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GLOBAL ASYMPTOTIC STABILITY IN A DIFFERENTIAL DELAY EQUATION MODELING MEGAKARYOPOIESIS

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Abstract. Simple form differential delay equation modeling megakaryopoiesis and other physiological processes is considered. Sufficient conditions are established for the global asymptotic stability of the unique positive equilibrium. The conditions are given in terms of induced interval maps, one set being delay independent conditions and another one involving the delay.

Key Words. Scalar differential delay equations, Negative feedback, Slowly oscillating solutions, Global asymptotic stability, Induced interval maps, Mathematical model of megakaryopoiesis

AMS(MOS) subject classification. 34K20, 34K11, 37E05, 92B05

Introduction. This paper deals with simple form scalar differential delay equation

$$x'(t) = -\mu x(t) + f(x(t))\, g(x(t-\tau)), \tag{1}$$

where continuous functions f and g are positive and decreasing on the positive semi-axis $\mathbb{R}_+$, delay $\tau > 0$ is positive, and $\mu > 0$ is a parameter. The equation was proposed in [3] as a mathematical model of megakaryopoiesis (platelet production). It is of a more general form than many of the well-known delay equations such as Mackey-Glass physiological model [18], Nicholson's blowflies population models [2], Wazewska-Lasota red blood cell production model [22], and some others [8, 9, 17, 20]. Each of those models can be obtained from (1) when $f(x) \equiv c$ - const and $g(x)$ is of a particular shape.

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The principal problem we address in this paper is about the global asymptotic stability of the unique positive equilibrium of equation (1). We use and further develop some basic approaches and techniques from our earlier papers [16, 15, 13] to study this equation. Several sufficient conditions are derived for the global asymptotic stability. One is a delay independent criterion in terms of an underlying interval map defined by functions f and g; the other one is dependent on the delay $\tau > 0$, parameter μ, and system's nonlinearities.

1. Preliminaries.

1.1. Assumptions and Linearization. In this section we describe standing assumptions on equation (1) as well as its basic and well known properties which are used in subsequent exposition in Chapter 2.

Throughout the paper we make the following assumptions on functions and parameters of equation (1):

- (A_1) Functions f, g are defined, continuous, and positive on the positive semiaxis $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$;
- (A_2) Both f and g are strictly decreasing on $\mathbb{R}_+$;
- (A_3) The decay rate μ and the delay τ are positive, $\mu > 0, \tau > 0$.

For some of the results derived additional differentiability properties of f and g will be needed. Therefore, in those instances we will be assuming the following enhanced assumption to (A_1):

- (A_1^*) Functions f, g are defined, continuously differentiable, and positive on the positive semiaxis $\mathbb{R}_+$.

Phase space for equation (1) is the set

$$\mathbb{X} = \{\varphi(s) \in C([-\tau, 0], \mathbb{R}_+) \mid \varphi(s) \geq 0 \;\forall\; s \in [-\tau, 0]\}$$

We shall assume that for arbitrary $\varphi \in \mathbb{X}$ there exists a unique solution $x = x(t, \varphi)$ to equation (1) defined for all $t \geq 0$. A sufficient condition for such existence can be e.g. the uniform Lipschitz continuity of $f(x)$ on $\mathbb{R}_+$ (thus the enhanced assumption (A_1^*) is also sufficient). Further conditions for the existence of global solutions for all $t \geq 0$ can be found in [1, 7, 12]. One can easily show that for every $\varphi \in \mathbb{X}$ the corresponding solution $x = x(t, \varphi)$ satisfies $x(t) > 0, \forall t > 0$. See e.g. [15, 14, 3] for proof and further details. Thus the set $\mathbb{X}$ is invariant under the shift operator along solutions of differential delay equation (1) in forward times.

Under the standing assumptions $(A1)$-$(A3)$ differential delay equation (1) has a unique positive equilibrium $x_* > 0$; it is found as the unique positive solution of the scalar non-linear equation $\mu x = f(x)\, g(x)$. Since the function

$f(x)g(x)$ is continuous and decreasing on $\mathbb{R}_+$, there is exactly one solution $x = x_* > 0$ of the latter equation.

The linearized equation (1) about the equilibrium x_* has the form

$$y'(t) = [-\mu + f'(x_*)g(x_*)]y(t) + f(x_*)g'(x_*)y(t-\tau) := -Ay(t) - By(t-\tau). \tag{2}$$

In order to derive (2) one makes an additional assumption that $f'(x_*)$ and $g'(x_*)$ exist. Since $f(x_*), g(x_*) > 0$ and $f'(x_*), g'(x_*) \le 0$ one generally has that $A > 0$ and $B \ge 0$. We shall be making a generic assumption that $f'(x_*) < 0$ and $g'(x_*) < 0$ are satisfied implying that the inequalities $A > 0, B > 0$ are valid.

