Reprinted From

A Modern Course on
STATISTICAL DISTRIBUTIONS IN SCIENTIFIC WORK

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Published by
D. Reidel Publishing Company
38 Papertspad, P. O. Box 17, Dordrecht, Holland
936 Dartmouth Street, Boston, Massachusetts 02116
STATISTICAL DISTRIBUTIONS IN SCIENTIFIC WORK

Based on the Nato Advanced Study Institute
A Modern Course on Statistical Distributions in Scientific Work
and The International Conference on Characterizations
of Statistical Distributions With Applications

Held at
The University of Calgary, Calgary, Alberta, Canada
July 29-August 10, 1974

Sponsored by
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The Pennsylvania State University
The University of Calgary
Indian Statistical Institute

With the Support of
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CHANCE MECHANISMS FOR DISCRETE DISTRIBUTIONS IN SCIENTIFIC MODELING

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KEY WORDS

Discrete distributions; chance mechanisms; sampling; waiting times; heterogeneity; clustering; population growth; group sizes; queueing processes.

1. INTRODUCTION AND SUMMARY

This paper gives a broad overview of models for major discrete distributions such as binomial, negative binomial, Borel-Tanner and lost-games. Because the need lies not only in prediction problems but, perhaps even more so, in developing understanding and insight of natural phenomena, there has been increasing interest in identifying and developing models which give rise to various distributions. For example, see Bates and Heyman (1952), Haight (1961, 1966), Kemp and Kemp (1968, 1969, 1971), Boswell and Patil (1970, 1971, 1972), Patil and Boswell (1972), and Janardan (1973).

In this paper we give a suitable collection of examples showing how the mechanisms result in various distributions. The collection is far from being comprehensive, but it can serve to illustrate the major mechanisms in a common notation. We assume the reader is familiar with the more common mechanisms. Thus in Section 2 we give some examples of univariate waiting time mechanisms resulting in "newer" distributions. Section 3 gives multivariate sampling mechanisms. Section 4 contains multivariate mechanisms for heterogeneity. Examples of multivariate mechanisms for clustering are few; Section 5 concentrates on uni-
2. Waiting Time Mechanisms

A waiting time mechanism is simply sampling or counting until some event occurs. This is also known as inverse sampling. A classical example is counting the number of failures before the k-th success in a sequence of Bernoulli trials. As everyone knows the negative binomial distribution results.

2.1. Waiting time for random walks

The lost-games distribution arises in the classical problem of gambler's ruin. Let \( N_p \) be the number of games played until ruin, which occurs with probability one if the probability, \( p \), of taking a step to the right (or winning) is less than \( q = 1 - p \), where \( r \) is the fortune (dollars) the ruined gambler starts with. Let \( Y_p \) be the number of games lost and \( X_p \) be the number won. Then \( X_p + Y_p = N_p \). Further \( X_p = Y_p + r \). Kemp and Kemp (1968) start with Feller's solution (Feller, 1968, p. 351) for the distribution of \( N_p \) and find the density for the lost-games distribution

\[
f_X(x) = \binom{2x-r}{x} p^{x-r} q^{r/(2x-r)}, \quad x = r, r + 1, \ldots
\]

Note we are using \( p \) as the probability of a step to the right; Kemp and Kemp use \( q = 1 - p \). Kemp and Kemp also point out that this distribution arises in the context of a simple epidemic. The total size of an epidemic is a waiting time. Reporting the work of McKendrick (1926) and Irwin (1963), they give the lost-games distribution with parameter \( r = 1 \) and \( p^k/(k+1) \) for the total size where the infection started from a single individual, \( k \) is the infection rate and the recovery rate is \( l \).

2.2. Waiting time for a single server queue with Poisson input

Suppose customers arrive at a queue according to a Poisson process with parameter \( \lambda \). Let \( X \) be the number of customers served
before the queue first vanishes, when there are \( r \) customers in the system.

**Example 2.2.1.** Let the service time for each customer be constant, say \( b \). Then \( X \) has the density

\[
f_X(x) = e^{-\lambda} \lambda^{x-r} x^{r-1} r/(x-r)! , \quad x = r, r+1, ... \]

For details see Haight and Breuer (1960).

