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On global dynamics in a periodic differential equation with deviating argument

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ABSTRACT

Several aspects of global dynamics and the existence of periodic solutions are studied for the scalar differential delay equation $x'(t) = a(t)f(x([t-K]))$, where $f(x)$ is a continuous negative feedback function, $x \cdot f(x) < 0, x \neq 0$, $0 < a(t)$ is continuous ω-periodic, $[\cdot]$ is the integer part function, and the integer $K \geqslant 0$ is the delay. The case of integer period ω allows for a reduction to finite-dimensional difference equations. The dynamics of the latter are studied in terms of corresponding discrete maps, including the partial case of interval maps ($K = 0$).

1. Introduction

Differential equations with deviating arguments (DEDAs) are an important part of modern theory and applications of nonlinear dynamical systems. Their theoretical fundamentals and multiple areas of applications were summarized in early works of the 60s and 70s, see e.g. monographs [1,11,27]. Most comprehensive theoretical achievements and numerous areas of applications were developed a few decades later and can be found in e.g. monographs [10,17]. The significance of new theoretical developments in the field and their wide applicability to modeling various real life phenomena have been overwhelmingly demonstrated in the past 20 years or so. The surge in research output during this time, both theoretical and numerical, was largely driven by applications, as can be seen from some recent review papers and monographs [12,21,22,31], where further related references can also be found. DEDAs are indispensable mathematical tools for modeling real life phenomena where after effects are intrinsic features of their functioning [13].

In this paper we study the global dynamics and, in particular, the existence and stability of periodic solutions, for a specific differential equation with piece-wise constant argument (see Eq. (1) below). This equation can be viewed as an approximation of a periodic differential delay equation with constant delay. Such an approximation allows for a reduction of its dynamics to that of an associated finite-dimensional map. Though the reduction is rather straightforward, the dynamics of the resulting map are highly non-trivial and not yet completely understood or studied. In a simpler partial case the dynamics of the DEDA are equivalent to that of a one-dimensional map. This case allows for a comprehensive study, the results of which are summarized in the present paper.

Consider the following differential equation with deviating argument

$$x'(t) = a(t)f(x([t-K])), \quad t \geqslant 0, \tag{1}$$

where the $[\cdot]$ is the integer value function, and the non-negative integer K is the delay. We shall assume throughout the paper that f is a continuous real-valued function satisfying the negative feedback condition

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$$x \cdot f(x) < 0 \quad \text{for all} \quad x \neq 0, \tag{2}$$

and is bounded from one side

$$f(x) \leqslant M \text{ or } f(x) \geqslant -M \text{ for all } x \in \mathbf{R} \text{ and some } M > 0. \tag{3}$$

The coefficient $a(t)$ is a continuous positive periodic function with integer period $\omega > 0$

$$a(t) > 0 \quad \text{and} \quad a(t+\omega) = a(t) \quad \text{for all} \quad t \in \mathbf{R}. \tag{4}$$

Eq. (1) can also be viewed as a differential equation with periodic delay.

Eq. (1) is closely related to the differential delay equation

$$x'(t) = a(t)f(x(t-\tau)), \tag{5}$$

with the same f and a, and where $\tau > 0$ is a constant delay. It can be viewed as a discrete version of Eq. (5). While the problem of global dynamics and the existence of periodic solutions for general Eq. (5) is quite difficult to approach, Eq. (1) appears to be somewhat simpler to study in this regard.

Eq. (1) falls within the class of differential equations with piecewise constant argument. Such equations have attracted a significant interest in recent years for their qualitative features and range of applications. As indicated by many authors, in general, numerical approximations of differential delay equations can give rise to DEDAs, see papers [5,6,15,24]. References to other applications can be found e.g. in [7,16,19,20]. Various aspects of their dynamics have been studied by many authors. Among those related to the present work we would like to mention papers [2,3,6,15,26,32].

When $a(t) = a_0 > 0$ is a constant Eq. (5) is equivalent to the well studied autonomous equation

$$x'(t) = G(x(t-1)). \tag{6}$$

It is well known that when G also satisfies the negative feedback condition (2), it is one-sided bounded in the sense of (3), and $G'(0) < -\pi/2$, then the differential delay Eq. (6) has a slowly oscillating periodic solution [10,17,18,25,28]. The proof of this fact is rather non-trivial; it constitutes a part of an established theory for the existence of periodic solutions of functional differential equations called ejective fixed point techniques [10,17,28].

It is a natural next step to look for the existence of non-trivial periodic solutions in similar but periodic functional differential equations of the form (5). As our initial approaches and analysis have indicated, the use of the standard techniques of the ejective fixed point theory does not appear immediately applicable to this case. New approaches and techniques seem to be necessary. Our first step in this direction is to study periodic solutions and other dynamical properties of the somewhat simpler differential delay Eq. (1). Another worthwhile avenue of possible approach to the study of periodic solutions of Eq. (5) is an extension and adaptation of techniques from [14] developed for a similar class of equations with almost periodic coefficients.

The assumption that the period ω is an integer is crucial for all principal considerations of the paper. It simplifies the dynamics of solutions of Eq. (1) significantly: they are essentially reduced to the dynamics of finite-dimensional discrete maps (which can be quite complex by themselves). At present we do not have a clear workable idea how to approach the case when the period ω is not commensurable with the delay in the equation (K for Eq. (1) and τ for Eq. (5)). Even in the simplest case of $K = 0$ and ω being irrational, the corresponding Eq. (1) seems to allow in some cases for the existence of quasi-periodic solutions.

2. Preliminaries

We shall be using throughout the paper the standard notions and definitions related to functional differential and difference equations, as well as to interval maps, most of which can be found in monographs [4,8–10,17,29,30].

For arbitrary initial function $\varphi \in \mathcal{C} := C([-K,0],\mathbf{R})$ the corresponding solution $x = x(t,\varphi)$ of Eq. (1) is easily found by successive integration for $t \geqslant 0$. One has

$$x(t) = \varphi(0) + f(\varphi(-K))\int_0^t a(s)ds \quad \text{for all} \quad t \in [0,1),$$

with

$$x(1) = \varphi(0) + a_1 \cdot f(\varphi(-K)), \quad \text{where} \quad a_1 := \int_0^1 a(s)ds.$$

Likewise

$$x(t) = x(1) + f(\varphi(-K+1))\int_1^t a(s)ds \quad \text{for all} \quad t \in [1,2),$$

with

$$x(2) = x(1) + a_2 f(\varphi(-K+1)), \quad \text{where} \quad a_2 := \int_1^2 a(s)ds,$$

and so on. Thus one can easily see that the solution $x(t,\varphi), t \geqslant 0$, depends only on the values $\varphi(-K), \varphi(-K+1), \dots, \varphi(-1), \varphi(0)$ of the initial function $\varphi \in \mathcal{C}$. In Section 3, based on the above calculations, we derive the explicit form for the translation operator S^{ω} along the solutions in the case of integer values ω as a discrete finite-dimensional map on the set of initial values $\mathbf{x}_0 = [x_0, x_{-1}, \dots, x_{-K}] = [\varphi(0), \varphi(-1), \dots, \varphi(-K)]$.

