) = \sum a(x_1, \ldots, x_k) \cdot \theta_i^{x_i} \cdot \prod_{k=1}^{k} \theta_k^{r_k} \frac{f(\theta_1, \ldots, \theta_k)}{f(\theta_i, \ldots, \theta_k)} \]

where the summation is taken over the constant \(x_i\). Clearly, one may write

\[ p_i(x_i) = a_i(\theta_1, \ldots, \theta_k) \cdot \theta_i^{x_i} \cdot \prod_{k=1}^{k} \theta_k^{r_k} \cdot f(\theta_i, \ldots, \theta_k) \quad \ldots \quad (4) \]

Thus \(x_i\) has a GPSD with the series function \(f(\theta_1, \ldots, \theta_k)\) as expanded in powers of \(\theta_i\), other \(\theta_i\)'s treated as constants. It follows from the well-known properties of the GPSD that

\[ \mu_i = E(x_i) = \theta_i \cdot \frac{\partial}{\partial \theta_i} \log f(\theta_1, \ldots, \theta_k) \]

\[ \sigma^2(x_i) = \theta_i \cdot \frac{\partial^2}{\partial \theta_i^2} \log f(\theta_1, \ldots, \theta_k) \]

and so on.

(iv) The \(r\)-dimensional random variable \(x_{i_1}, \ldots, x_{i_r}\), \(1 \leq r \leq k\), has a multivariate GPSD with the series function \(f(\theta_1, \ldots, \theta_k)\) as expanded in powers of \(\theta_{i_1}, \ldots, \theta_{i_r}\), other \(\theta_i\)'s treated as constants.

(v) The \(r\)-dimensional conditional random variable \(x_{i_1}, \ldots, x_{i_r}\) with the condition that \(x_\alpha\) is given, where \(\alpha \neq i_j, j = 1, 2, \ldots, r\), has a multivariate GPSD with the series function \(a_\alpha(\theta_1, \ldots, x_\alpha, \ldots, \theta_k)\) as expanded in powers of \(\theta_{i_1}, \ldots, \theta_{i_r}\), where \(a_\alpha(\theta_1, \ldots, x_\alpha, \ldots, \theta_k)\) is defined as in (4).
(vi) \( \sigma_{ij} = \text{covariance} (x_i, x_j) = \partial_i \theta_j \frac{\partial \mu_i}{\partial \theta_j} \).

(vii) The crude moment of order \( r_1, r_2, \ldots, r_k \) is given by
\[
m_{(r_1, r_2, \ldots, r_k)} = E(x_1^{r_1} \cdots x_k^{r_k}) = \frac{1}{f} \left[ \prod_{i=1}^{k} \left( \theta_i \frac{\partial f}{\partial \theta_i} \right) \right]^{r_i} f
\]
where \( f \) stands for \( f(\theta_1, \ldots, \theta_k) \).

(viii) A recursion relation between crude moments is available as
\[
m_{(r_1, r_2, \ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots, r_k)} = \theta_i \frac{\partial m_{(r_1, \ldots, r_{i-1}, r_i, \ldots, r_k)}}{\partial \theta_i} + \mu_{i} m_{(r_1, \ldots, r_k)}.
\]

(ix) Writing \( x^{(r)} = x(x-1) \cdots (x-r+1) \), the factorial moment of order \( r_1, \ldots, r_k \) is given by
\[
m_{(r_1, \ldots, r_k)} = E(x_1^{[r_1]} \cdots x_k^{[r_k]}) = \frac{1}{f} \left[ \prod_{i=1}^{k} \theta_i^{r_i} \prod_{i=1}^{k} \left( \frac{\partial f}{\partial \theta_i} \right)^{r_i} f \right]
\]
from which follows a recursion relation
\[
m_{(r_1, \ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots, r_k)} = \theta_i \frac{\partial m_{(r_1, \ldots, r_{i-1}, r_i, \ldots, r_k)}}{\partial \theta_i} + (\mu_i - r_i) m_{(r_1, \ldots, r_k)}.
\]

(x) Writing the central moment of order \( r_1, \ldots, r_k \) as
\[
\mu_{(r_1, \ldots, r_k)} = E(x_1^{(r_1)} \cdots x_k^{(r_k)})
\]
a recursion relation is available as
\[
\mu_{(r_1, \ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots, r_k)} = \theta_i \frac{\partial \mu_{(r_1, \ldots, r_{i-1}, r_i, \ldots, r_k)}}{\partial \theta_i} + \sum_{j=1}^{k} r_j \sigma_{ij} \mu_{(r_1, \ldots, r_j-1, \ldots, r_k)}
\]

(xi) On the lines of Khatri [2], one has the result that the multivariate
GPSD is uniquely determined by its vector of means and the variance-covariance
matrix given as functions of the parameters \( \theta_1, \theta_2, \ldots, \theta_k \).

(xii) A truncated multivariate GPSD is a multivariate GPSD in its own
right and the properties that hold for a multivariate GPSD hold automatically
for the truncated GPSD.