**Example 2.2.2.** Let the service time have a negative exponential distribution with average service time \( \mu \). Haight (1965) showed for this case that \( X \) has the lost-games distribution with parameters \( r \) and \( p = \lambda/(\lambda+\mu) \). Haight also gives results for general service time and even gives some results when the only assumption is stationary service times. The results are very similar. It is surprising what can be obtained with so few assumptions. In addition, Haight studies queues with balking and queues with feedback.

3. SAMPLING MECHANISMS

There are many sampling schemes such as direct and inverse sampling with or without replacement or with modified replacement via the various urn schemes. Many of the resulting distributions are special cases of what Janardan and Patil (1972) call the unified multivariate hypergeometric (UMH) distribution. Let \( \mathbf{X} = (X_1, X_2, \ldots, X_s) \) be a random vector. Then \( \mathbf{X} \) has a \((s\text{-variate})\) UMH distribution with parameters \( n \) and \((a_0, \ldots, a_s)\)

\[
f_{\mathbf{X}}(\mathbf{x}) = \binom{n}{\mathbf{x}} \prod_{i=0}^{s} \frac{a_i}{x_i},
\]

where \( x_0 = n - x_1 - \cdots - x_s \), \( a = a_1 + \cdots + a_s \). Note in this formulation that \( a_i \)'s and \( n \) are allowed to be either positive or negative real numbers, and the \( x_i \)'s are restricted depending on the signs of the \( a_i \)'s and \( x_i \)'s. To make the binomial coefficient \( \binom{n}{\mathbf{x}} \) well defined Janardan and Patil define

\[
(-a)! = (-1)^{a-1}/(a-1)! = (-1)^{a-1}/\Gamma(a), \quad \text{for } a = 1, 2, \ldots \quad \text{and} \quad a! = \Gamma(a+1) \text{ if } a > 0.
\]

The following two examples discuss multivariate versions of classical urn schemes. In a series of recent papers some authors have used urn schemes in a gambling situation resulting in winning or losing. For references see Consul (1973) and

Example 3.1. Polya's urn scheme is well known (see Feller, 1968, p.120). The multivariate version samples repeatedly from an urn containing \( s \) different colors; \( n \) balls are drawn from an urn containing \( k_0 \) white, \( k_1, \ldots, k_s \) of \( s \) other colors. After each draw the ball is replaced along with \( c \) balls of the same color. The resulting distribution has a density function

\[
f(x) = \sum_{x=0}^{n} \binom{n}{x_1, \ldots, x_s} \frac{(-k_0/c)}{n} \frac{(-k_1/c)}{x_1} \cdots \frac{(-k_s/c)}{x_s} , \quad x_0 = x - x_1 - \cdots - x_s = n \]

where \( x = (x_0, \ldots, x_s) \), and \( x_0 = n - x_1 - \cdots - x_s \).

It is interesting to observe that \( X = x_1 + \cdots + x_s \) has a Polya density function, since the count \( X \) would result if \( x \) were drawn by a person who could not tell colors 1 through \( s \) apart.

Example 3.2. An urn scheme which can be thought of as a modification of the (univariate) Polya's urn scheme is suggested by Sibuya, Yoshimura, and Shimizu (1964). The colored balls were split up into various types or colors. Each time a colored ball is drawn it is replaced along with \( c = c_1 + \cdots + c_s \) colored balls, \( c_1 \) of type 1, \( \ldots, c_s \) of type \( s \), and each time a white ball is drawn it is replaced along with \( c \) white balls. Then the vector \( X = (x_1, \ldots, x_s) \) of the number of balls of the \( s \) colors drawn in \( n \) turns has a density function

\[
f(x) = \prod_{i=1}^{s} \binom{n}{x_i} \frac{(-k_0/c)}{n} \frac{(-k_i/c)}{x_i} = \prod_{i=1}^{s} \frac{n!}{x_i!} \frac{(-k_0/c)}{n} \frac{(-k_i/c)}{x_i} , \quad x_0 = n - x_1 - \cdots - x_s \]

where \( x = x_1 + \cdots + x_s \), and \( x_0 = n - x \).

This can also be thought of as a modified multivariate Polya's urn scheme. Clearly \( X = x_1 + \cdots + x_s \) has a Polya distribution.