The oscillation of solutions of Eq. (1) is meant in a standard sense. A solution $x(t)$ is called eventually positive (negative) if there exists $T \geqslant 0$ such that $x(t) > 0$ ($x(t) < 0$) for all $t > T$. A nontrivial solution $x(t)$ ($x(t) \not\equiv 0$ for all $t \geqslant T$) is called oscillatory if it is not eventually positive or negative. For every $T \geqslant 0$ any oscillating solution $x(t)$ of Eq. (1) changes sign on the interval $[T, \infty)$. This is easily seen from the positivity of $a(t)$ and the negative feedback assumption (2) on f.

The oscillatory nature of solutions of differential delay equations is an important characteristic which can lead to certain implications such as the existence of nontrivial periodic solutions. It is a significant component of the ejective fixed point techniques used to prove the existence of slowly oscillating periodic solutions to Eq. (6).

Proposition 2.1 (*Eventual Uniform Boundedness*). *Suppose that nonlinearity f satisfies assumptions (2) and (3), and $a(t)$ satisfies (4). There is a constant M_0 such that for an arbitrary initial function $\varphi \in \mathcal{C}$ there exists a time moment t_φ such that the corresponding solution $x = x(t,\varphi)$ of Eq. (1) satisfies $|x| \leqslant M_0$ for all $t \geqslant t_\varphi$.*

Proof. The proof essentially follows from the fact of one-sided boundedness of the nonlinearity f, the negative feedback assumption (2), and the periodicity of $a(t)$ (therefore, its boundedness). Consider two potential possibilities for any solution x: (i) it does not have zeros on an interval $[T, \infty)$ for some $T \geqslant 0$, and (ii) it oscillates on $\mathbf{R}^+$.

In case (i), to be specific, one can assume that $x > 0$ in $(t_0, t_0 + K + 1]$, where t_0 is the largest zero of x in $\mathbf{R}^+$. Then $x(t)$ is decreasing in $[t_0 + K + 1, +\infty)$ with $\lim_{t\to\infty} x(t) = 0$. The other possibility of $x < 0$ is similarly treated.

In case (ii), to be definite, assume that f is bounded from below, $f(x) \geqslant -M$ for all $x \in \mathbf{R}$ and some $M > 0$. Set $a^* := \max\{a(t), t \in [0,\omega]\} > 0$. Let $t_0 \geqslant 0$ be any zero of the solution x. Then $x'(t) \geqslant -a^*M$, and therefore $x(t) \geqslant -a^*M(t - t_0)\ \forall t \geqslant t_0$.

Let $t_1 \geqslant 0$ be the first zero of the solution x. We shall show first that $x(t)$ is uniformly bounded from below for all $t \geqslant t_1$. In view of the above lower estimate on x, for all $t \in [t_1, t_1 + K + 1]$, the following inequality holds

$$x(t) \geqslant -Ma^*(K+1). \tag{7}$$

Let t_2 be the leftmost zero of the solution x such that $t_2 \geqslant t_1 + K + 1$. We claim that inequality (7) holds for all $t \in [t_1, t_2]$. Indeed, the case $t_2 = t_1 + K + 1$ is trivial. If $x(t_1 + K + 1) > 0$ then $x(t) > 0$ for all $t \in [t_1 + K + 1, t_2)$, so inequality (7) holds as well. Assume next that $x(t_1 + K + 1) < 0$, so that $x(t) < 0$ for all $t \in [t_1 + K + 1, t_2)$. Let $t_1^* \geqslant t_1$ be the rightmost zero of $x(t)$ on the interval $[t_1, t_1 + K + 1]$. If $t_1^* + K + 1 \geqslant t_2$, then inequality (7) holds on the claimed interval for the very same reason as it does for the zero t_1 on the interval $[t_1, t_1 + K + 1]$. Assume the other possibility, that $t_1^* + K + 1 < t_2$. Then $x(t) < 0$ for all $t \in (t_1^*, t_2)$. As with the zero t_1, for all $t \in [t_1^*, t_1^* + K + 1]$ the following estimate holds $0 \geqslant x(t) \geqslant -Ma^*(K+1)$. Due to the negative feedback condition (2) and the fact that $x(t_1^* + K + 1) < 0$, Eq. (1) implies that $x(t)$ is increasing in $[t_1^* + K + 1, t_2]$. Therefore, inequality (7) holds for all $t \in (t_1, t_2)$.

By using induction, one can indicate a sequence of zeros $\{t_j, j = 1, 2, 3, \dots\}$ for the solution x such that $t_{j+1} \geqslant t_j + K + 1$ and $x(t) \geqslant -Ma^*(K+1)\ \forall t \in [t_j, t_{j+1}], j \in \mathbf{N}$. This proves the uniform boundedness of the solution x from below for all $t \geqslant t_1$.

Since the solution x is bounded from below by $-Ma^*(K+1), \forall t \geqslant t_1$, the negative feedback assumption (2) shows that its derivative is bounded from above, $x'(t) \leqslant a^* M_1$, where $M_1 = \max\{f(x), x \in [-Ma^*(K+1), 0]\}$. This implies that the solution x is uniformly bounded from above, $x(t) \leqslant M_1 a^*(K+1)$ for all $t \geqslant t_2$. The reasoning is similar to that above for the case of boundedness from below (the details are left to the reader). By setting $M_0 := a^*(K+1)(M + M_1)$ one completes the proof.

The following two propositions are closely related; they essentially reflect the fact that all solutions of Eq. (1) oscillate when either the coefficient $a(t)$ or the derivative $|f'(0)|$ is sufficiently large.

Proposition 2.2 (*Oscillation*). *Let $f'(0) = f_0 < 0$ be fixed. There exists $a_0 > 0$ such that for an arbitrary ω-periodic function $a(t)$ with $a(t) \geqslant a_0, t \in [0,\omega]$, all non-zero solutions of Eq. (1) oscillate.*

Proposition 2.3 (*Oscillation*). *Let the ω-periodic function $a(t) > 0$ be fixed. There exists $f_0 < 0$ such that for an arbitrary function $f(x)$ with $f'(0) < f_0$, all non-zero solutions of Eq. (1) oscillate.*

Proof. The proofs of both Propositions 2.2 and 2.3 are straightforward and similar. They use a simple comparison argument which has been used multiple times in other papers. We provide its outline here for the sake of completeness.