The characteristic equation of the linear differential delay equation (2) has the form

$$\lambda + A + B \exp\{-\lambda\tau\} = 0. \tag{3}$$

Many properties of the characteristic equation (3) are associated with various properties of solutions of both linear equation (2) and the original non-linear equation (1) [1, 7, 12].

1.2. Invariance and Persistence. The invariance and persistence are simple and well-known properties of solutions of equation (1). As already mentioned, the phase space of equation (1) is chosen as $\mathbb{X} := C([-\tau, 0], \mathbb{R}_+)$ due to its invariance along solutions in forward times. It is proved that for arbitrary initial function $\varphi \in \mathbb{X}$ the corresponding solution $x = x(t, \varphi)$ satisfies $x(t) > 0$ $\forall t > 0$. Moreover, there are two positive constants $0 < m \le M < \infty$, dependent on f, g and independent of the initial function φ, such that there exists a time moment $T = T(\varphi)$ such that the corresponding solution $x(t, \varphi)$ satisfies $m \le x(t) \le M$ for all $t \ge T$. One can extract the proof of the invariance and uniform persistence properties from other published work, in particular, from papers [3, 14, 15]. In this paper we provide explicit expressions for the positive constants m and M in terms of functions f, g and the parameter μ. They are given in subsection 2.1.

1.3. Oscillation. In this subsection we summarize some of the known results about the oscillatory behavior of all solutions to the differential delay equation (1). The oscillating nature of solutions will be used explicitly in subsection 2.2 for the derivation of stability conditions.

As usual, a solution of equation (1) is called oscillatory (about the equilibrium x_*) if $x(t) - x_*$ has an infinite unbounded sequence $\{z_n\}$ of zeros, $x(z_n) = x_*$ with $\lim_{n\to\infty} z_n = \infty$. In differential delay systems with negative feedback, which is the case of equation (1), the oscillation of a solution $x(t)$

is equivalent to the existence of an infinite sequence $\xi_n : \xi_1 < \xi_2 < \xi_3 < \xi_4 < \cdots < \xi_{2n-1} < \xi_{2n} \cdots \to \infty$ such that $x(\xi_{2n}) > x_*$ and $x(\xi_{2n-1}) < x_*$.

Oscillation criteria for solutions of equation (1) are abundant and well known (see e.g. monograph [10] for further references and details). The following result follows from Theorem 1 of paper [4].

PROPOSITION 1.1. *Suppose that functions f and g are C^2 smooth in a neighborhood of the equilibrium x_*. If the characteristic equation (3) of its linearization (2) has no real zeros then all solutions to equation (1) oscillate.*

A simple analysis of the characteristic equation (3) shows that it has no real solutions when the inequality $B\tau = -f(x_*)g'(x_*)\tau > 1/e$ is satisfied.

Sometimes one can use a weaker but more transparent sufficient condition for the oscillation of all solutions to equation (1).

PROPOSITION 1.2. *Suppose that functions f and g are C^1 smooth in a neighborhood of the equilibrium x_*. If the inequality $1-B\,\tau = 1+f(x_*)g'(x_*)\,\tau < 0$ holds then all solutions to equation (1) oscillate.*

This is a standard result that can be easily proved by using the linearized criterion for the oscillation, when the oscillation of all solutions to equation (1) follows from the oscillation of all solutions to equation (2) (see [10] for more details). Assuming the latter has a non-oscillatory solution $x \geq 0$, one can conclude that $x(t) > 0$ and $x'(t) < 0$ for all large t. By integrating (2) over the interval $[t-\tau, t]$ then and using the decreasing nature of $x(t)$ on the interval $[t-2\tau, t]$ one arrives at the contradictory inequality

$$x(t) + (\mu + A)\int_{t-\tau}^{t} x(s)\,ds \leq x(t-\tau)\,(1 - B\tau) < 0 \quad \text{for all large} \quad t.$$

The following statement is a stronger sufficient condition for the oscillatory behavior of all solutions to equation (1).

PROPOSITION 1.3. *Suppose that functions f and g are C^1 smooth in a neighborhood of the equilibrium x_*. If the inequality $\tau B\exp\{A\tau\} > 1/e$ holds then all solutions to equation (1) oscillate.*

This result is derived by using the methods of linearized oscillation [10]. It is given as Theorem 4.1 in [3].