2. A CHArACTERIZATION OF THE MULTIVARIATE GPSD

If \( K_{(r_1, \ldots, r_k)} \) denotes the cumulant of order \( r_1, \ldots, r_k \) of the multivariate GPSD
defined by (2), then a recursion relation is given by
\[
K_{(r_1, \ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots, r_k)} = \theta_i \frac{\partial K_{(r_1, \ldots, r_k)}}{\partial \theta_i}
\]
which has been obtained by Guldberg [1] and Wishart [7] for multinomial
and negative multinomial distributions whereas Khatri [2] has obtained it for
multivariate power series distributions. Here we shall derive it by a different method and in a slightly different form and further show that the multivariate GPSD is characterized by this recursion relation. Let \( \theta_i = e^{\omega_i} \), \( i = 1, 2, \ldots, k \) and write
\[
g(\omega_1, \ldots, \omega_k) = f(e^{\omega_1}, \ldots, e^{\omega_k}). \tag{5}
\]
Then we prove the following:

**Theorem 1:** \( K(\tau_1, \ldots, \tau_{i+1}, \ldots, \tau_k) = \frac{\partial K(\tau_1, \ldots, \tau_k)}{\partial \omega_i} \)

**Proof:** The cumulant generating function for the multivariate GPSD with parameters \( \omega_1, \omega_2, \ldots, \omega_k \) is given by
\[
K(\tau_1, \tau_2, \ldots, \tau_k; \omega_1, \omega_2, \ldots, \omega_k) = \sum_{\tau_1} K(\tau_1, \ldots, \tau_k) \prod_{i=1}^{k} \frac{\tau_i^{\omega_i}}{\omega_i!} \tag{6}
\]
where the summation extends over \( \tau_1, \tau_2, \ldots, \tau_k \) \( eI \times I \times \cdots \times I \) and one has after little algebra
\[
K(\tau_1, \ldots, \tau_k; \omega_1, \ldots, \omega_k) = \log [g(\tau_1 + \omega_1, \ldots, \tau_k + \omega_k)] - \log [g(\omega_1, \ldots, \omega_k)] \tag{7}
\]
where \( g(\omega_1, \ldots, \omega_k) \) is defined by (5).

Partial differentiation of (7) w.r.t. \( \tau_1 \) gives
\[
e(\tau_1, \tau_2, \ldots, \tau_k; \omega_1, \omega_2, \ldots, \omega_k) = \frac{\partial}{\partial \tau_1} \{ K(\tau_1, \ldots, \tau_k; \omega_1, \ldots, \omega_k) \} \tag{8}
\]
\[
= \frac{\partial}{\partial \tau_1} \{ g(\tau_1 + \omega_1, \ldots, \tau_k + \omega_k) \} / g(\omega_1, \ldots, \omega_k) = h(\tau_1 + \omega_1, \ldots, \tau_k + \omega_k) = h, \text{ say.} \tag{9}
\]

Because of symmetry in \( h(\tau_1 + \omega_1, \ldots, \tau_k + \omega_k) \) of \( \tau_i \) and \( \omega_i \) for all \( i \), one has
\[
\frac{\partial^{\tau_1 + \cdots + \tau_k}}{\partial \tau_1^{\tau_1} \cdots \partial \tau_k^{\tau_k}} h = \frac{\partial^{\tau_1 + \cdots + \tau_k}}{\partial \omega_1^{\tau_1} \cdots \partial \omega_k^{\tau_k}} h = h(\tau_1, \ldots, \tau_k)(\omega_1, \ldots, \omega_k).
\]

\[
K(\tau_1, \ldots, \tau_k; \omega_1, \omega_2, \ldots, \omega_k) = \left[ \frac{\partial^{\tau_1 + \cdots + \tau_k}}{\partial \omega_1^{\tau_1} \cdots \partial \omega_k^{\tau_k}} K(\tau_1, \ldots, \tau_k; \omega_1, \ldots, \omega_k) \right]_{\omega_1 = \cdots = \omega_k = 0} \tag{10}
\]
\[
= h(\tau_1 - 1, \ldots, \tau_k)(\omega_1, \ldots, \omega_k).
\]

One gets therefore that
\[
K(\tau_1, \ldots, \tau_{i+1}, \ldots, \tau_k; \omega_1, \omega_2, \ldots, \omega_k) = \frac{\partial K(\tau_1, \ldots, \tau_k)}{\partial \omega_i} \tag{11}
\]
5.5] MULTIVARIATE GENERALISED POWER SERIES DISTRIBUTIONS 187

Theorem 2: A discrete k-dimensional probability distribution is a multivariate
GPSD if and only if the recursion relation (11) holds between its cumulants.

Proof: The necessity part is proved in Theorem 1. Sufficiency is proved
as follows.

Partial differentiation of (6) w.r.t. $t_i$ gives, together with (8),

$$
s(t_1, \ldots, t_k; \omega_1, \ldots, \omega_k) = \frac{\partial}{\partial t_i} \left\{ K(t_1, \ldots, t_k; \omega_1, \ldots, \omega_k) \right\} = \sum_{r=1}^{k} K_{(r_1, \ldots, r_k)} \prod_{i=1}^{k} \frac{t_i^{r_i}}{r_i!} \quad \ldots \quad (12)
$$

which reduces to

$$
\sum_{\omega_1^1 \cdots \omega_k^k} \frac{\partial^{r_1 + \cdots + r_k}}{\partial \omega_1^{r_1} \cdots \partial \omega_k^{r_k}} [s(0, \ldots, 0; \omega_1, \ldots, \omega_k)] \prod_{i=1}^{k} \frac{t_i^{r_i}}{r_i!} \quad \ldots \quad (13)
$$

because one has from (11)

$$
K_{(r_1^1, \ldots, r_k^k)} = \frac{\partial^{r_1 + \cdots + r_k}}{\partial \omega_1^{r_1} \cdots \partial \omega_k^{r_k}} K_{(1,0,\ldots,0)}
$$

and by definition

$$
K_{(1,0,\ldots,0)} = s(0, \ldots, 0; \omega_1, \ldots, \omega_k).
$$

But one has the expansion

$$
s(t_1, \ldots, t_k; \omega_1, \ldots, \omega_k) = \sum_{\omega_1^1 \cdots \omega_k^k} \left[ \frac{\partial^{r_1 + \cdots + r_k}}{\partial \omega_1^{r_1} \cdots \partial \omega_k^{r_k}} s(t_1, \ldots, t_k, \omega_1, \ldots, \omega_k) \right] \prod_{i=1}^{k} \frac{t_i^{r_i}}{r_i!} \quad \ldots \quad (14)
$$