Indeed, assuming say $x(t) > 0$ for all $t \geqslant t_0$, one sees that $x(t)$ is decreasing for $t \geqslant t_0 + K + 1$ with $\lim_{t\to+\infty} x(t) = 0$. Comparing Eq. (1) with its linearization about $x(t) \equiv 0$ one concludes that for arbitrary $\varepsilon > 0$ there exists $t_* \geqslant t_0 + 2K + 2$ such that for all $t \geqslant t_*$ the following holds

$$x'(t) < a_*[f'(0) + \varepsilon]x(t - K) \quad \text{where} \quad a_* = \min\{a(t),\ t \in [0,\omega]\}.$$

Integrating the last inequality on the interval $[t-K, t]$ one arrives at the estimate $x(t) < [1 + Ka_*(f'(0)+\varepsilon)]x(t-K)$. The latter one contradicts the positiveness of $x(t)$ when $Ka_*f'(0) < -1$. The case of $x(t)$ being eventually negative is treated in a completely analogous way.

Note that Propositions 2.1, 2.2 are valid for the general Eq. (5). The proofs are similar to those above and are left to the reader.

3. Main results

3.1. Shift by time ω operator: integer period

In this subsection we shall explicitly calculate the form of the shift-by-period operator along solutions of the differential equation with deviating argument (1) in the case when the period ω is a positive integer, $\omega = N$.

Define positive numbers $a_i, i = 1, 2, \dots, N$, by the following integral values over the coefficient $a(t)$

$$a_i = \int_{i-1}^{i} a(t)dt. \tag{8}$$

Introduce next the sequence of maps F_i of the Euclidean space $\mathbf{R}^{K+1}$ into itself by

$$F_i : [u_0, u_1, \dots, u_{K-1}, u_K] \mapsto [u_0 + a_i f(u_K), u_0, u_1, \dots, u_{K-1}], \tag{9}$$

with the composite map F defined by

$$F = F_N \circ F_{N-1} \circ \dots \circ F_2 \circ F_1.$$

Theorem 1 (*Existence and stability of periodic solutions*). *Differential delay Eq. (1) has a non-trivial periodic solution with the period a multiple of N if and only if the map F has a non-trivial cycle. It has a periodic solution with period N if and only if the map F has a fixed point different from* $\mathbf{u}_* = [0, \dots, 0]$. *Moreover, the stability of any such periodic solution is the same as the stability of the corresponding cycle of F.*

Proof. The proof is straightforward since the shift operator along solutions of Eq. (1) in the case $\omega = N$ is equivalent to the map F. Indeed, for arbitrary $t_0 \in [0, \omega]$ one finds the solution satisfying the initial condition $x(t_0) = x_0$ by

$$x(t) = x_0 + \int_{t_0}^{t} a(s) f(x([s-K]))ds.$$

In particular, for any integer point $t_0 = i \in [0, \omega]$ one has

$$x(t) = x(i) + \left(\int_i^t a(s)ds\right) f(x(i-K)), \quad \forall t \in [i, i+1). \tag{10}$$

Let an initial function $\varphi(s) \in \mathcal{C} = C([-K, 0], \mathbf{R})$ be given. Set $\varphi(0) = x_0, \varphi(-1) = x_{-1}, \dots, \varphi(-K) = x_{-K}$. As it is shown in the introduction, the corresponding solution $x(t, \varphi)$ depends on the values $\{x_0, x_{-1}, \dots, x_{-K}\}$ only, and does not depend on values of $\varphi(t)$ at other non-integer times $t \in [-K, 0]$. By using (10) one easily finds for $t \in [0, 1)$

$$x(t) = x_0 + \left(\int_0^t a(s)ds\right) f(x_{-K}).$$

At $t = 1$, by the continuity of the solution, one has

$$x(1) = x_1 = x_0 + \left(\int_0^1 a(t)dt\right) f(x_{-K}) = x_0 + a_1 f(x_{-K}).$$

The shift operator along the solution $x(t, \varphi)$ by time $T = 1$ is now defined as

$$\begin{aligned} &x_0 \mapsto x_0 + a_1 f(x_{-K}) = x'_0 \\ &x_{-1} \mapsto x_0 = x'_{-1} \\ &x_{-2} \mapsto x_{-1} = x'_{-2} \\ &\quad \dots \quad \dots \\ &x_{-K} \mapsto x_{-K+1} = x'_{-K}, \end{aligned}$$

which is the map F_1 applied to the point $[x_0, x_{-1}, \dots, x_{-K}]$.

Likewise, for $t \in [1, 2)$ one has

$$x(t) = x_1 + \left(\int_1^t a(s)ds\right) f(x_{-K+1}),$$

with

$$x(2) = x_2 = x_1 + \left(\int_1^2 a(t)dt\right) f(x_{-K+1}) = x_1 + a_2 f(x_{-K+1}).$$

Therefore, the shift along the solution by time $T = 2$ is given by the map

$$\begin{aligned} &x_1 \mapsto x_1 + a_2 f(x_{-K+1}) \\ &x_0 \mapsto x_1 \\ &x_{-1} \mapsto x_0 \\ &\quad \dots \quad \dots \\ &x_{-K+1} \mapsto x_{-K+2}, \end{aligned}$$

which is the map F_2 applied to the point $[x_1, x_0, x_{-1}, \dots, x_{-K+1}]$, that is $F_2 \circ F_1([x_0, x_{-1}, \dots, x_{-K}])$.

By continuing this step-by-step integration procedure, one finds that the shift along the solution by period $\omega = N$ is given by

$$F([x_0, x_{-1}, \dots, x_{-K}]) = F_N \circ \dots \circ F_2 \circ F_1([x_0, x_{-1}, \dots, x_{-K}]).$$

If there exists an initial vector $\mathbf{u}_* = [u_0, u_1, \dots, u_K] \neq \mathbf{0}$ such that $F(\mathbf{u}_*) = \mathbf{u}_*$ then any initial function $\varphi \in \mathcal{C}$ with $\varphi(-i) = u_i, i = 0, 1, \dots, K$ generates a non-trivial N-periodic solution of Eq. (1).

