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# On the Discretization of a Delay Differential Equation

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The dynamics of the delay difference equation $\mu[\Delta x_n + \alpha\Delta x_{n-N}] = -x_{n+1} + f(x_{n-N})$ as $n \to \infty$ is studied for small positive $\mu$. The equation is shown to possess stable periodic solutions that correspond to hyperbolic attracting cycles of the one-dimensional map $f$.

*Keywords:* Singularly perturbed delay differential equations; Euler discretizations; Difference-delay equations; Interval maps; Attracting hyperbolic cycles; Stable periodic solutions of difference equations

*AMS Subject Classification:* 34K; 39

## 1 INTRODUCTION

This paper deals with the difference equation of the form

$$\mu[\Delta x_n + \alpha\Delta x_{n-N}] = -x_{n+1} + f(x_{n-N}), \tag{1.1}$$

where $0 < \mu$ is a (small) parameter, $\alpha \in \mathbf{R}$, $f: \mathbf{R} \to \mathbf{R}$ is a continuous function, and $\Delta x_n := x_{n+1} - x_n$. Equation (1.1) is easily solved for $x_{n+1}$

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so one has its equivalent form

$$x_{n+1} = \frac{\mu}{\mu+1} x_n + \frac{\alpha\mu}{\mu+1}(x_{n-N} - x_{n-N+1}) + \frac{1}{\mu+1} \cdot f(x_{n-N}). \qquad (1.2)$$

Some basic preliminaries for Eqs. (1.1) and (1.2) are described below in Section 3.

Equation (1.1) arises from Euler discretizations of the delay differential equation

$$\varepsilon[\dot{x}(t) + \alpha\dot{x}(t-1)] = -x(t) + f(x(t-1)). \qquad (1.3)$$

The details of the discretization procedure of Eq. (1.3) and reduction to (1.1) are given in Section 2. Equation (1.1) can also be looked upon as a discrete analog of Eq. (1.3). Thus in these regards Eqs. (1.1) and (1.3) are closely connected.

Equation (1.3) appears in a number of important applications and it has been a subject of study, especially its particular case $\alpha = 0$, in a number of recent publications, see for example [1,2,9,10,12–15,17] and further references therein. One of the most significant applications, in our opinion, is that Eq. (1.3) comes as an exact reduction of certain boundary value problems for hyperbolic partial differential equations with the singular term $\varepsilon[\dot{x}(t) + \alpha\dot{x}(t-1)]$ representing the small viscosity effects in the original physical models [3,14,15,17]. Note that the presence of the neutral term in Eq. (1.3) is essential, that is $\alpha \neq 0$.

Even in a simpler case of the retarded type equation (1.3) when $\alpha = 0$,

$$\varepsilon\dot{x}(t) = -x(t) + f(x(t-1)) \qquad (1.4)$$

it is a very nontrivial object to study. A natural approach to investigate (1.4) for small $\varepsilon > 0$ has been to compare its dynamics with the dynamics of the limiting case $\varepsilon = 0$, the difference equation

$$x(t) = f(x(t-1)). \qquad (1.5)$$

The latter is completely determined by the dynamics of the one-dimensional map $f$. In spite of a number of recent publications on Eq. (1.4) (see for example [1,2,10,12,13] and further references therein) only partial results on its dynamics have been obtained. There is a number of important problems about its relation to the dynamics of the map $f$ that still remain unanswered.

Virtually nothing is known about dynamics as $t \to +\infty$ of Eq. (1.3) with $\alpha \neq 0$. Though there is a closeness within finite time intervals between solutions of (1.3) and (1.5) [4,5], no other rigorous results about asymptotical behavior of solutions of Eq. (1.3), to the best of our knowledge, are available. This leads to another approach to study the dynamics of the processes behind Eq. (1.3) – to look at its discrete versions or to study it numerically. In this way one comes down to the difference equation (1.1), and it is our major motivation for its study.

We compare the dynamics as $n \to \infty$ of solutions of Eq. (1.1) with the dynamics of the interval map $f$ when the parameter $\mu > 0$ is sufficiently small. The principal result we derive in this paper is that whenever the map $f$ has an attracting hyperbolic cycle the corresponding Eq. (1.1) has stable periodic solutions "close" to the cycle (Theorem 4.3).

This has as we think several important implications:

1. The dynamics of the continuous time delay differential equation (1.3) and its discrete version, Eq. (1.1), are, generally speaking, different. For example, Eq. (1.3) with $\alpha = 0$ (that is Eq. (1.4)) may exhibit a simple dynamical behavior of all solutions (for example, global stability) even though the dynamics of $f$ is more complicated (for example, the map $f$ has attracting cycles), see examples in [10]. In this latter situation of attracting cycles Eq. (1.1) will still have stable periodic solutions that are "close" to the cycles.