$$
t_1 = \cdots = t_k = 0.
$$

The identity between (13) and (14) gives for all $r_1$

$$
\frac{\partial^{r_1 + \cdots + r_k}}{\partial \omega_1^{r_1} \cdots \partial \omega_k^{r_k}} [s(0, \ldots, 0; \omega_1, \ldots, \omega_k)]
$$

$$
= \left[ \frac{\partial^{r_1 + \cdots + r_k}}{\partial \omega_1^{r_1} \cdots \partial \omega_k^{r_k}} s(t_1, \ldots, t_k; \omega_1, \ldots, \omega_k) \right]_{t_1=\cdots=t_k=0} \quad \ldots \quad (15)
$$

Substituting $\omega_1 = \omega_2 = \cdots = \omega_k = 0$ in (15) and using the results on the equivalence of two power series expansions, one has

$$
s(0, \ldots, 0; \omega_1, \ldots, \omega_k) = s(\omega_1, \ldots, \omega_k; 0, \ldots, 0) = m(\omega_1, \ldots, \omega_k), \text{ say.}
$$

Effecting the transformation $\omega_i = \omega_i + \omega_i'$ with $\omega_i$ as arbitrary constants and now following the analysis given in Paoli [5], one obtains

$$
K(t_1, \ldots, t_k; \omega_1, \ldots, \omega_k) = M(t_1 + \omega_1, \ldots, t_k + \omega_k) - M(\omega_1, \ldots, \omega_k) \quad \ldots \quad (16)
$$

where $M$ is some functional form,
It can be now verified, using properties of transforms, that a distribution with cumulant generating function of the form (16) is multivariate exponential-type which when discrete is the multivariate GPD.

3. **Maximum Likelihood Estimation for The Multivariate GPD**

To estimate the parameters \( \theta_1, \theta_2, ..., \theta_k \) by the method of maximum likelihood on the basis of a random sample of size \( n \) with \( x_{ij}, i = 1, 2, ..., k; j = 1, 2, ..., n \) drawn from the multivariate GPD given by (2), the likelihood function is given by

\[
L(\theta_1, ..., \theta_k; x_{ij}, i = 1, ..., k; j = 1, 2, ..., n) = \prod_{j=1}^{n} p(x_{ij}, ..., x_{kj}; \theta_1, ..., \theta_k).
\]

Since

\[
\frac{\partial}{\partial \theta_k} \{ \log p(x_1, ..., x_k; \theta_1, ..., \theta_k) \} = \frac{x_k - \mu_k}{\theta_k},
\]

the likelihood equations become

\[
\frac{\partial}{\partial \theta_1} \{ \log L \} = \frac{\sum_{j=1}^{n} x_{ij} - \mu_i}{\theta_i} = 0
\]

where \( \bar{x}_i = \frac{1}{n} \sum_{j=1}^{n} x_{ij} \) is the sample mean of the \( i \)-th component.

Thus the maximum likelihood estimators for \( \theta_1, ..., \theta_k \) are available by solving the set of \( k \) equations

\[
x_i = \mu_i(\theta_1, ..., \theta_k)
\]

which incidentally are also the equations provided by the method of moments. Thus the moment estimates and the maximum likelihood estimates are identical here as in the case of univariate GPD's. That the maximum of the likelihood is achieved at the solution \( \theta_1, ..., \theta_k \) of the equations is demonstrated by the validity of a sufficient condition that the matrix

\[
\left( \frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} \right)
\]

is negative definite since in this case

\[
\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} = \frac{n}{\theta_i} \left( \frac{\partial \mu_i}{\partial \theta_j} \right) = -n \sigma_{ij} \theta_i \theta_j.
\]

Finally, the information matrix is given by

\[
I = (I_0)
\]

where \( I_0 = n \sigma_{ij} \theta_i \theta_j \). The asymptotic variance-covariance matrix of the maximum likelihood estimators \( \theta_1, ..., \theta_k \) is then given by \( V = I^{-1} \).
4. Notation and Terminology

(i) Let \( A = \{0, a_1, a_2, \ldots, a_n\} \) be a subset of the set \( I \) of the non-negative integers. The Schnirelmann density \( d(A) \) of the set \( A \) is defined by

\[
d(A) = g.l.b. \frac{A(m)}{m}
\]

where \( A(m) \) denotes the number of positive integers in the set \( A \) which do not exceed \( m \). It is easy to see that \( 0 \leq A(m) \leq m \) and therefore \( 0 \leq d(A) \leq 1 \). Note that the Schnirelmann density is defined only for such sets in \( I \) which contain zero.

(ii) Let \( A^{(i)} = \{a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}\}, i = 1, 2, \ldots, n \) be arbitrary subsets of vectors of a \( k \)-dimensional space. The sum \( A_n = \sum_{i=1}^{n} A^{(i)} \) of the given subsets is defined as the set of all vectors of the form \( \sum_{i=1}^{n} a_i^{(i)} \) where \( a_i^{(i)} \in A^{(i)} \). If \( A^{(i)} = A \), \( A_n = \sum A \) is denoted by \( n[A] \). The difference \( A^{(1)} - A^{(2)} \) is defined as the set of all vectors of the form \( a^{(1)} - a^{(2)} \) where \( a^{(1)} \in A^{(1)} \) and \( a^{(2)} \in A^{(2)} \).

(iii) A set \( A \) is called a basis of the set \( B \) if \( n[A] = B \) for some \( n \). In such case \( n \) is called the order of the basis \( A \) of the set \( B \).