If an initial function $\varphi \in \mathcal{C}$ results in an N-periodic solution $x = p(t)$ of Eq. (1) then the vector $\mathbf{u}^* = [u_0, \dots, u_K]$ with $u_i = \varphi(-i), i = 0, 1, \dots, K$ is a fixed point of the map F. Small perturbations ψ of the initial function φ in $\mathcal{C}$, $||\varphi - \psi||_{\mathcal{C}} < \delta$, yield small perturbations of the vector $\mathbf{u}^*$ in $\mathbf{R}^{K+1}$, $||\mathbf{u}^* - \mathbf{u}||_{\mathbf{R}^{K+1}} < \delta$. And vise versa: small perturbations of the fixed point (vector) $\mathbf{u}^*$ in $\mathbf{R}^{K+1}$ can be translated into small perturbations of the corresponding initial function $\varphi \in \mathcal{C}$ for the periodic solution $x = p(t)$. In fact, their norms in respective spaces are related by the inequalities $||\mathbf{u}||_{\mathbf{R}^{K+1}} \leqslant ||\psi||_{\mathcal{C}} \leqslant P||\mathbf{u}||_{\mathbf{R}^{K+1}}$ for some $P \geqslant 1$ along solutions of Eq. (1). This follows from the continuous dependence of solutions of Eq. (1) on initial conditions and the representation (10). Therefore, the stability of the N-periodic solution of DDE (1) and of the corresponding fixed point of the map F are the same. They are both either stable, or asymptotically stable, or unstable.

The same reasoning about existence and stability applies to cycles of the map F and the corresponding periodic solutions of differential delay Eq. (1).

3.2. Case $K = 0$

In this subsection we consider a special case of Eq. (1) when $K = 0$ and $\omega = N$ is a positive integer

$$x'(t) = a(t) f(x([t])). \tag{11}$$

We shall also assume throughout this subsection that $f(x)$ is differentiable at $x = 0$ with $f'(0) < 0$.

Given an initial value $x(0) = x_0$ one easily solves Eq. (11) by consecutive step-by-step integration for all $t \geqslant 0$, as described above.

Introduce the following auxiliary functions:

$$F_i(x) := x + \left(\int_{i-1}^i a(t)dt\right) f(x) := x + a_i f(x), \quad i = 1, 2, \dots, N,$$

and set

$$F = F_N \circ F_{N-1} \circ \dots \circ F_1, \tag{12}$$

where the $\circ$ stands for the composition of functions.

It is easy to see that $x = 0$ is a fixed point of the map F which corresponds to the trivial solution $x(t) \equiv 0$ of differential delay Eq. (11). Eq. (11) has a non-trivial periodic solution with the period a multiple of N if and only if map F has a nontrivial cycle of any period (including a fixed point). The stability of a cycle of map F and the stability of the corresponding periodic solution of differential delay Eq. (11) are the same.

We shall indicate and derive certain basic properties of the map F which are based on the properties of function f as a one-dimensional map.

Theorem 2 (*Existence of a globally attracting interval*). *Suppose that the nonlinearity f satisfies assumptions (2) and (3), and coefficient $a(t)$ satisfies (4). Then map F has a finite globally attracting interval $I_0 = [\alpha_0, \beta_0], \alpha_0 \leqslant 0 \leqslant \beta_0$, such that*

$$F(I_0) = I_0 \text{ and } \cap_{i \geqslant 0} F^i(\mathcal{U}) = I_0 \text{ for every open bounded set } \mathcal{U} \supset I_0.$$

Note that there is a possibility of interval I_0 being a single point, $\alpha_0 = \beta_0 = 0$. In this case the only fixed point $x_* = 0$ of the map F is globally attracting. The corresponding trivial solution $x(t) \equiv 0$ of differential delay Eq. (11) is then globally asymptotically stable.

Proof of Theorem 2. We shall prove the theorem in several steps, starting with the simplest case $N = 1$. The principal idea is that any finite number of iterations of maps of the form $F_a = x + af(x)$, where $a > 0$ is a parameter, possesses the same basic property described by the theorem as the single map F_a does.

Case $N = 1$. For arbitrary fixed $a > 0$ map F_a has a closed globally attracting invariant interval I_a such that

$$F_a(I_a) = I_a \quad \text{and} \quad \cap_{n\geqslant 1} F_a^n(J) = I_a,$$

for any interval J (open or closed) such that $I_a \subseteq J$.

To be specific, assume that $f(x) \geqslant -M$ for some $M > 0$ and all $x \in \mathbf{R}$. The case $f(x) \leqslant M$ can be treated similarly (and therefore, it is left to the reader).

Due to the negative feedback assumption (2) one has that $F_a(x) > x$ for all $x < 0$ and $F_a(x) < x$ for all $x > 0$. Besides, $F_a(x) \geqslant x - Ma$. Therefore, for every $x \geqslant Ma$ one has $x > F_a(x) \geqslant 0$.

Assume first that $F_a(x) \geqslant 0$ also holds for all $x \in [0, Ma]$. Then for every $x_0 \geqslant 0$ the sequence of its consecutive iterations $x_n := F_a^n(x_0)$ is decreasing with $x_{n+1} \leqslant x_n, n \neq 0$. Therefore, $\lim_{n\to\infty} x_n = 0$.

The particular shape of $F_a(x)$ in $(-\infty, 0]$ is now of no importance: the fixed point $x = 0$ is globally attracting. Indeed, for any $x_0 < 0$ consider the sequence of its consecutive iterations, $x_n := F_a^n(x_0)$. If $x_n < 0$ for all $n \geqslant 0$ then $\lim_{n\to\infty} x_n = 0$. If $x_{n_0} > 0$ for some $n_0 \geqslant 1$ then the sequence $y_n := F_a^n(x_{n_0})$ is nonnegative for all $n \geqslant 0$ and decreasing with $\lim_{n\to\infty} y_n = 0$.

Suppose next that $F_a(x)$ can assume negative values in the interval $[0, Ma]$. Set $m = \min\{F_a(x), x \in [0, Ma]\}$, and let $L := \max\{F_a(x), x \in [m, 0]\} \geqslant 0$.

If $L = 0$ then $F_a(x) \leqslant 0$ for all $x \in [m, 0]$. Therefore, for every $x_0 \in [m, 0]$ the sequence of its consecutive iteration $x_n := F_a^n(x_0)$ is increasing with $\lim_{n\to\infty} x_n = 0$. Thus, for every point x_0 such that $x_0 \geqslant 0$ or $x_0 \in [m, 0]$ one has $\lim_{n\to\infty} F_a^n(x_0) = 0$. Choose next arbitrary x_0 with $x_0 < m$. Consider the sequence of its consecutive iterations $x_n := F_a^n(x_0)$. If $x_n \leqslant 0$ for all $n \geqslant 0$ then $\lim_{n\to\infty} x_n = 0$. If $x_{n_0} > 0$ for some n_0 then for the sequence $y_n := F_a^n(x_{n_0}), \lim_{n\to\infty} y_n = 0$, by the above reasoning. Thus, $x = 0$ is a globally attracting fixed point in the case $L = 0$.