We summarize the above description of the oscillatory criteria in the following statement

THEOREM 1.4. *Every nontrivial solution to equation (1) oscillates about equilibrium x_* if anyone of the following assumptions is satisfied:*

(a) The characteristic equation (3) of the linearized equation (2) has no real solutions (with the additional assumption that f and g are C^2-continuous in a neighborhood of $x = x_*$);

(b) The inequality $1 + f(x_*)g'(x_*) < 0$ holds;

(c) The inequality $\tau\, f(x_*)g'(x_*)\,\exp\{[\mu - f'(x_*)g(x_*)]\tau\} < -1/e$ *holds.*

Note that condition (c) of the theorem is the strongest one - if it is satisfied then either one of (a) or (b) is satisfied as well.

1.4. Negative feedback, slow oscillation, monotone solutions. System (1) exhibits a negative feedback property with respect to the positive equilibrium x_*. This in particular implies that initial functions with the range entirely above or below the equilibrium result in either slowly oscillating solutions or in monotone solutions approaching the equilibrium. Such behavior of solutions is well known and common for scalar equations with nonlinearities of negative feedback type [7, 11, 12] or for cyclic systems with an overall negative feedback [4, 5, 13]. Recall that a solution is defined as slowly oscillating (with respect to x_*) if the distance between any two zeros of $x(t) - x_*$ is greater than the delay τ.

The following statement is an implication of the negative feedback structure in equation (1).

Proposition 1.5. *Suppose that an initial function* $\varphi \in \mathbb{X}$ *is such that* $\varphi(s) > x_*$ *(or* $\varphi(s) < x_*$*)* $\forall s \in [-\tau, 0]$ *. Then the corresponding solution* $x = x(t,\varphi)$ *to equation (1) is either slowly oscillating for* $t \geq 0$ *or eventually monotone with* $\lim_{t\to\infty} x(t,\varphi) = x_*$.

The proof of this statement is straightforward when one rewrites the equation in the form

$$
\begin{aligned}
x'(t) &= f(x(t))\left[-\tfrac{\mu x(t)}{f(x(t))} + g(x(t-\tau))\right] \\
&:= f(x(t))\,[g(x(t-\tau)) - F(x(t))]
\end{aligned}
\tag{4}
$$

where $F(x) := \mu\, x/f(x)$ is a strictly increasing function on $\mathbb{R}_+$. Since $f(x) > 0\ \forall x \in \mathbb{R}_+$ the negative feedback in the equation is achieved by interaction between the two terms, $F(x(t))$ and $g(x(t-\tau))$ with $g(x)$ being strictly decreasing.

When $\varphi(s) > x_*\ \forall s \in [-\tau, 0]$ the corresponding solution is strictly decreasing in some right neighborhood of $t = 0$. Then one of the two possibilities holds: either (i) $x(t) > x_*\ \forall t \geq 0$ with $x(t)$ decreasing, or (ii) there exists first positive zero z_1 of $x - x_*$ such that $x(t) - x_* > 0\ \forall t \in [0, z_1)$. In the first case the finite limit $\lim_{t\to\infty} x(t) = \bar{x}$ exists and satisfies the equation $f(\bar{x})(g(\bar{x}) - \mu\bar{x}/f(\bar{x})) = 0$, implying $\bar{x} = x_*$. In the second case, consider the solution $x(t,\varphi)$ on the interval $[z_1 - \tau, z_1]$ as a new initial function φ_1. Zero z_1 of $x(t) - x_*$ is a simple one since $g(x(z_1 - \tau)) < F(x_*)$. Then $x(t) < x_*$ holds in some right neighborhood $(z_1, z_1 + \delta), \delta > 0$. In fact, the inequality $x(t) < x_*$ remains valid on the entire interval $(z_1, z_1 + \tau]$. This follows from the fact that for arbitrary $t_1 \in (z_1, z_1 + \delta)$ and $x_1 < x_*$ the solution of the initial value problem $x'(t) = [g(\varphi_1(t-\tau)) - F(x(t))]f(x(t)), x(t_1) = x_1$ satisfies $x(t) < x_* \ \forall t \in (z_1, z_1+\tau]$ (see papers [15, 14] for additional details). Therefore, there exists $\delta_1 > 0$ such that the solution satisfies $x(t) - x_* < 0 \ \forall t \in (z_1, z_1+\tau+\delta_1]$. Due to the symmetry reasons, for the continuation of the solution $x(t)$ for $t \geq z_1+\tau+\delta_1$ one concludes that either there exists the second simple zero $z_2 \geq z_1+\tau+\delta_1$ or the solution satisfies $x(t) < x_* \ \forall t \geq z_1+\tau$ and is monotone increasing. By induction, one builds a sequence of simple zeros $\{z_n, n \in \mathbb{N}\}$ of the function $x(t) - x_*$ such that $z_{n+1} - z_n > \tau$ and $x(t) < x_* \ \forall t \in (z_{2n-1}, z_{2n})$ and $x(t) > x_* \ \forall t \in (z_{2n}, z_{2n+1})$. If the sequence $\{z_n\}$ is infinite then the solution $x(t)$ is slowly oscillating about x_*. If it is finite, then $x(t) - x_*$ is of definite sign for sufficiently large t, and therefore eventually monotone and convergent to x_* as $t \to \infty$.