Example 3.3. Polya's urn scheme with inverse sampling is the same as explained in Example 3.1 except sampling continues until there are \( x_0 \) balls of the 0-th type in the sample. The resulting density function is

\[
f(x) = \prod_{i=1}^{s} \binom{n}{x_i} \frac{(k_0/c)}{x_i} \frac{(k_i/c)}{x_i} = \prod_{i=1}^{s} \frac{n!}{x_i!} \frac{(k_0/c)}{x_i} \frac{(k_i/c)}{x_i} , \quad x_0 = x - x_1 - \cdots - x_s \]

where \( x = x_0 + x_1 + \cdots + x_s \). Note that \( x_0 \) is a parameter of this
distribution.

4. MECHANISMS INVOLVING HETEROGENEITY

Suppose for a particular population, with known parameters, the count \( X = (X_1, \ldots, X_s) \) has some distribution with parameter \( \lambda \), but from place to place or from species to species the count has the same distribution while the parameter \( \lambda \) varies. If one samples one of these populations the count has parameter \( \Lambda \) which is a random variable; \( \Lambda \) can be a vector of random variables.

Example 4.1. Bates and Neyman (1952) proposed a model for accident proneness. They hypothesized that some individuals are more prone than others to different types of accidents. The number of accidents an individual has depends on his exposure to the risk of the various types of accidents. Consider a types of accidents and let \( a_1, \ldots, a_s \) denote the levels of exposure to risk. Let \( \lambda \) be the accident proneness. Then the number of accidents \( X = (X_1, \ldots, X_s) \) an individual has of the \( s \) different types can be taken to be a multiple Poisson distribution with parameters \( (a_1 \lambda, \ldots, a_s \lambda) \). Assuming the accident proneness, \( \lambda \), varies from individual to individual according to a gamma distribution with parameters \( \alpha \) and \( \beta^{-1} \) gives the negative multinomial distribution with

\[
\mathbb{P}(x) = \binom{\alpha + x - 1}{x} \prod_{i=0}^{s} p_i^{x_i} \frac{\Gamma(\alpha)}{\Gamma(x + 1)},
\]

where \( x_0 = n \), \( p_0 = (\alpha + 1)^{-1} \), \( p_i = a_i \beta / (a_0 + 1) \), \( a = a_1 + \ldots + a_s \) and \( x = x_1 + \ldots + x_s \), \( i = 1, \ldots, s \).

We observe that given \( \Lambda = \lambda \), \( X = X_1 + \ldots + X_s \) has a Poisson distribution with parameter \( \lambda \), thus \( X \) has the negative binomial distribution with parameters \( k = \alpha \) and \( p = (a_0 + 1)^{-1} \).

Example 4.2. Moessnna (1963) proposed a model for use by the pollen analyst. In cores taken from dry lake beds pollen is counted in the various strata. Let \( X = (X_1, \ldots, X_s) \) be the numbers of \( s \) types from \( n + 1 \) pollen types observed when a count of \( n \) is taken. Then, assuming the pollen rain is independent but in constant proportion, \( X \) has the multinomial distribution, with density

\[
\mathbb{P}(x | \pi) = \binom{s}{x_1} \prod_{i=0}^{s} p_i^{x_i} \frac{1}{x_1! \ldots x_s!} \quad x_1 + \ldots + x_s = n,
\]
where \( x_0 = n - \sum x_i \), \( \ldots, x_n, p_0 = 1 - p_1 - \ldots - p_n \), with parameters \( n \) and \( p = (p_1, \ldots, p_n) \).

In the various strata the ratios of the pollen density would change due to variation in the vegetation types. Assume \( \theta = (p_1, \ldots, p_n) \) has a Dirichlet distribution

\[
    f_{\theta}(\theta) = \frac{\Gamma(\theta_1) \cdots \Gamma(\theta_n)}{\Gamma(\theta_1 + \cdots + \theta_n)} \theta_1^{n-1} \cdots \theta_n^{n-1}, 0 < p_1 < 1, p_0 < \sum p_i = 1,
\]

\( \theta_1 > 0, \theta_0 > 0, \theta_i > 0, i = 1, \ldots, n \) where \( p_0 = 1 - p_1 - \ldots - p_n \),

\( \theta = \theta_0 + \cdots + \theta_n \).

The resulting distribution is the \( s \)-variate-negative-hypergeometric distribution with density

\[
    f_X(x) = \frac{\left( \begin{array}{c} -\theta_0 \\ n \end{array} \right) \cdots \left( \begin{array}{c} -\theta_n \\ n \end{array} \right)}{\left( \begin{array}{c} \sum x_i \\ n \end{array} \right)^n},
\]

where \( \theta = \theta_0 + \cdots + \theta_n \), and \( x = x_1 + \cdots + x_n \).