(iv) The set \( [A]/r \) is defined to be the set of all vectors of the form \( a/r \) where \( a \in A \) and \( r \) is any non-zero scalar.

(v) Set \( A = \{a\} \) denotes the singleton, the set of only one member, namely \( a \).

(vi) The set \( A \) of \( k \)-dimensional space is said to be stable relative to the vector \((r_1, \ldots, r_k)\) if \((x_1, \ldots, x_k) \in A\) implies that \((x_1 + r_1, \ldots, x_k + r_k) \in A\). Equivalently, \( T + \{(r_1, \ldots, r_k)\} \subseteq T \). Or, in words, the set \( T \) when pushed through the vector \((r_1, \ldots, r_k)\) remains a subset of \( T \).

(vii) Let \( T(a_1, \ldots, a_k; r_1, \ldots, r_k) \) denote the set of vector points of \( T \) lying on the \( k \)-dimensional line which passes through the point \((a_1, \ldots, a_k)\) in \( T \) and has \((r_1, \ldots, r_k)\) as the vector of its direction numbers. Then \( T(a_1, \ldots, a_k; r_1, \ldots, r_k) \) is said to be the linear set of \((a_1, \ldots, a_k)\) in \( T \) with direction vector \((r_1, \ldots, r_k)\).

(viii) \( T_i(a_1, \ldots, a_k; r_1, \ldots, r_k) = \{x_i : (x_1, \ldots, x_k) \in T(a_1, \ldots, a_k; r_1, \ldots, r_k)\} \) is to denote the set of \( i \)-th components of the projection of \( T(a_1, \ldots, a_k; r_1, \ldots, r_k) \) on the \( i \)-th axis. In other words, \( T_i(a_1, \ldots, a_k; r_1, \ldots, r_k) \) denotes the set of \( i \)-th co-ordinates of the vectors in \( T(a_1, \ldots, a_k; r_1, \ldots, r_k) \).

(ix) The linear set \( T(a_1, \ldots, a_k; r_1, \ldots, r_k) \) is said to have the vector \((c_1, \ldots, c_k)\) as its source point if \((c_1, \ldots, c_k) \in T(a_1, \ldots, a_k; r_1, \ldots, r_k) \) and

\[
c_i = \begin{cases} 
\min T_i(a_1, \ldots, a_k; r_1, \ldots, r_k) & \text{if } r_i > 0 \\
\max T_i(a_1, \ldots, a_k; r_1, \ldots, r_k) & \text{if } r_i < 0 \\
\text{if } r_i = 0
\end{cases}
\]

Clearly, in this case \( T(a_1, \ldots, a_k; r_1, \ldots, r_k) = T(c_1, \ldots, c_k; r_1, \ldots, r_k) \) which we call the linear set of the source point \((c_1, \ldots, c_k)\) in \( T \) with direction vector \((r_1, \ldots, r_k)\).
(x) \( C(T; r_1, \ldots, r_k) \) is to denote the set of all source points of linear sets in
\( T \) with direction vector \((r_1, \ldots, r_k)\). It is obvious that \( C(T; r_1, \ldots, r_k) \) is not vacuous.

(xi) A subset \( T \) of a \( k \)-dimensional space is said to be the index-set of the
function \( f(\theta_1, \theta_2, \ldots, \theta_k) \) if

\[
f(\theta_1, \ldots, \theta_k) = \sum \alpha(x_1, \ldots, x_k)\theta_1^{x_1} \cdots \theta_k^{x_k}
\]

where \( \alpha(x_1, \ldots, x_k) \neq 0 \), \((x_1, \ldots, x_k) \in T \) and is denoted by

\[
T = W[f(\theta_1, \ldots, \theta_k)].
\]

Clearly, the range of a multivariate GPSD \( \) is also the index-set of its series function.

(xii) A real-valued parameter \( g(\theta_1, \ldots, \theta_k) \) is called MVU estimable if it
has minimum variance unbiased estimator based on a random sample of some size \( n \).

5. MVU Estimation for the Multivariate GPSD

For a parameter to be MVU estimable we shall obtain here conditions in
terms of the structure of the range \( T \) of a multivariate GPSD. We have the following.

Theorem 3: A necessary and sufficient condition for the parameter
\( g(\theta_1, \theta_2, \ldots, \theta_k) = \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k} \) of the multivariate GPSD given by (2) to be MVU estimable
on the basis of a single observation is that \( T \) is stable relative to the vector \((r_1, \ldots, r_k)\).

Proof: Since \((x_1, \ldots, x_k)\) is complete and sufficient for \((\theta_1, \ldots, \theta_k)\), it follows
that an unbiased estimator \( h(x_1, \ldots, x_k) \), if any, of \( g(\theta_1, \ldots, \theta_k) \) is the MVU estimator
of \( g(\theta_1, \ldots, \theta_k) \). Conversely, if \( g(\theta_1, \ldots, \theta_k) \) is not MVU estimable on the basis of a
single observation, \( g(\theta_1, \ldots, \theta_k) \) does not have an unbiased estimator either.

To prove the necessity, suppose \( h(x_1, \ldots, x_k) \) denotes an unbiased estimator
for \( g(\theta_1, \ldots, \theta_k) = \prod_{i=1}^{k} \theta_i^{x_i} \). One has therefore for all \((\theta_1, \ldots, \theta_k) \in H\) the identity

\[
\Sigma h(x_1, \ldots, x_k) a(x_1, \ldots, x_k)\theta_1^{x_1} \cdots \theta_k^{x_k} = \Sigma' a(x_1-r_1, \ldots, x_k-r_k)\theta_1^{x_1} \cdots \theta_k^{x_k}
\]

where \( \Sigma \) extends over \((x_1, \ldots, x_k) \in T \) and \( \Sigma' \) extends over \((x_1, \ldots, x_k) \in T +\{(r_1, \ldots, r_k)\}\).