Assume next that $L > 0$. Set $\alpha_a := m$ and $\beta_a := \max\{L, Ma\}$. Then the interval $[\alpha_a, \beta_a]$ is mapped by F_a into itself. This is evident from its construction. Set next $I_a := \cap_{n\geqslant 0} F_a^n([\alpha_a, \beta_a])$. I_a is a closed invariant interval (possibly degenerating into a single point $\{0\}$) which attracts all points from $[\alpha_a, \beta_a]$. We shall show that it is also a global attractor, i.e. it attracts all points from $\mathbf{R} \setminus [\alpha_a, \beta_a]$. Indeed, for every point $x_0 \geqslant Ma$ there exists $n_0 \in \mathbf{N}$ such that $F_a^{n_0}(x_0) \in [m, L]$. This is due to the fact that $x > F_a(x) \geqslant x - Ma$ for $x \geqslant Ma$. For every $x_0 < m$ the sequence of its iterations $x_n := F_a^n(x_0)$ is either monotone increasing with $x_n \leqslant 0$ for all n, or $x_{n_0} > 0$ for some $n_0 \in \mathbf{N}$. In the first case, $\lim_{n\to\infty} x_n = 0$. In the second case, a subsequent finite iteration of $y_0 = x_{n_0} > 0$ belongs to the interval $[m, L]$, due to the reasoning above. Therefore, every finite interval $J \supset I_a$ (closed or open) has the property that $F_a^{n_0}(J) \subset [\alpha_a, \beta_a]$ for some $n_0 > 0$. Thus $\cap_{n\geqslant 0} F_a^n(J) = I_a$.

Case $N = 2$. Consider any two maps F_a and F_b, where $a > 0$ and $b > 0$ are arbitrary fixed values. Let $F(x) := F_b \circ F_a(x) = F_b(F_a(x))$.

Note that in general $F(x)$ does not satisfy the inequalities $F(x) > x$ for all $x < 0$ and $F(x) < x$ for all $x > 0$ used in case $N = 1$, even though both F_a and F_b do. However, it retains the following two basic properties that every F_a has:

(i) there exists $x_+ \geqslant 0$ such that for all $x \geqslant x_+$ function $F(x)$ satisfies

$$x > F(x) \geqslant x - M_{ab}, \quad \text{where} \quad M_{ab} = M(a + b);$$

(ii) there exists $x_- \leqslant 0$ such that for all $x \leqslant x_-$ one has $F(x) > x$.

Indeed, since $x - Ma \leqslant F_a(x) < x$ for all $x \geqslant Ma$, then for all sufficiently large x the following inequalities hold

$$F_a(x) - Mb \leqslant F_b(F_a(x)) < F_a(x) < x.$$

Which in turn implies that

$$x - M(a + b) \leqslant F_b(F_a(x)) < x,$$

for all $x > x_+$ with some $x_+ \geqslant 0$. This proves (i).

In order to prove (ii), assume on the contrary that there exists a sequence $x_n \to -\infty$ such that $F(x_n) = F_b(F_a(x_n)) \leqslant x_n$ for all $n \in \mathbf{N}$. If $F_a(x_n)$ is bounded from below, $c \leqslant F_a(x_n) := y_n$ for all $n \in \mathbf{N}$ and some $c \in \mathbf{R}$, then the sequence $F_b(y_n)$ is also bounded from below. Therefore, $F_b(F_a(x_n)) > x_n$ for all sufficiently large n, a contradiction. If $F_a(x_n)$ is not bounded from below, then there is a subsequence x_{n_k} such that $y_k = F_a(x_{n_k}) \to -\infty$. Without loss of generality one can assume that $y_n := F_a(x_n) \to -\infty$ together with $x_n \to -\infty$ as $n \to \infty$. But then, for sufficiently large n, one has $F(x_n) = F_b(F_a(x_n)) = F_b(y_n) > y_n = F_a(x_n) > x_n$, since both F_b and F_a satisfy (ii). This contradiction completes the proof of (ii).

The properties (i) and (ii) are sufficient to prove the existence of an invariant globally attracting interval for the map F. To show this, set $\max\{F(x), x \in [x_-, x_+]\} := F^+$ and $\min\{F(x), x \in [x_-, x_+]\} := F^-$. Let $\alpha := \min\{x_-, F^-\}$ and $\beta := \max\{x_+, F^+\}$. Then the interval $[\alpha, \beta]$ is mapped into itself, $F([\alpha, \beta]) \subset [\alpha, \beta]$. This is evident from its definition. Define $I_0 = [\alpha_0, \beta_0] = \cap_{n\geqslant 0} F^n([\alpha, \beta])$.

I_0 is a closed invariant interval (possibly coinciding with a single point $\{0\}$). We claim that I_0 is also globally attracting on $\mathbf{R}$, that is $\cap_{n\geqslant 0}F^n(J) = I_0$ for every interval $J \supset I_0$. Indeed, if the initial value x_0 is such that $x_0 \geqslant \beta$ then the sequence $x_n := F^n(x_0)$ is decreasing as long as $x_n \geqslant \beta$. Since there are no fixed points of F in $[x_+, \infty])$, there exists n_0 such that $x_{n_0} \in [\alpha, \beta]$. Likewise, for any initial value $x_0 \leqslant \alpha$ the sequence $x_n := F^n(x_0)$ is increasing as long as $x_n < \alpha$. Then, for some $n_0 \geqslant 1$, either $x_{n_0} \in [\alpha, \beta]$, or $x_{n_0} > \beta$. The former means the invariance; for the latter case the first reasoning above should be applied again.

Case $N \geqslant 3$. The proof is done by induction, by repeating the reasoning of the case $N = 2$, as any finite composition of maps of the F_a-type possesses the two properties (i) and (ii). This completes the proof.

Remark. As it can be seen from the proof of the case $N = 2$ of Theorem 2, the negative feedback condition (2) does not have to hold for all $x \in \mathbf{R}$. Therefore, a globally attracting interval $I_0 := [\alpha_0, \beta_0]$ will always exist for the differential delay Eq. (11) if the nonlinearity f is bounded from one side and the negative feedback condition is satisfied for all sufficiently large $x, x \cdot f(x) < 0$ for all $|x| \geqslant x_0$ and some $x_0 > 0$.