The numerical simulations of Eqs. (1.3) or (1.4) may be showing in some cases dynamics which are irrelevant to the actual dynamics in these equations. That is, such dynamics may simply not exist for the delay differential equations. The results of the present paper demonstrate this fact if one uses the Euler discretization for the singularly perturbed delay equations. The phenomenon is known more generally for the numerical solution of ordinary differential equations, and is well understood for example for the occurrence of the spurious fixed points in Runge–Kutta methods (see e.g. [6,16,18] and further references therein).

2. The delay differential equation (1.3) was derived as a continuous time model of certain physical processes that takes into account the small viscosity effects. Mathematically the singular terms in Eq. (1.3) represent the small viscosity while the nonlinear map $f$ is expected to

determine the asymptotical behavior of solutions as $t \to \infty$. See papers [3,9,17] and monograph [15] for the description of physical processes and their mathematical models as well as for further details. As examples and rigorous mathematical results show, the dynamics of solutions of Eq. (1.3) as $t \to \infty$ can be essentially different from that defined by the interval map $f$ (see [10], Sections 4 and 5, case $\alpha = 0$). The continuous time model is actually derived via a limiting transition from a discrete one, which is in essence a difference equation. Therefore, in some cases the difference equation (1.1) can probably be a better mathematical model, at least as far as following the dynamics of the map $f$ is concerned.

## 2 EULER DISCRETIZATIONS

The simplest way to discretize Eq. (1.3) is to use the Euler discretization scheme.

Assume that the delay 1 is a multiple of the discretization step, that is $h := 1/N$ for some positive integer $N$. By using the forward discretization one has $\dot{x}(t) \approx [x(t+h) - x(t)]/h$, and Eq. (1.3) is replaced by

$$\varepsilon\left[\frac{x(t+h) - x(t)}{h} + \alpha \cdot \frac{x(t-1+h) - x(t-1)}{h}\right] = -x(t) + f(x(t-1)).$$

By introducing new notations: $\varepsilon/h := \mu$, $x(t+h) := x_{n+1}$, $x(t) := x_n$, where $t = t_n = nh$, one finds that $x(t-1) = x_{n-N}$, $x(t-1+h) = x_{n-N+1}$. Therefore the latter difference equation becomes

$$\mu[(x_{n+1} - x_n) + \alpha(x_{n-N+1} - x_{n-N})] = -x_n + f(x_{n-N}) \tag{2.1}$$

or

$$\mu[\Delta x_n + \alpha \Delta x_{n-N}] = -x_n + f(x_{n-N}),$$

where $\Delta x_n = x_{n+1} - x_n$.

Equation (2.1) is easily solved for $x_{n+1}$:

$$x_{n+1} = x_n + \alpha(x_{n-N} - x_{n-N+1}) + \frac{1}{\mu}[-x_n + f(x_{n-N})]. \tag{2.2}$$

Since we are interested in the dynamics as $\varepsilon \to 0+$ and $\mu = \varepsilon/h$, the difference equation (2.2) shows that the numerical solutions may easily "blow up" as $\varepsilon$ becomes much smaller than the chosen step $h$ of the discretization.

Another complication with the forward Euler discretization is that in the simpler case of the retarded type equation (1.3), when $\alpha=0$, some of its basic properties are not preserved. For example if the map $f$ has a closed invariant interval $I$ the corresponding solutions of Eq. (1.4) remain within the interval $I$ provided the initial functions were chosen there (see [10] for precise statements). Since the difference equation (2.2) in this case becomes

$$x_{n+1} = x_n + \frac{1}{\mu}\cdot[-x_n + f(x_{n-N})], \tag{2.3}$$

again because of the blow-up phenomenon for small $\mu$, this basic invariance property is not preserved if one replaces the differential delay equation (1.3) with $\alpha=0$ by the difference equation (2.3).

This difficulty with the forward discretization is evident from the fact that $\dot{x}(t)$ and $x(t)$ at the same time $t$ produce two terms involving $x_n$ and a single forward "singular" term $\mu x_{n+1}$. As we show below this problem can be fixed if one uses the backward difference $[x(t)-x(t-h)]/h$ for $\dot{x}(t)$. The difficulty is explained by the fact that the forward Euler scheme for the singular equation (1.3) is unstable unless the step $h$ of the discretization is sufficiently small compared with $\varepsilon$, while the backward Euler scheme is A-stable (see e.g. [11] for more details).