It follows that

\[
h(x_1, \ldots, x_k) = a(x-r_1, \ldots, x_k-r_k)/a(x_1, \ldots, x_k)
\]

for \((x_1, \ldots, x_k) \in T +\{(r_1, \ldots, r_k)\}\) and \( = 0 \) otherwise and further \( T +\{(r_1, \ldots, r_k)\} \subset T \),

because, otherwise, it contradicts the fact that \( a(x_1, \ldots, x_k) \geq \) 0 for \((x_1, \ldots, x_k) \in T \).

The sufficiency follows if one considers \( h(x_1, \ldots, x_k) \) as defined above to be an estimator
for \( g(\theta_1, \ldots, \theta_k) \).

Theorem 4: A necessary and sufficient condition for the parameter
\( g(\theta_1, \ldots, \theta_k) = \theta_1^{x_1} \cdots \theta_k^{x_k} \) of the multivariate GPSD given by (2) to be MVU estimable is
that \( n[T] \) is stable relative to the vector \((r_1, \ldots, r_k)\) for some value of \( n \).

Proof: To prove the theorem, one has only to see that \( z_i = \sum_{j=1}^{n} x_{ij} \), where
\( x_{ij}, i = 1, 2, \ldots, n \) is a random sample of size \( n \) drawn from the multivariate GPSD
given by (2), follow a multivariate GPSD with range \( n[T] \) and the series function

\[
f_n(\theta_1, \ldots, \theta_k) = [f(\theta_1, \ldots, \theta_k)]^n = \sum b(x_1, \ldots, x_k; n)\theta_1^{x_1} \cdots \theta_k^{x_k}
\]

(21)
where the summation extends over \( n[T] \) and \( b(x_1, \ldots, z_k; n) \) is the coefficient of \( \prod_{i=1}^{k} \theta_i^{z_i} \) in the expansion of \( f(x_1, \ldots, z_k) \). Clearly, \( b(z_1, \ldots, z_k; n) > 0 \) for \((x_1, \ldots, z_k) \in n[T]\). Also \((x_1, \ldots, z_k)\) is complete and sufficient for \( \theta_1, \ldots, \theta_k \). Because of Theorem 1, the theorem under consideration now follows.

**Corollary 1:** \( \prod_{i=1}^{k} \theta_i^{z_i} \) is not MVU estimable if \( T \) is finite. This follows from the finiteness of \( n[T] \) inherited from \( T \).

**Corollary 2:** \( \prod_{i=1}^{k} \theta_i^{z_i} \) is not MVU estimable if \( r_i \) is fractional or negative for some \( i \). This follows from the fact that \( T \) is assumed for the purposes of this paper to be a set of vectors with non-negative integral co-ordinates.

**Corollary 3:** When it exists, the MVU estimator of \( \prod_{i=1}^{k} \theta_i^{z_i} \) is given by

\[
h(z_1, \ldots, z_k; n) = \begin{cases} 
b(z_1, \ldots, z_k; n)/b(z_1, \ldots, z_k; n)(z_1, \ldots, z_k) \in n[T] + (r_1, \ldots, r_k) \\ 0 \text{ otherwise} \end{cases}
\]

where symbols carry usual sense.

Theorem (2) brings out that the MVU estimability of \( \prod_{i=1}^{k} \theta_i^{z_i} \) depends only on the structure of the range \( T \) of the multivariate GFS and it is curious to note that it has nothing to do with the specific form of the multivariate GFS as determined by the coefficients \( a(x_1, \ldots, x_k) \). A few natural and fruitful statistical questions in this connection concerning univariate GFS's are discussed in Patil [6]. It is shown there that the questions of statistical interest turn out to be some of those that have been of interest to the specialists of additive number theory. We study their multivariate counterpart in the following section.

### 6. A Characterization of the MVU Estimability of \( \prod_{i=1}^{k} \theta_i^{z_i} \)

**In Terms of the Range \( T \)**

Let \((r_1, \ldots, r_k)\) be the fixed vector as given. In terms of the notation and terminology of Section 4, we have the following lemmas which we state without proof.

**Lemma 1:** The set \( T \) is stable relative to the vector \((r_1, \ldots, r_k)\) if and only if the class of linear sets of source points in \( T \) with direction vector \((r_1, \ldots, r_k)\) form a partition of \( T \). In symbols, the necessary and sufficient condition can be expressed as

\[
T = UT(c_1, \ldots, c_k; r_1, \ldots, r_k)
\]

where the union is taken over the source points \((c_1, \ldots, c_k) \in C(T; r_1, \ldots, r_k)\).

**Lemma 2:** The set \( T \) is stable relative to the vector \((r_1, \ldots, r_k)\) if and only if, for all \( i \) corresponding to non-zero \( r_i \) the set

\[
[T(c_1, \ldots, c_k; r_1, \ldots, r_k) - [c_i]]/r_i
\]

is the basis of \( I \) of order \( 1 \) for every source point \((c_1, \ldots, c_k) \in C(T; r_1, \ldots, r_k)\), i.e. the set

\[
[T(c_1, \ldots, c_k; r_1, \ldots, r_k) - [c_i]]/r_i = I.
\]
It may be mentioned here that the two lemmas need not hold if $T$ was of the additive number theory which states that a necessary and sufficient condition for a set of non-negative integers to be identical with $I$ is that $d(A) = 1$, where $d$ stands for the Schmirnau density as defined in (i) of Section 4.

Theorem 7: If $\Pi_{i=1}^n \theta_i$ is MVU estimable for a sample of size $n$, it is MVU estimable for sample size $n+1$ and thus for every sample size exceeding $n$.