Corollary 3.1 (*Uniform boundedness of solutions*). *Suppose f satisfies assumptions (2) and (3), and $a(t)$ satisfies (4). Then all solutions of Eq. (11) are bounded. Moreover, for arbitrary $\varepsilon > 0$ and every initial function $\varphi \in \mathcal{C}$ there exists time $t_\varphi \geqslant 0$ such that the corresponding solution satisfies*

$$\alpha_0 - \varepsilon \leqslant x(t) \leqslant \beta_0 + \varepsilon \quad \text{for all} \quad t \geqslant t_\varphi.$$

The proof is straightforward from the existence of the globally attracting interval $I_0 = [\alpha_0, \beta_0]$ and its stability properties. The corollary is a more refined version of the general Proposition 2.1.

One can derive certain information about the global dynamics in differential delay Eq. (11) based on the size of the periodic function $a(t)$. Some of it is given by the following statements.

Proposition 3.2 (*Global asymptotic stability*). *Suppose $f(x)$ satisfies assumptions (2) and (3), $f'(0) < 0$, and $a(t)$ satisfies (4). There exists $a_0 > 0$ such that if $a(t) \leqslant a_0\ \forall t \in \mathbf{R}$ then the zero solution of differential delay Eq. (11) is globally asymptotically stable.*

Proof. The existence of $f'(0) < 0$ and the one-sided boundedness of $f(x), f(x) \geqslant -M$, imply that $f(x) \geqslant -kx$ for some $k > 0$ and all $x \geqslant 0$. This in turn shows that for all sufficiently small $a, 0 < a \leqslant a_0$, any map $F_a(x)$ has the property that $x > F_a(x) > 0$ for all $x > 0$. But then any composition map $F := F_{a_1} \circ F_{a_2}, 0 < a_1, a_2 \leqslant a_0$ also has the same property. This implies that $x_* = 0$ is a globally attracting fixed point. See the case $N = 1$ of the proof of Theorem 2 for additional related details.

The existence of nontrivial periodic solutions of period ω is now given by the following.

Proposition 3.3 (*Existence of periodic solutions of period N*). *Differential delay Eq. (11) has a nonzero periodic solution of period N if and only if the map F given by (12) has a fixed point different from $x = 0$. The stability of such a periodic solution is determined by the stability of the corresponding fixed point.*

Proposition 3.4 (*Existence of periodic solutions of period N*). *Suppose that $F'(0) > 1$. Then differential delay Eq. (11) has at least two periodic solutions with period N.*

Proof. Recall that the interval $I_0 = [\alpha_0, \beta_0] = \cap_{n\geqslant 0}F^n(I)$ is invariant and the global attractor, where I is a sufficiently large invariant interval of the map F (see details in the proof of Theorem 2). Note that in this case $\alpha_0 < \beta_0$ in view of $F'(0) > 1$. Since $F(\alpha_0) \geqslant \alpha_0$ there exists a point $x^\alpha \in [\alpha_0, 0)$ such that $F(x^\alpha) = x^\alpha$. Likewise, there exists $x^\beta \in (0, \beta_0]$ such that $F(x^\beta) = x^\beta$.

Proposition 3.5 (*Existence of periodic solutions of period 2N*). *Suppose that $F'(0) < -1$. Then differential delay Eq. (11) has a periodic solution of period 2N.*

Proof. The differential delay Eq. (11) has a periodic solution of period $2N$ if and only if the map F has a cycle of period 2. Since interval I is invariant under F, and $x = 0$ is a repelling fixed point with the negative feedback condition satisfied locally

$$x \cdot F(x) < 0 \quad \text{for all} \quad x \in [-\delta, \delta] \quad \text{and some} \quad \delta > 0,$$

its instability implies the existence of a cycle of period two [4,8,30]. Note that the stability of such periodic solution is the same as the stability of the two-cycle.

The value of $F'(0)$ is easily calculated as

$$F'(0) = [1 + a_1 f'(0)] \cdot [1 + a_2 f'(0)] \cdot \ldots \cdot [1 + a_N f'(0)] := \lambda.$$

Based on Propositions 3.4 and 3.5 we can state the following result on the existence of periodic solutions to Eq. (11).

Theorem 3 (*Existence of periodic solutions*).

(i) Eq. (11) has at least two nonzero periodic solutions with period N when $\lambda > 1$;
(ii) Eq. (11) has a nonzero periodic solution with period 2N when $\lambda < -1$.

The following corollary provides sufficient conditions for the existence of periodic solutions of Eq. (11) when either $a(t)$ or $|f'(0)|$ is sufficiently large. Set $a_* := \min\{a_i, i = 1, \dots, N\} > 0$ where a_i are defined by formula (8).

Corollary 3.6 (*Existence of periodic solutions*). *Suppose that $a_* \cdot f'(0) < -2$. Then*

(i) when N is even, differential delay Eq. (11) has at least two periodic solutions with period N;
(ii) when N is odd, differential delay Eq. (11) has a periodic solution with period 2N.

It is straightforward to conclude that Corollary 3.6 holds when $a_m = \min\{a(t), t \in [0, N]\} > 0$ satisfies $a_m > -2/f'(0)$.

Remark. It is easy to see that for respective even or odd values of N map F can be made such that $\lambda > 1$ holds and it has exactly two additional non-zero fixed points. Likewise, map F can be made such that $\lambda < -1$ is satisfied and it has exactly one cycle of period two.

3.3. Example: Wright-type periodic DEDA

Consider the following DEDA

$$z'(t) = -a(t)z([t])(1 + z(t)), \tag{13}$$

where $a(t)$ is a positive continuous periodic function with an integer period N, and $z > -1$. Eq. (13) can be viewed as a generalization of the well-known Wright differential delay equation, $z'(t) = -az(t-1)[1 + z(t)]$, where $a > 0$ is a constant [33]. By applying a standard change of variables, $1 + z = \exp(-x)$, Eq. (13) is reduced to the following DEDA of the form of Eq. (11)

$$x'(t) = a(t)f(x([t])) \quad \text{where} \quad f(x) = \exp(-x) - 1. \tag{14}$$

Note that the substitution shows that $z > -1$ must hold for Eq. (13), which is the same range for z as in the original work [33]. It is easily seen that $f(x) = \exp(-x) - 1$ is a bounded from below strictly decreasing function satisfying the negative feedback condition $x \cdot f(x) < 0, x \neq 0$.

To demonstrate the applicability of results of SubSection 3.2 to Eq. (14) we consider two partial cases for the value of period N of function $a(t), N = 1$ and $N = 2$, and briefly mention several possibilities for the case $N \geqslant 3$.