Remark. It can easily be shown that every initial function $\varphi \in \mathbb{X}$ satisfying $\varphi(s) \geq x_*$ or $\varphi(s) \leq x_*$ for all $s \in [-\tau, 0]$ and $\varphi \not\equiv x_*$ gives rise to a solution which is eventually either slowly oscillating or monotone for large t and convergent to x_* at $t \to \infty$. By simple but multi-case considerations one can show that any such initial function produces a solution which range becomes strictly above or below the equilibrium level x_* on a time interval of the delay length τ. One can apply then the reasoning of Proposition 1.5.

2. Global asymptotic stability. In this section we apply some of the ideas, methods, and techniques in our earlier papers [15, 14] to differential delay equation (1). Paper [15] deals with the equation of the form

$$\begin{aligned} x'(t) &= \begin{vmatrix} f_1(x(t)) & g_1(x(t-\tau)) \\ f_2(x(t)) & g_2(x(t-\tau)) \end{vmatrix} \\ &= f_1(x(t))g_2(x(t-\tau)) - f_2(x(t))g_1(x(t-\tau)). \end{aligned} \tag{5}$$

Clearly the latter includes equation (1) as a partial case when $f_2(x) = \mu x$ and $g_1 \equiv 1$. Though some of the techniques developed in [15] are straightforward to apply to equation (1), some others require refinement and further development in view of its special form. All these are done in the current section.

Paper [14] concerns the delay differential equation of the form

$$x'(t) = G(x(t-\tau)) - F(x(t)). \tag{6}$$

Some of the properties of solutions of equation (6) derived in [14] can be extended to equation (1). Many properties of both equations (5) and (6) are similar or related through the representation

$$\begin{aligned} x'(t) &= f_1(x(t))g_2(x(t-\tau)) - f_2(x(t))g_1(x(t-\tau)) \\ &= f_1(x(t))g_1(x(t-\tau))\left[\tfrac{g_2(x(t-\tau))}{g_1(x(t-\tau))} - \tfrac{f_2(x(t))}{f_1(x(t))}\right] \\ &:= f_1(x(t))g_1(x(t-\tau))\left[G(x(t-\tau)) - F(x(t))\right], \end{aligned}$$

where the term $f_1(x)g_1(y) > 0$ is positive $\forall x, y \in \mathbb{R}_+$ and functions G, F have properties similar to those respective ones in equation (6). The latter shows that the change of the sign of the derivative of a solution is determined by the relative sizes of the two terms $F(x(t))$ and $G(x(t-\tau))$ in both equations. This fact is essentially used in several places in the present paper.

2.1. Delay independent stability. In this subsection we derive sufficient conditions for the global asymptotic stability of the unique constant solution $x(t) \equiv x_*$ of equation (1) which do not depend on the size of delay $\tau \geq 0$. The conditions are derived in terms of the global attractivity of the fixed point $x = x_*$ for a one-dimensional interval map which is built from given nonlinearities f and g. Some considerations and analysis in this part are similar to those in our earlier papers [13, 14, 15, 16].

By changing the independent variable by $t = s\tau$ equation (1) is reduced to the form

$$\frac{1}{\tau}\frac{dy}{ds} = -\mu y(s) + f(y(s))g(y(s-1)), \quad \text{where} \quad y(s) = x(t) = x(\tau s). \tag{7}$$

By taking the formal limit as $\tau \to \infty$ the latter produces the implicit difference equation

$$\mu y(s) = f(y(s))g(y(s-1)), \quad \text{or} \quad F(y_n) := \frac{\mu y_n}{f(y_n)} = g(y_{n-1}),$$

where $y(s) := y_n, y(s-1) := y_{n-1}, n \in \mathbb{N}$. Since function $F(y) = \mu y/f(y)$ is strictly increasing on $\mathbb{R}_+$ the last difference equation can be explicitly solved for y_n to produce:

$$y_n = F^{-1}(g(y_{n-1})) := \Phi(y_{n-1}),\ n \in \mathbb{N}. \tag{8}$$

The dynamics of solutions of difference equation (8) is entirely determined by the corresponding properties of the interval map $\Phi = F^{-1} \circ g$. They are well known and can be found in numerous sources; we refer the reader to the two monographs [6, 19].