Proof: We remark that for all $(x_1, x_2, ..., x_n) \in T$, we have $T^{[n]}(c_1, ..., c_n; r_1, ..., r_n) + \{x_1, ..., x_n\}$ to be a subset of the linear set of $(c_1 + x_1, ..., c_n + x_n)$ in $T^{[n+1]}$ with direction vector $(r_1, ..., r_n)$. Now by the hypothesis of the theorem and the validity of Lemma 1, we have the partition

$$T^{[n]} = UT^{[n]}(c_1, ..., c_n; r_1, ..., r_n),$$

where the union ranges over $(c_1, ..., c_n) \in C(T^{[n]}; r_1, ..., r_n)$. Therefore, if $(c_1 + x_1, ..., c_n + x_n) \in C(T^{[n+1]}; r_1, ..., r_n)$, then $T^{[n]}(c_1, ..., c_n; r_1, ..., r_n) + \{x_1, ..., x_n\}$ is a linear set of the source point $(c_1 + x_1, ..., c_n + x_n)$ in $T^{[n+1]}$ with direction vector $(r_1, ..., r_n)$. Further, if $(c_1 + x_1, ..., c_n + x_n) \in C(T^{[n+1]}; r_1, ..., r_n)$, then $T^{[n]}(c_1, ..., c_n; r_1, ..., r_n) + \{x_1, ..., x_n\}$ is a proper subset of $T^{[n+1]}(d_1 + y_1, ..., d_n + y_n; r_1, ..., r_n)$ for some $(d_1, ..., d_n) \in T^{[n]}$ and $(y_1, ..., y_n) \in T$ so that $(d_1 + y_1, ..., d_n + y_n) \in C(T^{[n+1]}; r_1, ..., r_n)$.

It follows that we have the partition

$$T^{[n+1]} = UT^{[n+1]}(c_1, ..., c_n; r_1, ..., r_n),$$

where the union now ranges over $(c_1, ..., c_n) \in C(T^{[n+1]}; r_1, ..., r_n)$ from which we conclude because of Theorem 4 and Lemma 1 that $\Pi_{i=1}^n \theta_i$ is MVU estimable for sample size $n+1$.

7. The MVU Estimation of an Arbitrary Function of $(\theta_1, ..., \theta_n)$

Let $g(\theta_1, ..., \theta_n)$ be a given function of $(\theta_1, ..., \theta_n)$ which is such that $g(\theta_1, ..., \theta_n) \cdot f_n(\theta_1, ..., \theta_n)$ admit a power series expansion in $\theta_1, ..., \theta_n$ where $f_n(\theta_1, ..., \theta_n)$ is defined by (21). Proceeding on the same lines as in Section 5, we have the following:

Theorem 7: A necessary and sufficient condition for $g(\theta_1, ..., \theta_n)$ to be MVU estimable on the basis of a random sample of size $n$ from the multivariate GPSD given by (2) is that

$$W[g(\theta_1, ..., \theta_n) \cdot f_n(\theta_1, ..., \theta_n)] \subseteq W[f_n(\theta_1, ..., \theta_n)],$$

where $W$ is defined by (20). Also, whenever it exists, the MVU estimator for $g(\theta_1, ..., \theta_n)$ is given by

$$h(x_1, ..., x_n; n) = c(x_1, ..., x_n; n)/b(x_1, ..., x_n; n) \quad \quad (23)$$

for $(x_1, ..., x_n) \in W[g(\theta_1, ..., \theta_n) \cdot f_n(\theta_1, ..., \theta_n)]$ and $= 0$ otherwise, where $c(x_1, ..., x_n; n)$ is the coefficient of $\theta_1^{x_1} \theta_2^{x_2}$ in the expansion of $g(\theta_1, ..., \theta_n) \cdot f_n(\theta_1, ..., \theta_n)$ and $(x_1, ..., x_n)$ and $b(x_1, ..., x_n; n)$ are defined as in Section 5.

One may observe that the variance of $h(x_1, ..., x_n; n)$ is given by $E[(h(x_1, ..., x_n; n))^2] - (g(\theta_1, ..., \theta_n))^2$. Writing $g_n(\theta_1, ..., \theta_n) = g(\theta_1, ..., \theta_n)^2$, the MVU estimator of $g_n(\theta_1, ..., \theta_n)$ to be denoted by $h_n(x_1, ..., x_n; n)$ can be written down on the usual
5.5] Multivariate Generalised Power Series Distributions

It is clear that, because \((x_1, \ldots, x_k)\) is complete sufficient for \((\theta_1, \ldots, \theta_k)\), the MVU estimator of the variance of the MVU estimator of \(g(\theta_1, \ldots, \theta_k)\) is given by \(\hat{h}(x_1, \ldots, x_k; n)^2 - \hat{h}(x_1, \ldots, x_k; n)\) which, when \(g(\theta_1, \ldots, \theta_k) = \prod \theta_i^t\) reduces to \(h(x_1, \ldots, x_k; n)[h(x_1, \ldots, x_k; n) - h(x_1 - r_1, \ldots, x_k - r_k; n)]\) where \(h(x_1, \ldots, x_k; n)\) is defined by (22). The advantage of this form is obvious.