Example 4.3. Moolan (1963) gives a model similar to the above except one samples inversely until \( n \) of the same th type are observed. Then \( X = (X_1, \ldots, X_n) \) has the negative multinomial distribution with

\[
    f_X(x | p) = \left( \begin{array}{c} x+n-1 \\ x \end{array} \right) \left( \begin{array}{c} \sum x_i \\ n \end{array} \right)^n, p_0 \left( \begin{array}{c} \sum x_i \\ n \end{array} \right)^n
\]

where \( x = x_1 + \cdots + x_n \), and the \( p_1 \)'s are relative frequencies of the \( s+1 \) types of pollen in the pollen rain.

Now if \( P \) has a Dirichlet distribution as in the above example, then \( X \) has the \( s \)-variate-negative-inverse-hypergeometric distribution with

\[
    f_X(x) = \frac{\left( \begin{array}{c} -\theta_0 \\ -x+n \end{array} \right) \cdots \left( \begin{array}{c} -\theta_n \\ -n \end{array} \right) \left( \begin{array}{c} \theta_0 \\ x_1 \end{array} \right)^n}{\left( \begin{array}{c} \sum x_i \\ n \end{array} \right)^n}, x_i = 0, 1, \ldots,
\]

where \( x = x_1 + \cdots + x_n \) and \( \theta = \theta_0 + \cdots + \theta_n \).

Example 4.4. Suppose \( X = (X_1, \ldots, X_n) \) has a multinomial distribution with parameters \( N, p = (p_1, \ldots, p_n) \) where \( N \) is a
random variable with a binomial distribution with parameters \( m \) and \( q \). Then \( X \) has a multinomial distribution with parameters \( m \) and \( q = (q_1, \ldots, q_s) \). It is educational to see intuitively how this might come about. Suppose \( m \) items are looked at sequentially and classified as good or bad. Then the good items are looked at again and classified as to which of \( s+1 \) types they are. The same result could have been obtained by looking at all \( m \) and classifying as to both good and type for \( s \) types or as bad.

Example 4.5. Suppose \( X = (X_1, \ldots, X_s) \) has a multivariate hypergeometric distribution

\[
f(X | m) = \left[ \prod_{i=0}^{s} \frac{M_i!}{X_i!} \right] / m!
\]

where \( X_0 = n - X_1 - \cdots - X_s \) and \( M = M_1 + \cdots + M_s \). Further suppose \( M = (M_1, \ldots, M_s) \) has a multinomial distribution with parameters \( m \) and \( p = (p_1, \ldots, p_s) \); then \( X \) has a multinomial distribution with parameters \( m \) and \( p \). This example can also be seen intuitively. Consider a sequence of \( m \) independent trials resulting in \( X \) with the multinomial distribution. Then from the resulting \( X_0, \ldots, X_s \) of \( s+1 \) types a sample of \( n \) is chosen without replacement. Clearly each item chosen could be any of the \( s+1 \) types. Thus the sample results in a multinomial distribution.

There are many more mixtures (heterogeneity mechanisms). A few sources for these are Patil and Joshi (1958), Johnson and Kotz (1969, 1970, 1972), Cohen (1971), and Ord (1972).

5. MECHANISMS INVOLVING CLUSTERING

Suppose a certain population tends to form groups or clusters and the size of the clusters varies according to some chance mechanisms. Then the population count, \( X = X_1 + \cdots + X_M \), is a sum of a random number \( N \) of groups. If the clusters are distributed at random, according to a Poisson process, the count is called a Poisson sum, with stepping random variable \( N \). In general the population could consist of individuals of various types in which case \( X \) would be a vector.

Example 5.1. Kagan (1973) proposes a Poisson sum in his study of seismic processes. He asserts earthquakes come in groups composed of a main shock, foreshocks and aftershocks. Due to strain relief when a major shock occurs the groups cannot be independent as required by the Poisson assumption, but Kagan hypothesizes that if major quakes are ignored the clusters should
be nearly independent. He proposes a Poisson process in 4-dimensional space composed of Euclidean space and time for the cluster centers. Let \( L \), the energy level of the main shock, have a uniform distribution on \((k_1, k_\infty)\). Then assuming the number of members in a cluster with an energy level greater than \( k \) has a geometric distribution with parameter \( p = e^{-\mu(\delta-k)} \) for given \( L > \delta \), Kagan obtains the negative binomial with parameters 
\[
-p(k-m)
\]
\( \nu/\mu \) and \( p = e^{-\mu(\delta-k)} \) for the number of shocks with energy level \( k \) in a large region (so that clusters can be assumed to be points) and concludes the total events in a cluster will follow a logarithmic series distribution.