Recall that an interval $I \subseteq \mathbb{R}_+$ is called invariant under Φ if $\Phi(I) \subseteq I$. Real value $x_0 \in \mathbb{R}_+$ is called a fixed point of map Φ if $\Phi(x_0) = x_0$. There is only one fixed point $x = x_*$ of the map Φ since the function $y = \Phi(x)$ is strictly decreasing. Fixed point $x = x_*$ is called attracting if there is its neighborhood U such that $\Phi(U) \subset U$ and $\lim_{n\to\infty} \Phi^n(x) = x_*$ for all $x \in U$. Here Φ^n stands for the n-th iteration of map Φ. The largest interval U_0 with such property is called the domain of immediate attraction of the fixed point x_*. Fixed point x_* will be called globally attracting if $U_0 = \mathbb{R}_+$. Note that x_* is globally attracting if and only if map Φ has no cycles of period two [6, 19].

Given an interval $I \subseteq \mathbb{R}_+$ define $\mathbb{X}_I$ as a subset of initial functions from $\mathbb{X}$ which range is within the interval I: $\mathbb{X}_I := \{\varphi \in \mathbb{X} \,|\, \varphi(s) \in I \,\forall s \in [-\tau, 0]\}$.

Proposition 2.1. *(Invariance Property) Suppose that a closed interval $I \subset \mathbb{R}_+$ is invariant under map Φ: $\Phi(I) \subseteq I$. Then the set $\mathbb{X}_I$ is invariant under the shift operator along solutions of differential delay equation (1):*

$$\forall \varphi \in \mathbb{X}_I : \; x(t, \varphi) \in I \;\forall t \geq 0.$$

Proposition 2.1 includes the meaning of the invariance for an interval map in the broad sense, when either one of the two possibilities $\Phi(I) = I$ or $\Phi(I) \subset I, \Phi(I) \neq I$ holds. In the latter case a stronger version of the invariance property can be given as the following statement.

Proposition 2.2. *(Squeezing Property) Suppose that a closed interval $I \subset \mathbb{R}_+$ is mapped into itself under Φ, $\Phi(I) = I_1 \subset I$, and none of the endpoints of interval I_1 is a fixed point. Then for arbitrary $\varphi \in \mathbb{X}_I$ and its corresponding solution $x = x(t, \varphi)$ there exists $T = T(\varphi)$ such that $x(t) \in I_1 \;\forall t \geq T$.*

Propositions 2.1 and 2.2 follow from corresponding statements in our earlier papers [15, 14]; see Lemmas 1, 2 of [15] and Propositions 2.1, 2.2, 2.3 of [14]. They can also be derived directly for equation (1) when it is represented as

$$\begin{aligned} x'(t) &= \tfrac{1}{f(x(t))}\left[-\tfrac{\mu x(t)}{f(x(t))} + g(x(t-\tau))\right] \\ &:= \tfrac{1}{f(x(t))}\left[g(x(t-\tau)) - F(x(t))\right]. \end{aligned} \tag{9}$$

Since $f(x) > 0$ the sign of the derivative of the solution is determined by the deviation between the two terms $F(x(t)) = \mu x(t)/f(x(t))$ and $g(x(t-\tau))$. Details of the corresponding proofs in paper [14], where the equation $x'(t) = g(x(t-\tau)) - F(x(t))$ is considered, can then easily be carried over to equation (9).

The squeezing property Proposition 2.2 allows one to establish a uniform persistence property for solutions of equation (1) in terms of given nonlinearities f, g and parameter μ only. Indeed, since $g(\mathbb{R}_+) = (g(+\infty), g(0)]$, where $g(+\infty) := \lim_{x\to+\infty} g(x)$, this establishes the upper bound for $I_1 = \Phi(I), I := \mathbb{R}_+$, as $F^{-1}(g(0))$. In the case when $g(+\infty) := g_0 > 0$, the reasoning of Proposition 2.2 also gives the lower bound of I_1 as $F^{-1}(g_0) > 0$.

When $g_0 = 0$, one has to consider the second iteration of $\mathbb{R}_+$ under Φ (the image $\Phi(I_1)$). This gives $\Phi^2(0) = F^{-1}(g(F^{-1}g(0)))$ as the lower bound of the set $\Phi^2(\mathbb{R}_+)$. Therefore, as an easy implication of Proposition 2.2 one obtains the following uniform persistence property of solutions to differential delay equation (1).

PROPOSITION 2.3. *(Uniform Persistence) There are positive constants* $0 < m \le M < \infty$ *such that for every solution* $x = x(t, \varphi), \varphi \in \mathbb{X}$ *there exists a finite* $T = T(\varphi)$ *such that* $m \le x(t) \le M$ *holds for all* $t \ge T$. *One can choose* $M = \Phi(0)$ *and* $m = \Phi(M)$ *where* $\Phi = F^{-1} \circ g$.