8. The MVU Estimation for Multinomial and Negative Multinomial Distributions

It can be easily verified that the multivariate GPSD given by (2) with the series function \(f(\theta_1, \ldots, \theta_k) = (1 + \theta_1 + \ldots + \theta_k)^m\) reduces to the multinomial distribution with probability function

\[
p(x_1, \ldots, x_k; p_1, \ldots, p_k) = \frac{m!}{x_1! \cdots x_k! (m - \Sigma x_k)!} p_1^{x_1} \cdots p_k^{x_k} (1 - \sum p_i)^{m - \Sigma x_k} \quad \cdots (24)
\]

where \(p_i = \theta_i/(1 + \Sigma \theta_i), \ i = 1, 2, \ldots, k\), with \(0 < \Sigma p_i < 1\) and \(0 < \Sigma x_i \leq m\). Using Section 7, we get after some algebra that for the multinomial distribution defined by (24) with parameters \(p_1, \ldots, p_k\) and \(m\), a function \(g(p_1, \ldots, p_k)\) is MVU estimable on the basis of a single observation if and only if it can be expanded as a polynomial in \(p_1, \ldots, p_k\) of degree not exceeding \(m\). Also, since \(f_n(\theta_1, \ldots, \theta_k) = (1 + \theta_1 + \ldots + \theta_k)^{mn}\), the vector \((x_1, \ldots, x_k)\) has a multinomial distribution with parameters \(p_1, \ldots, p_k\) and \(mn\). Therefore it follows that on the basis of a random sample of size \(n\) from the multinomial distribution defined by (24) a function \(g(p_1, \ldots, p_k)\) is MVU estimable if and only if it can be expanded as a polynomial in \(p_1, \ldots, p_k\) of degree not exceeding \(mn\).

Next, it may be verified that the multivariate GPSD given by (2) with the series function \(f(\theta_1, \ldots, \theta_k) = (1 - \theta_1 - \ldots - \theta_k)^{-m}\) reduces to the negative multinomial distribution with probability function

\[
p(x_1, \ldots, x_k; p_1, \ldots, p_k) = \frac{(m + x_1 + x_2 + \ldots + x_k - 1)!}{x_1! \cdots x_k! (m - 1)!} p_1^{x_1} \cdots p_k^{x_k} (1 - \sum p_i)^{m - \Sigma x_k} \quad \cdots (26)
\]

where \(p_i = \theta_i\), with \(0 < \Sigma p_i < 1\) and \((x_1, \ldots, x_k) \in \mathbb{I} \times \mathbb{I} \times \ldots \times \mathbb{I}\). Following the analysis of Section 7, we get that for the negative multinomial distribution defined by (26) with parameter \(p_1, \ldots, p_k\) and \(m\), a function \(g(p_1, \ldots, p_k)\) is MVU estimable on the basis of a single observation if and only if it admits a power series expansion about the origin. In particular, a polynomial in \(p_1, \ldots, p_k\) of every degree is MVU estimable on the basis of a single observation and also more since \((x_1, \ldots, x_k)\) has a negative multinomial distribution with parameters \(p_1, \ldots, p_k\) and \(mn\).

9. The MVU Estimation of the Probability and Distribution Functions of a Multivariate GPSD

Let \(g(\theta_1, \ldots, \theta_k) = p(r_1, \ldots, r_k; \theta_1, \ldots, \theta_k)\) where \((r_1, \ldots, r_k) \in \mathbb{T}\) is known and \(p(x_1, \ldots, x_k; \theta_1, \ldots, \theta_k)\) is defined by (2). One has \(W[g(\theta_1, \ldots, \theta_k) \cdot f_n(\theta_1, \ldots, \theta_k)] = (n - 1)[T] + [r_1, \ldots, r_k]\), whereas \(W[f_n(\theta_1, \ldots, \theta_k)] = n[T]\). Clearly the conditions of Theorem 7 are satisfied. Thus the probability function of a multivariate GPSD has a MVU estimator for every sample size \(n\) and it can be verified that it is given by

\[
h(x_1, \ldots, x_k; r_1, \ldots, r_k; n) = c(x_1, \ldots, x_k; r_1, \ldots, r_k; n)/\hat{h}(x_1, \ldots, x_k; n),
\]

for \((x_1, \ldots, x_k) \in (n - 1)[T] + \{(r_1, \ldots, r_k)\} \quad \text{and} \quad = 0, \quad \text{otherwise},

where

\[
c(x_1, \ldots, x_k; r_1, \ldots, r_k; n) = a(r_1, \ldots, r_k)\hat{h}(x_1 - r_1, \ldots, x_k - r_k; n - 1). \quad \cdots (26)
\]
Further, the MVU estimator of the variance of the MVU estimator under consideration always exists and is given by

\[ h(z_1, \ldots, z_k; r_1, \ldots, r_k; n) \]

\[ = h(z_1, \ldots, z_k; r_1, \ldots, r_k; n) - h(z_1 - r_1, \ldots, z_k - r_k; r_1, \ldots, r_k; n - 1) \]  \hspace{1cm} \ldots (27)

The advantage of this form is obvious.

In order to estimate the distribution function, the comments in Patil [6] apply here and the MVU estimator can be written down accordingly.

One may record below the applications of the results of this section to the multinomial and negative multinomial distributions.

(i) **Multinomial distribution**: One has

\[ a(r_1, \ldots, r_k) = mn!r_1! \ldots r_k!(m - \Sigma r_i)! \]

with usual restrictions on \( r_i \); and

\[ b(z_1, \ldots, z_k; n) = (mn)!z_1! \ldots z_k!(mn - \Sigma z_i)! \].

Therefore the MVU estimator for \( p(r_1, \ldots, r_k; \theta_1, \ldots, \theta_k) \) as defined by (24) can be obtained upon some rearrangement of terms as a multivariate hypergeometric probability

\[ h(z_1, \ldots, z_k; r_1, \ldots, r_k; n) = \left( \begin{array}{c} z_1 \\ r_1 \end{array} \right) \ldots \left( \begin{array}{c} z_k \\ r_k \end{array} \right) \left( \frac{mn - \Sigma z_i}{m - \Sigma r_i} \right) \left( \frac{mn}{m} \right). \]  \hspace{1cm} \ldots (28)

(ii) **Negative multinomial distribution**: One has

\[ a(r_1, \ldots, r_k) = (m + r_1 + \ldots + r_k - 1)!r_1! \ldots r_k!(m - 1)! \]

with usual restrictions on \( r_i \) and further

\[ b(z_1, \ldots, z_k; n) = (mn + z_1 + \ldots + z_k - 1)!z_1! \ldots z_k!(mn - 1)! \].