It is interesting to observe that when Kagan formulated this as a birth process, an event could give birth to an event which actually preceded it in time. That is, the major event gave birth to fore-shocks which in turn gave birth to other fore-shocks. The birth process again is in 4-dimensional space. The major events are immigrants, and any event can give birth to any other event anywhere in the time-space space. That is the process involves birth-with-immigration.

**Example 5.2.** Consider in the above example in the following Poisson sum. Let \( N \) have a Poisson distribution with parameter \( \lambda \), and let \( X_i, i=1,2, \ldots \) be independent random variables with the logarithmic series distribution with parameter \( \theta \). Then \( X = X_1 + \ldots + X_n \) has the negative binomial distribution with parameters \( k = -\lambda/\ln(1-\theta) \) and \( p = 1-\theta \) (see also Bowers and Patil, 1970).

**Example 5.3.** Let \( N \) have a Poisson distribution with parameter \( \lambda \), and let \( X_i, i=1,2, \ldots \) be independent Bernoulli random variables with parameter \( p \). Then \( X = X_1 + \ldots + X_n \) has a Poisson distribution with parameter \( \lambda p \). This can be interpreted as individuals distributed at random and surviving independently with probability \( p \).

**Example 5.4.** A less well-known distribution which can occur as the result of a clustering mechanism is the lost-games distribution or a translation of it. Using the notation of Section 2, and subtracting \( r \) results in a random variable taking non-negative integer values (this distribution can be interpreted as the won-games distribution). Let \( Y_j = X - r + j \) be a further translation taking values \( j, j=1, \ldots, \). Then we say \( Y_j \) has a translated-lost-games distribution with parameters \( r, j, \) and \( p \). See Kemp and Kemp (1969, 1971), for more details.

Let \( N \) have a binomial distribution with parameters \( n \) and \( (1-p) \), and let \( X_i, i=1,2, \ldots \) be independent random variables
with a translated-lost-games distribution with parameters \( r=2, \)
\( j=1, \) and \( p \) (same \( p \)). Then the population count \( X = X_1^+ \ldots + X^N \)
has the lost-games distribution with parameters \( n \) and \( p \).

**Example 5.5.** Using the notation of Example 5.4 let \( N \) have
a negative binomial distribution with parameters \( k \) and \((1-p)\) and
let \( X_1, X_2, \ldots \) be independent random variables with a translated-
lost-games distribution with parameters \( r=2, j=1 \) and \( p \) (same \( p \)).
Then Kemp and Kemp (1969) observe that \( Y = Y_1^+ \ldots + Y^N \) has the
lost-games distribution with parameters \( k \) and \( p \).

**Example 5.6.** Using the notation of Example 5.4 let \( N \) have
a Poisson distribution with parameter \( \lambda = -r \ln(p) \), and let
\( X_1, X_2, \ldots \) be independent random variables with the
logarithmic series distribution with parameter \( \theta = 1-p \). Further
let \( Y_1 \) have the lost-games distribution with parameters \( 1 \) and \( p \).
Then Kemp and Kemp (1969) observe that \( Y = Y_1^+ \ldots + Y^N \) has the
lost-games distribution with parameters \( r \) and \( p \), where
\( X = X_1^+ \ldots + X^N \). It is interesting to see how the parameters
of the various distributions are interrelated. The above example
shows incidentally that the lost-games distribution is
infinitely divisible since \( Y \) can be written as a Poisson sum.