The above uniform persistence property can be refined and its boundary estimates improved when one considers the consecutive iterations of the interval $I_0 := [0, F^{-1}(g(0))]$ under map Φ. Define $I_* = [a_*, b_*]$ as the intersection of all the iterated intervals $I_n := \Phi^n(I_0), n \in \mathbb{N}$: $I_* = \cap_{n\ge0} I_n$. In case $a_* < b_*$ the uniform persistence property of Proposition 2.3 holds with $m = a_*, M = b_*$. In case $a_* = b_* = x_*$ one necessarily obtains that $\lim_{t\to\infty} x(t, \varphi) = x_*$ for every initial function $\varphi \in \mathbb{X}$. Therefore, the latter implies that the following global asymptotic stability result holds for solutions of equation (1).

THEOREM 2.4. *(Global Asymptotic Stability I) Suppose that the unique fixed point* x_* *of the map* Φ *is globally attracting on* $\mathbb{R}_+$*:* $\forall x_0 \in \mathbb{R}_+, \Phi^n(x_0) \to x_*$ *as* $n \to \infty$. *Then the constant solution* $x(t) \equiv x_*$ *of differential delay equation (1) is globally asymptotically stable:* $\lim_{t\to\infty} x(t, \varphi) = x_*,\ \forall \varphi \in \mathbb{X}$.

The proof of Theorem 2.4 and of the refined uniform persistence property follows by induction when one repeatedly applies the squeezing property Proposition 2.2 to first the set $I = \mathbb{R}_+$, and then to the set $I_0 = [0, \Phi(0)] \supset \Phi(\mathbb{R}_+)$.

2.2. Delay dependent stability. In this subsection we establish sufficient conditions for the global asymptotic stability of the equilibrium $x(t) \equiv x_*$ of equation (1) which are dependent on the size of delay τ. As in the previous subsection 2.1 they are given in terms of the global attractivity of the unique fixed point of an underlying interval map.

Given arbitrary $x_0 \in \mathbb{R}_+$ consider the initial value problem

$$x'(t) = -\mu x(t) + f(x(t))g(x_0),\ x(0) = x_*. \tag{10}$$

Let $x^*(t) = x^*(t, x_0)$, $t \ge 0$, be its solution. Define the following one-dimensional map Ψ on $\mathbb{R}_+$ by

$$\forall x_0 \ge 0\ :\ \Psi(x_0) := x^*(\tau) := x_1.$$

The solution to the initial value problem (10) is given by the integral equation

$$\int_{x_*}^{x(t)} \frac{dx(t)}{-\mu x(t) + f(x(t))g(x_0)} = t, \quad t \geq 0.$$

Therefore, the value $x_1 = \Psi(x_0)$ is defined implicitly by

$$\int_{x_*}^{x_1} \frac{du}{-\mu u + f(u)g(x_0)} = \tau. \tag{11}$$

The following statements describe some basic but elementary properties of the interval map Ψ.

Proposition 2.5. *Map Ψ has the following properties:*

(i) x_* is its only fixed point;

(ii) Ψ has the negative feedback property with respect to the fixed point x_*:

$$x_1 = \Psi(x_0) < x_* \text{ if } x_0 > x_* \text{ and } x_1 = \Psi(x_0) > x_* \text{ if } x_0 < x_*;$$

(iii) $\Psi(x_0)$ is strictly decreasing for $x_0 \in \mathbb{R}_+$;

(iv) There exists a finite limit $\lim_{x_0\to\infty}\Psi(x_0) = \Psi_\infty$, which is also bounded away from zero $\Psi_\infty > 0$. Hence, $\Psi(\mathbb{R}_+) = (\Psi_\infty, \Psi(0)]$.

(i) This property is obvious. The uniqueness will follow from (iii).

(ii) When $x_0 > x_*$ then $x'(0) < 0$; the corresponding solution to (10) is such that $x(t) < x_*$ and monotone decreasing on $(0,\tau]$, implying $x_1 = \Psi(x_0) = x(\tau) < x_*$. A symmetric reasoning applies when $x_0 < x_*$.

(iii) If initial values are such that $x_0^1 > x_0^2 > x_*$ then $g(x_0^1) < g(x_0^2)$, and the respective solutions $x_*^1(t)$ and $x_*^2(t)$ of the initial value problem (10) satisfy $x_*^1(t) < x_*^2(t)$ $\forall t \in [0,\tau]$ (see basic related comparison statements for solutions of ordinary differential equations in [21]). Therefore, $x_1^1 = \Psi(x_0^1) < x_1^2 = \Psi(x_0^2)$. A symmetric reasoning applies when $x_0^1 < x_0^2 < x_*$.

(iv) The value $\Psi_\infty > 0$ is found from solving the initial value problem (10) when $g_0 = \lim_{x_0\to\infty} g(x_0) \geq 0$.