Subcase $N = 1$. Let $a := \int_0^1 a(t)\,dt > 0$. We shall use a as a parameter. The map $F_a(x) = x + af(x)$ has the form $F_a(x) = x + a[\exp(-x) - 1]$. One easily sees that F_a is a unimodal map: $F_a(x)$ is decreasing in $(-\infty, c)$ and increasing in (c, ∞), where $c = \ln a$. Also, $F_a(x)$ is concave up for all real values of x. Besides, its Schwarzian derivative $(SF_a)(x) = F_a'''(x)/F_a'(x) - \frac{3}{2}[F_a''(x)/F_a'(x)]^2$ is negative for $x \neq \ln a : (SF_a)(x) = [-2a\exp(-x) \; -a^2\exp(-2x)]/ \; [2(1 - a\exp(-x))^2]$

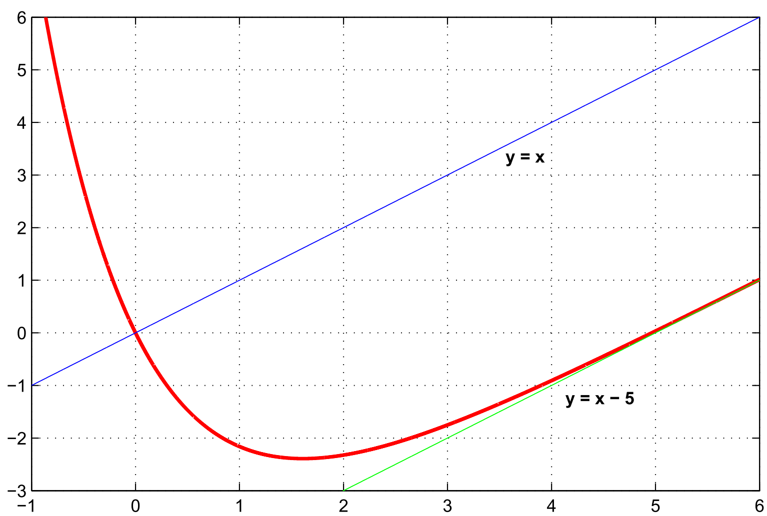

Fig. 1. Function $F_a(x)$, $a = 5$.

< 0. Therefore, map F_a has all the significant properties that unimodal maps with negative Schwarzian possess [4,8,30]. A typical function $F_a(x)$ is depicted in Fig. 1 (for the parameter value $a = 5$).

The following list of statements illustrates some of the global dynamics of the map F_a depending on the values of the parameter a. They are immediately translated into the analogous properties of solutions of DEDA (14).

Map F_a has the following properties:

(i) For any $0 < a \leqslant 2$ the unique fixed point $x = 0$ is a global attractor;
(ii) For any $a > 2$ the fixed point $x = 0$ is repelling. There exists a unique cycle of period two;
(iii) For every $a > 2$ the map F_a has an invariant interval $I_a = [\alpha, \beta]$, where $\alpha := F_a(\ln a)$ and $\beta := \max\{F_a(x), x \in [F_a(\ln a), F_a^2(\ln a)]\}$. It attracts all other points: $F_a^n(x) \in I_a$ for every x and some $n \in \mathbf{N}$. There exists a nontrivial minimal globally attracting interval $I_* : I_* := \cap_{n \geqslant 0} F_a(I_a)$;
(iv) There exists a sequence of parameter values $2 = a_1 < a_2 < a_3 < \ldots < a_n < a_{n+1} < \ldots$ with $\lim_{n\to\infty} a_n = a_\infty$ and such that for all $a \in (a_{n-1}, a_n]$ map F_a has a unique cycle of period 2^n which is a global attractor. It attracts all points from $\mathbf{R}$ except a countable set consisting of the unique repelling cycles of periods $2^i, i < n$, and all their pre-images;
(v) Our numerical calculation shows that $a_2 = 2.5322\ldots, a_3 = 2.6553\ldots, a_4 = 2.6841\ldots, a_5 = 2.6899\ldots$ and $a_\infty = 2.6925\ldots$;
(vi) For every $a > a_\infty$ the map F_a is chaotic;
(vii) There exists a parameter value $a^{(3)}$ such that for every $a \geqslant a^{(3)}$ the map F_a has a cycle of period three, and it does not have a cycle of period three for any $a < a^{(3)}$. Map F_a has cycles of all periods for $a \geqslant a^{(3)}$;

Remark 1. The above properties are well-known facts for unimodal interval maps with a negative Schwarzian, which proof can be found e.g. in monographs [4,8,30]. Some of them are highly nontrivial; however, the scope of the present paper does not allow for further detailed explanation and justification within the corresponding theory of interval maps. The chaos can be meant in either the sense of Li-Yorke, or Denaney [9], or some other refined definitions; in either case it follows from the existence of a cycle with a period not a power of two for the map F_a.

Remark 2. The dynamics of solutions of Eq. (14) in the case $N = 1$ are equivalent to that of the equation $u'(t) = \mu u(t)(1 - u([t])), u(0) = c_0, t \geqslant 0$, with constant coefficient $\mu > 0$, which was considered in [2]. One can easily see that the latter equation is equivalent to our Eq. (13) via a constant shift. The authors of [2] derive an underlying interval map $G_\mu(u) = u \exp(\mu(1 - u))$, which is conjugate to our map $F_a(x)$ via $u = \exp(-x)$. Therefore, the dynamics of the two maps G_μ and F_a are equivalent for the same values of the parameters μ and a. They indicate a possibility for the dynamics to be chaotic, which they relate to the existence of an attracting cycle of period three for the map G_μ.

Subcase $N = 2$. Let $a := \int_0^1 a(t)dt > 0$ and $b := \int_1^2 a(t)dt > 0$. We shall use a and b as parameters. Consider the composite map $F = F_b \circ F_a$. When $(a - 1)(b - 1) > 1$ the fixed point $x = 0$ is repelling. Map F then also has two other nontrivial fixed points $x_1^* < 0 < x_2^*$. Thus the corresponding DEDA (14) has exactly two periodic solutions with period 2. A typical map F is shown in Fig. 2 for the parameters values $a = 2.4, b = 6.1$. In general, the dynamics of the map F can vary depending on the values of the parameters a and b. As the parameters vary the dynamics can be changing from simple, with the two fixed points being (semi) global attractors, to their instability and period doubling bifurcations, and to further transition to chaotic behaviors of different kinds. In the case shown in Fig. 2 the map F has two non-trivial repelling fixed points $x_1^* = -1.25862\ldots, F'(x_1^*) = -7.07189\ldots$ and $x_2^* = 1.67173\ldots, F'(x_2^*) = -3.86981\ldots$. On the interval $[0, z_2]$, where $z_2 = 2.10863\ldots$ is the first positive solution of the equation $F(x) = 0$, the map F has a chaotic repeller homeomorphic to the Cantor set. Likewise, map F has a similar chaotic repeller on the interval $[z_1, 0]$ where $z_1 = -1.41742\ldots$ is the only negative zero of function F.