Therefore the MVU estimator for \( p(r_1, \ldots, r_k; \theta_1, \ldots, \theta_k) \) as defined by (25) can be obtained upon some rearrangement of terms as

\[ h(z_1, \ldots, z_k; r_1, \ldots, r_k; n) = \frac{m}{m + r_1 + \ldots + r_k} \left( \begin{array}{c} z_1 \\ r_1 \end{array} \right) \ldots \left( \begin{array}{c} z_k \\ r_k \end{array} \right) \left( \frac{mn - 1}{m + r_1 + \ldots + r_k} \right) \left( \frac{mn}{m} \right). \]  \hspace{1cm} \ldots (29)

The MVU estimators of the variances of the MVU estimators of the above probability functions can be obtained by using the relevant formulae of this section.

**References**


5. **Unified Models and Inference**

<table>
<thead>
<tr>
<th>Section</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1. Estimation of the parameters of the hyper-Poisson distributions</td>
<td>Edwin L. Oroz and George E. Bardwell</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>John Gurland</td>
<td>141</td>
</tr>
<tr>
<td>5.3. Inverse factorial series as frequency distributions</td>
<td>J. O. Irwin</td>
<td>159</td>
</tr>
<tr>
<td>5.4. Unified treatment of a broad class of discrete probability</td>
<td>Leo Katz</td>
<td>175</td>
</tr>
<tr>
<td>distributions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.5. On multivariate generalized power series distribution and its</td>
<td>G. P. Patil</td>
<td>183</td>
</tr>
<tr>
<td>application to the multinomial and negative multinomial</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.6. Further results concerning expectation-inversion technique</td>
<td>M. C. K. Tweedie</td>
<td>195</td>
</tr>
</tbody>
</table>

6. **Some Classical Distributions**

<table>
<thead>
<tr>
<th>Section</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1. Selection and ranking procedures and order statistics for the</td>
<td>Shanti S. Gupta</td>
<td>219</td>
</tr>
<tr>
<td>binomial distribution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.2. Some invariants of some discrete distributions admitting</td>
<td>V. S. Huswadzaz</td>
<td>231</td>
</tr>
<tr>
<td>sufficient statistics for parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.3. Comments on estimation for the negative binomial</td>
<td>L. R. Shenton and R. Myers</td>
<td>241</td>
</tr>
<tr>
<td>distribution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.4. Sequential estimation of a binomial parameter</td>
<td>M. T. Wasan</td>
<td>263</td>
</tr>
</tbody>
</table>

7. **Contagious Distributions**

<table>
<thead>
<tr>
<th>Section</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1. A case of contagion in binomial distribution</td>
<td>Roshan L. Chaddha</td>
<td>273</td>
</tr>
<tr>
<td>7.2. Asymptotic expansions for some contagious distributions</td>
<td>J. B. Douglas</td>
<td>291</td>
</tr>
<tr>
<td>7.3. Analysis of contagious data through behavioral models</td>
<td>S. K. Katti and Lawrence E. Sly</td>
<td>303</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.4. On discrete distributions arising out of methods of</td>
<td>C. Radhakrishna Rao</td>
<td>320</td>
</tr>
<tr>
<td>ascertainment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5. Some comments on the question of identifiability of parameters</td>
<td>D. A. Sprott</td>
<td>333</td>
</tr>
<tr>
<td>raised by Rao</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.6. A class of contagious distributions and maximum likelihood</td>
<td>D. A. Sprott</td>
<td>337</td>
</tr>
<tr>
<td>estimation</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. **Inference for Mixtures of Distributions**

<table>
<thead>
<tr>
<th>Section</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1. Mixtures of discrete distributions</td>
<td>W. R. Blischke</td>
<td>351</td>
</tr>
</tbody>
</table>
CHAPTER

8.2. Estimation in mixtures of discrete distributions ... ... ... ... A. Clifford Cohen, Jr. ... 373

8.3. Some elementary tests for mixtures of discrete distributions ... ... ... ... J. Tiago de Oliveira ... 379

9. Certain Distributions in Biological Sciences

9.1. An analysis of some insect trap records ... C. I. Bliss ... 385

9.2. Maximum likelihood estimation for the complete and truncated logarithmic series distributions ... ... ... ... G. P. Patil and J. K. Wani ... 398

9.3. The concept of segregation pattern in ecology: some discrete distributions applicable to the run lengths of plants in narrow transects ... ... ... ... E. C. Pielou ... 410

10. Finite Populations

10.1. Quota fulfilment in finite populations ... N. L. Johnson ... 419

10.2. Distribution of the product of random samples from a finite population ... ... Chia Kuei Tsao ... 427

11. General Topics

11.1. Probability distributions, factorial moments, empty cell test ... ... ... ... A. G. Lawrent ... 437

11.2. The zeta distribution ... ... ... ... Paul R. Rider ... 443

11.3. Orthogonal statistics ... ... ... ... L. R. Shenton and R. Myers ... 445

11.4. General form of the probability function associated with paired-comparison experiments ... ... ... ... B. J. Trawinski ... 459

12. Bibliography

12.1. A proposed bibliography on discrete distributions ... ... ... ... G. P. Patil ... 465

12.2. A selected bibliography of statistical literature on classical and contagious discrete distributions ... ... ... ... G. P. Patil ... 469

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