Due to the monotone decreasing nature of map Ψ it can only have cycles of period two. The unique fixed point x_* is globally attracting on $\mathbb{R}_+$ if and only if map Ψ has no cycles of period two [6, 19].

Theorem 2.6. *(Global Asymptotic Stability II) Suppose that the unique fixed point x_* of the interval map Ψ is globally attracting: $\lim_{n\to\infty}\Psi^n(x_0) = x_*$ $\forall x_0 \in \mathbb{R}_+$. Then the constant solution $x(t) \equiv x_*$ of the differential delay equation (1) is globally asymptotically stable: $\lim_{t\to\infty} x(t,\varphi) = x_*$ $\forall \varphi \in \mathbb{X}$.*

Proof. Let $\varphi \in \mathbb{X}$ be arbitrary but fixed, and consider the corresponding solution $x = x(t,\varphi), t \geq 0$, to equation (1). If the solution is such that

$x(t) - x_*$ has a finite number of zeros on $t \geq 0$ then it is eventually of definite sign, $x(t) > x_*$ or $x(t) < x_*$ $\forall t \geq T$ for some $T \geq 0$. Therefore, $\lim_{t\to\infty} x(t, \varphi) = x_*$, in view of Proposition 1.5.

Assume next that $x(t, \varphi) - x_*$ has an infinite sequence of zeros $\{z_n, n \in \mathbb{N}\}$ on $t \geq 0$. Due to the autonomous nature of equation (1) one can assume that $\varphi(0) = x_*$. Define $m_0 = \min\{\varphi(s), s \in [-\tau, 0]\}, M_0 = \max\{\varphi(s), s \in [-\tau, 0]\}$. By using basic comparison results for solutions of ordinary differential equations [21] one easily concludes that the solution $x(t, \varphi)$ satisfies the inequalities

$$x^*(t, m_0) \geq x(t, \varphi) \geq x^*(t, M_0) \quad \forall t \in [0, \tau]. \tag{12}$$

Therefore,

$$\Psi(m_0) \geq \max\{x(t, \varphi), t \in [0, \tau]\} \geq \min\{x(t, \varphi), t \in [0, \tau]\} \geq \Psi(M_0).$$

Assume next that the initial function φ does not change sign on $[-\tau, 0]$, i.e. $\varphi(s) \geq 0$ or $\varphi(s) \leq 0$ for all $s \in [-\tau, 0]$. To be specific let $\varphi(s) > 0$ $\forall s \in [-\tau, 0), \varphi(0) = 0 := x(z_1, \varphi)$ (the latter implies that $m_0 = x_*$). Then the solution $x(t, \varphi)$ is slowly oscillating for $t \geq 0$ (see the related details and properties in Proposition 1.5). In particular, $x(t, \varphi) < x_*$ $\forall t \in (0, \tau]$ and $x(t, \varphi)$ is increasing on the interval $[\tau, z_2]$. Therefore $\min\{x(t, \varphi), t \in [z_1, z_2]\} \geq \Psi(M_0)$. Likewise, when $\varphi(s) < x_*$ $\forall s \in [-\tau, 0], \varphi(0) = 0$ is assumed one finds that $\max\{x(t, \varphi), t \in [z_1, z_2]\} \leq \Psi(m_0)$ is valid. By induction one concludes that the solution $x(t, \varphi)$ satisfies

$$\begin{aligned} &\min\{\Psi^n([m_0, M_0])\} \leq \min\{x(t, \varphi), t \in (z_n, z_{n+1})\} \\ &\leq \max\{x(t, \varphi), t \in (z_n, z_{n+1})\} \leq \max\{\Psi^n([m_0, M_0)])\},\ n \in \mathbb{N}. \end{aligned}$$

Since x_* is globally attracting fixed point on $\mathbb{R}_+$ one has $\Psi^n([m_0, M_0]) \to x_*$ as $n \to \infty$. Therefore, $\lim_{t\to\infty} x(t, \varphi) = x_*$.

Assume next that the solution $x(t, \varphi)$ is not slowly oscillating, but rapidly oscillating. It means there exists no time interval of the delay length τ on which the function $x(t, \varphi) - x_*$ is non-negative or non-positive. Therefore, every such interval contains at least one its zero. As in the case of slow oscillation, assume $\varphi(0) = x_* := \varphi(\xi_1)$, and let $\xi_2 \leq \tau$ be the largest zero of $x(t, \varphi) - x_*$ on the interval $[0, \tau]$. Consider the segment of the solution $x(\xi_2 + s, \varphi) := \varphi_1(s), s \in [-\tau, 0]$, as a new initial function $\varphi_1 \in \mathbb{X}$, with $m_1 = \min\{\varphi_1(s), s \in [-\tau, 0]\}, M_1 = \max\{\varphi_1(s), s \in [-\tau, 0]\}$. Then for the solution $x(t, \varphi)$ the inequalities (12) hold for $t \in [\xi_2, \xi_2 + \tau]$. This implies that

$$\Psi(m_1) \geq \max\{x(t, \varphi), t \in [\xi_2, \xi_2 + \tau]\} \geq \min\{x(t, \varphi), t \in [\xi_2, \xi_2 + \tau]\} \geq \Psi(M_1).$$