As with mixtures there are many random sums (clustering
mechanisms). A few sources of these are Patil and Joshi (1968),

5. MECHANISMS OF POPULATION GROWTH

5.1. Introduction

Counting processes where the count can increase by one, stay
the same, or decrease by one at any instant are called birth and
death processes. We formulate the problem in the multivariate
case. The population size changes with birth and death rates
which in general are functions of size and time.
Let \( X(t) = (X_1(t), \ldots , X^N(t)) \) be the population count where
there are \( s \) possible classifications. The birth rate \( \lambda_j(x,t) \) of
the \( j \)-th type is in general a function of the count in each
classification as well as time. Similarly let the death rate of
the \( j \)-th type be \( \mu_j(x,t) \). More general results than the following
material may be found in Boswell and Patil (1972) and Patil and
Boswell (1972).
6.2. Pure birth processes

In this case we assume the count can only increase. An increase occurs in one coordinate at a time. We can always assume the count at time zero is zero by counting the increases instead of the population sizes. Let

\[ \lambda_j = a_j (\gamma + \delta x) h(t) \text{ if } \gamma + \delta x > 0, \]

where \( \Sigma a_j = 1, \delta = -1, 0, \text{ or } +1, \gamma > 0, \) and where \( x = \Sigma x_j. \)

(1) If \( \delta = -1 \) and \( \gamma \) is an integer, then \( \lambda(t) \) has the multinomial distribution

\[
P(\lambda(t)) = \frac{\gamma!}{\lambda_1! \cdots \lambda_n!} \left( P_0(t) \right)^{\gamma - \lambda} \prod_{j=1}^{n} \left( a_j (1 - P_0(t)) \right)^{x_j}
\]

where \( P_0(t) = e^{-H(t)}. \)

(2) If \( \delta = 0, \) then \( \lambda(t) \) has the multiple Poisson distribution

\[
P(\lambda(t)) = \prod_{j=1}^{n} \left( a_j H(t) \right)^{x_j} \frac{e^{-H(t)}}{x_j!}.
\]

(3) If \( \delta = +1, \) then \( \lambda(t) \) has the negative multinomial distribution

\[
P(\lambda(t)) = \frac{\gamma!}{\lambda_1! \cdots \lambda_n!} \left( P_0(t) \right)^{\gamma - \lambda} \prod_{j=1}^{n} \left( a_j (1 - P_0(t)) \right)^{x_j}
\]

where \( P_0(t) = e^{-H(t)}. \) We observe that in each case

\[ \lambda = \lambda_1^+ \ldots + \lambda_n \]

has the corresponding univariate distribution.

6.3. Birth and death processes

Let \( \lambda_j (\lambda(t)) = a_j (\gamma + \delta x) h(t) \) and \( \mu_j (\lambda(t)) = a_j g(t) \) where \( \Sigma a_j = 1, \gamma > 0, \delta = -1, 0, \) or \( +1 \) and where \( x = \Sigma x_j. \)

(1) If \( \delta = -1, \) and \( \gamma \) is an integer, then \( \lambda(t) \) has the multinomial distribution

\[
P(\lambda(t)) = \frac{\gamma!}{\lambda_1! \cdots \lambda_n!} \left( P_0(t) \right)^{\gamma - \lambda} \prod_{j=1}^{n} \left( a_j (1 - P_0(t)) \right)^{x_j}
\]

with \( P_0(t) = 1 - E[X] / \gamma, \) where \( E[X] = E[\Sigma x_j(t)] \) is given below.
(2) If $\delta = 0$, then $X(t)$ has the multiple Poisson distribution

$$P(X(t) = n) = \frac{n!}{\prod_{j=1}^{n} (m_j(t))} \cdot \frac{\exp(-m_j(t))}{(m_j(t))^n}$$

with $m_j = a_j E[X]$.

(3) If $\delta = 1$, then $X(t)$ has the negative multinomial distribution

$$P(X(t) = n) = \binom{n}{k} \left[ \sum_{j=1}^{n} a_j (1 - P_0(t)) \right]^{k} \left[ \sum_{j=1}^{n} a_j P_0(t) \right]^{n-k}$$

with $P_0(t) = \gamma / (1 + E[X])$.

For each of the above cases

$$E[X] = \gamma e^{gH(t) - G(t)} \int_0^t h(u) e^{-[\gamma H(u) - G(u)]} du,$$

where as before $H(t) = \int_0^t h(u) du$ and $G(t) = \int_0^t g(u) du$. Again we observe that $X = X_1 + \cdots + X_6$ has the corresponding univariate distribution.

5.4. Pure death processes

A pure death process starts with say $n = (n_1, \ldots, n_6)$ individuals at $t = 0$, and they die or are removed as time progresses. This process can be formulated as a pure birth process if one counts the number of individuals of each type $X(t)$ which die instead of the number of individuals $Y(t)$ alive at time $t$. Observe $X(t) + Y(t) = n$. If the death rates are $\mu_j(y_j, t)$, then the birth rates for $X(t)$ are $\lambda_j(x_j, t) = \mu_j(n - x_j, t)$.