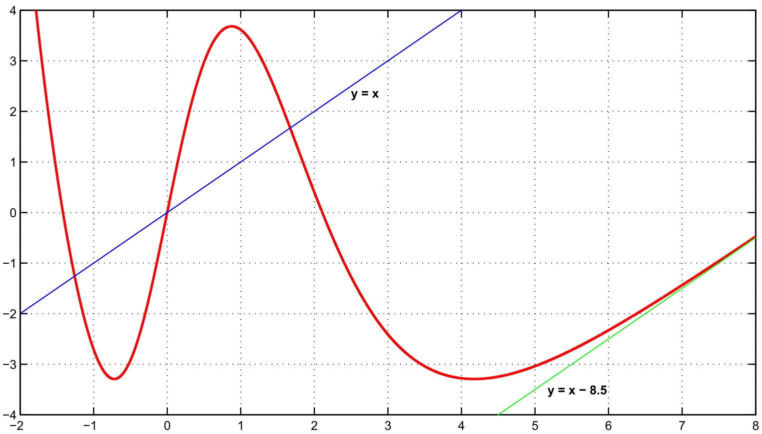

Fig. 2. Function $F(x) = F_b \circ F_a(x), a = 2.4, b = 6.1$.

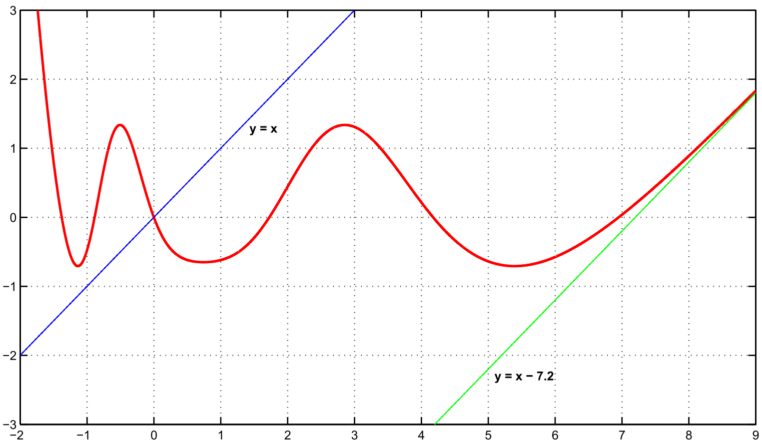

Fig. 3. Function $F(x) = F_c \circ F_b \circ F_a(x), a = 2.1, b = 2.4, c = 2.7.$

Subcase $N \geqslant 3$. The case of greater integer values of the period N can be treated analogously to the above two cases of $N = 1$ and $N = 2$. Two subcases of the case $N = 3$ are shown in Figs. 3 and 4. The composite map $F = F_c \circ F_b \circ F_a$ shown in Fig. 3 corresponds to the parameter values $a = 2.1, b = 2.4, c = 2.7$. One can see that there is no other fixed point of the map F but the trivial one $x = 0$. Therefore, DEDA (14) does not have periodic solutions with period 3. However, $F^2 = F \circ F$ does have two attracting fixed points $x_1 = -0.464186\ldots$ and $x_2 = 1.31953\ldots$ which correspond to an attracting cycle of period 2 for the map F. The multiplier $\lambda = F'(x_1)F'(x_2)$ of the cycle is $\lambda = -0.652781\ldots$ since $F'(x_1) = -0.86331\ldots$ and $F'(x_2) = 0.756138\ldots$. Therefore, DEDA (14) has an asymptotically stable periodic solution with period 6.

Fig. 4 shows the map $F = F_c \circ F_b \circ F_a$ for the parameter values $a = 3.6, b = 3.1, c = 4.2$. It has additional six repelling fixed points each of which yields an unstable period 3 solution of DEDA (14). There is also a repelling cycle of period 2 which corresponds to an unstable periodic solution of period 6 of the DEDA.

Remark 3. Note that in the case of arbitrary period N the stability conditions for Eq. (14) can be found in [26]. If $a_i = \int_{i-1}^{i} a(t)\,dt \leqslant 2$ for all $1 \leqslant i \leqslant N$ then its zero solution is globally asymptotically stable (see Theorem 1.1 of [26]). This conclusion can also be deduced from the fact that in this case the fixed point $x = 0$ is a global attractor for each map $F_i(x) = x + a_i(\exp(-x) - 1)$, and that the composite map $F = F_N \circ \ldots \circ F_1$ has a negative Schwarzian.

Finally note that this example demonstrates some profound differences between the dynamics of DEDA (13) and that of the original Wright's equation $z'(t) = -az(t-1)[1 + z(t)]$. While the former are rich and various, as much as general interval maps can be, the latter are rather simple. The original Wright's equation is conjectured to possess a unique slowly oscillating periodic solution, which is then known to be locally asymptotically stable and attracting almost all solutions. This is confirmed numerically for all $a > \pi/2$ and proved analytically for the larger part of this parameter domain (see details in [23]). Therefore, the typical asymptotic dynamics that one sees in the original Wright's equation are stable periodic motions.

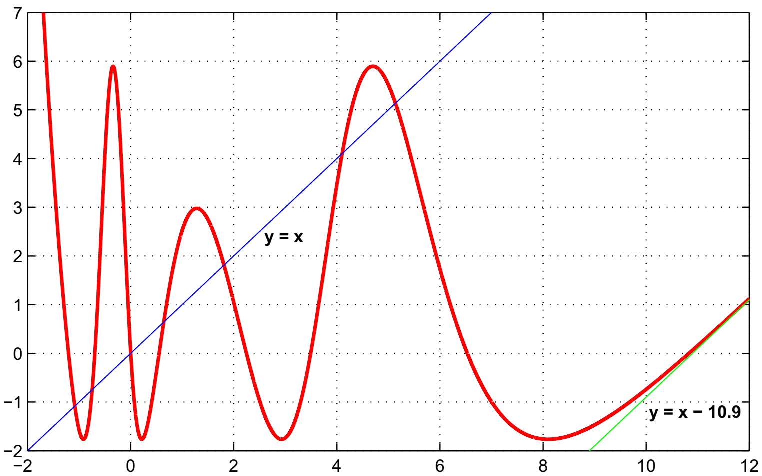

Fig. 4. Function $F(x) = F_c \circ F_b \circ F_a(x), a = 3.6, b = 3.1, c = 4.2.$

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.amc.2014.12.015.

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