Since x is rapidly oscillating there exists another zero ξ_3 in the interval $[\tau, \xi_2 + \tau]$. The last inequality together with the previous estimate (12) imply that the following estimates hold

$$\min\{\Psi^2([m_0, M_0])\} \leq \min\{x(t,\varphi), t \in [\xi_2, \xi_3]\}$$
$$\leq \max\{x(t,\varphi), t \in [\xi_2, \xi_3]\} \leq \max\{\Psi^2([m_0, M_0])\}.$$

By induction, one establishes the existence of an infinite sequence $\{\xi_n, n \in \mathbb{N}\}$ of zeros of $x(t) - x_*$ such that $\xi_{n+1} - \xi_n \geq \tau$ and the following inequalities hold

$$\min\{\Psi^n([m_0, M_0])\} \leq \min\{x(t,\varphi), t \in [\xi_n, \xi_{n+1}]\}$$
$$\leq \max\{x(t,\varphi), t \in [\xi_n, \xi_{n+1}]\} \leq \max\{\Psi^n([m_0, M_0])\}, \ n \in \mathbb{N}.$$

As in the sub-case of slow oscillation the latter implies $\lim_{t\to\infty} x(t,\varphi) = x_*$.

The proof of the theorem is complete. □

Next we derive sufficient conditions in terms of functions f, g and the decay coefficient μ under which the fixed point x_* of interval map Ψ is globally attracting. We further assume that the nonlinearity g is Lipschitz continuous on the interval I_1:

$$|g(x_1) - g(x_2)| \leq L_g |x_1 - x_2| \quad \forall x_1, x_2 \in I_1$$

for some $L_g > 0$. Note that in the case when $g(x)$ is continuously differentiable on I_1 the constant L_g can be chosen as $L_g := \sup\{|g'(x)|, x \in I_1\}$. Recall that $I_1 \subset \Phi(\mathbb{R}_+)$ is the first interval defining the persistence boundaries of all solutions (see details in Subsection 2.1).

Theorem 2.7. *Suppose that the following inequality is satisfied:*

$$\frac{1 - \exp\{-\mu\tau\}}{\mu} f(x_*) L_g < 1. \tag{13}$$

Then the fixed point x_* of the interval map Ψ is globally attracting on $\mathbb{R}_+$. Hence, the constant solution $x(t) \equiv x_*$ of differential delay equation (1) is globally asymptotically stable on $\mathbb{X}$.

Proof. We shall show that the interval map Ψ implicitly defined by integral equation (11) is a contraction on the interval I_1. Note that due to the uniform persistence property of subsection 2.1 we can assume that the initial value x_0 in (11) satisfies $x_0 \in I_1$.

To be definite, suppose first that $x_0 < x_*$. Then $x^*(t)$ is increasing on $[0, \tau]$ with $x_1 > x_*$. Due to the decreasing nature of $f(x)$ one has the inequality

$$\tau = \int_{x_*}^{x_1} \frac{dx}{-\mu x + f(x) g(x_0)} \geq \int_{x_*}^{x_1} \frac{dx}{-\mu x + f(x_*) g(x_0)}.$$

By solving the latter one derives the estimate

$$x_1 - x_* \leq \frac{1 - e^{-\mu\tau}}{\mu} f(x_*)[g(x_0) - g(x_*)].$$

By using the Lipschitz continuity of g one arrives at the inequalities

$$|x_1 - x_*| \leq \frac{1 - e^{-\mu\tau}}{\mu} f(x_*) L_g |x_0 - x_*| \leq q\, |x_0 - x_*|,$$

where $0 \leq q := \frac{1-e^{-\mu\tau}}{\mu} f(x_*) L_g < 1$. Likewise, one derives the same inequality when $x_0 > x_*$ and $x(t)$ is decreasing on $[0, \tau]$ with $x_1 < x_*$. This completes the proof. ∎

The following statement immediately follows from inequality (13) in the above theorem.

Corollary 2.8. *Given f, g and μ with g being Lipschitz continuous there exists a positive value $\tau_0 > 0$ such that the constant solution $x(t) \equiv x_*$ of equation (1) is globally asymptotically stable for every delay $\tau \in [0, \tau_0)$.*

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