When there is no interaction each population type dies independently. The usual assumption for the death rates is

$$\lambda_j(x_j, t) = \mu_j(n - x_j, t) = a_j(t) \cdot (n_j - x_j),$$

which by Section 6.2 yields independent binomials.

6.5. Bivariate Poisson process

If a simultaneous change in more than one coordinate can occur the process is not a birth and death process. With the
remaining assumptions of a birth and death process this is known as a Markov process. Handran (1973) considers a bivariate case where increases only can occur. The rate of increase in the $i$-th coordinate only is $\lambda_i - \lambda_{12}$, $i=1,2$. The simultaneous rate of increase in both coordinates is $\lambda_{12}$. By methods often applied to the Poisson process the author finds a differential equation for the generating function. The joint density is the bivariate Poisson density,

$$P_{X_1(t),X_2(t)} = P[X_1(t) = x_1, X_2(t) = x_2]$$

$$= e^{-(\lambda_1 + \lambda_2 - \lambda_{12})} \frac{\min(x_1, x_2)(\lambda_1 - \lambda_{12})^{x_1-1}(\lambda_2 - \lambda_{12})^{x_2-1}(\lambda_{12})^{x_1+x_2}}{(x_1-1)!(x_2-1)!}$$

where the summation symbol is inadvertently left out in Handran's paper.

V. GROUP SIZE MECHANISMS

In Section 4 we considered a population divided into $a$ types, here we consider various groups of individuals associating for whatever reason. For example, those who prefer types $t_1, \ldots, t_a$ of foodstuffs (undecided and no preference could be two groups). Whatever the reason the population is partitioned into disjoint groups and from time to time individuals change groups. We are interested in the number of groups $X_k(t)$ with $k$ individuals at time $t$.

Cohen (1971) has introduced an essentially deterministic model for the number of individuals in a randomly chosen sleeping group of monkeys. He assumes a closed system of $N$ monkeys and that changes in size are proportional to the number of individuals and groups involved (both an individual factor and a group factor are involved for either joining or leaving a group). Further these changes occur by a single individual either joining or leaving a group. Interpreting the resulting size at equilibrium as the expected size and the ratio of expected size to total population size as the probability he obtains the (truncated) negative binomial distribution. The details are messy and are not given here.

This model is modified in Rosewell and Pattil (1970) by making similar assumptions about the expected group size; the (truncated) binomial, Poisson, and negative binomial distributions result by the appropriate choice of the individual and group.
factors.

Instead of assuming a fixed population size, assume there are always single individuals (infinite number) and follow a single group to equilibrium. The rate at which a group of size $k$ increases to a group of size $k+1$ is $(a+bk)$, where $a$ is the group factor and $b$ is the individual factor. The rate at which a group of size $k$ decreases to a group of size $k-1$ is $(-cdk)$ where $c$ is the group factor and $d$ is the individual factor for losing an individual from the group. This then is nothing more than a birth and death process with linear birth and death rates. At equilibrium the distribution does not depend on the initial conditions. As explained in Section 6, this process (with suitable boundary conditions) results in the binomial, negative binomial or Poisson distributions.

8. QUEUING PROCESSES AT EQUILIBRIUM

Suppose a queue has been operating for some time; the number of people, $X$, in the queue, after a long time, is of interest. In the following we assume people arrive at random according to a Poisson process and join the queue with probability $c \cdot f(x)$, where $f(x)$ is some function of the queue length and $c$ is a constant. We also assume exponential service time. The details of the examples below can be found in Boswell and Patil (1970, p.17) and Boswell and Patil (1971, p.109).

Example 8.1. Let $X$ be the queue length at equilibrium in the above model with $f(x) = (x+1)/(x+1)$. Then $X$ has a negative binomial distribution with parameters $k$ and $p = (1 - c\lambda/\mu)$ where $\lambda$ is the arrival rate and $\mu$ is the service rate.

Example 8.2. Let $X$ be the queue length, as above, with $f(x) = x/(x+1)$. Then $X$ has a zero-modified logarithmic series distribution with parameter $\theta = c\lambda/\mu$ where $\lambda$ is the arrival rate and $\mu$ is the service rate.

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