Another minor problem that appears with the backward discretization at $\dot{x}(t-1)$ is that one obtains a term $x(t-1-h)$ which falls outside the initial interval $[-1,0]$ at $t=0$. $x(-1-h)$ will have to be defined then. An alternative possibility to avoid this is to use the forward difference for $\dot{x}(t-1)$. As it can easily be seen from what follows, both ways to discretize $\dot{x}(t-1)$ give essentially the same type difference equation. Therefore, the forward discretization at the delayed term $\dot{x}(t-1)$ does not create the same difficulty as it does at $\dot{x}(t)$. We proceed here by using the backward difference for $\dot{x}(t)$ and the forward difference for $\dot{x}(t-1)$. This results in the equation

$$\varepsilon\left[\frac{x(t)-x(t-h)}{h} + \alpha\cdot\frac{x(t-1+h)-x(t-1)}{h}\right]$$
$$= -x(t) + f(x(t-1)).$$

With the new notations as introduced above the latter becomes

$$\mu[(x_n - x_{n-1}) + \alpha(x_{n-N+1} - x_{n-N})] = -x_n + f(x_{n-N}),$$

which is equivalent, by associating $t$ with $(n+1)h$ and $t-1$ with $(n-N)h$, to

$$\mu[\Delta x_n + \alpha\Delta x_{n-N}] = -x_{n+1} + f(x_{n-N}). \tag{2.4}$$

The backward difference used at both $\dot{x}(t)$ and $\dot{x}(t-1)$ gives the following difference equation

$$\mu[(x_n - x_{n-1}) + \alpha(x_{n-N} - x_{n-N-1})] = -x_n + f(x_{n-N}),$$

which is equivalent to

$$\mu[\Delta x_n + \alpha\Delta x_{n-N}] = -x_{n+1} + f(x_{n-N+1}). \tag{2.5}$$

As it can be seen from the proof of our main results in Section 4, Eq. (2.5) exhibits the same dynamics as Eq. (2.4).

## 3 PRELIMINARIES FOR THE DELAY DIFFERENCE EQUATION

Solving Eq. (1.1) for $n > 0$ is straightforward if one uses its equivalent explicit form (1.2). The iterative scheme (1.2) requires that initial data, $\{x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0\}$, be defined. The latter will be called the *initial string* and denoted by $\bar{x}_0$. It is obvious that for every initial string $\bar{x}_0$ there exists a unique solution $\{x_1, x_2, x_3, \ldots\}$ of Eq. (1.2) defined for all $n > 0$. The segment of the solution made up of $\{x_1, x_2, \ldots, x_{N+1}\}$ will be called the *first string* and denoted by $\bar{x}_1$, the subsequent segment $\{x_{N+2}, \ldots, x_{2N+2}\}$ – the second string $\bar{x}_2$, etc. We say that a string $\bar{x}_0 = \{x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0\}$ belongs to a set $A \subset \mathbf{R}$, $\bar{x}_0 \in A$, if $x_i \in A$ for all $i \in \{-N, \ldots, -1, 0\}$.

Given two strings $\bar{x}_0 = \{x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0\}$ and $\bar{y}_0 = \{y_{-N}, y_{-N+1}, \ldots, y_{-1}, y_0\}$ we define the distance between them by $\|\bar{x}_0 - \bar{y}_0\| := \max\{|x_i - y_i|, -N \le i \le 0\}$. This makes the set of all strings, $S := \{\bar{x}_0 = \{x_{-N}, \ldots, x_{-1}, x_0\} \,|\, x_i \in \mathbf{R}\, \forall i \in \{-N, \ldots, -1, 0\}\}$, a normed space.

When $\mu = 0$ one gets the difference equation

$$x_{n+1} = f(x_{n-N}) \tag{3.1}$$

from Eq. (1.1) (or (1.2)). Likewise Eq. (3.1) requires an initial string $\bar{x}_0$ in order to have a solution defined for all $n > 0$. Given an initial string $\bar{x}_0$ Eq. (3.1) is then equivalent to $N+1$ difference equations $z_{n+1} = f(z_n)$, $n \geq 0$, $z_0 \in \bar{x}_0$. The dynamics of the latter, and therefore that of Eq. (3.1), is completely determined by the dynamics of the one-dimensional map $f$. We will also have to measure the distance between strings of Eqs. (1.2) and (3.1). This will be done in the obvious way in accordance with the above definition of the distance between strings.

We shall also use some standard terminology for the interval maps. We say that $x_* \in \mathbf{R}$ is a fixed point of the map $f$ if $f(x_*) = x_*$. A fixed point $x_*$ is called attracting if there exists an open interval $J \ni x_*$ such that $f(J) \subset J$ and $\lim_{n\to\infty} f^n(x) = x_*$ for every point $x \in J$. Here $f^n$ stands for the $n$th iteration of the map $f$, i.e. $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$. A maximal open interval about an attracting fixed point with this property is called the *domain of immediate attraction* of the fixed point.

A fixed point $x_*$ is called repelling if there exists an open interval $J \ni x_*$ such that for every point $x \in J$, $x \neq x_*$ there is a positive integer $n = n(x)$ such that $f^n(x) \notin J$.

A set of points $\beta := \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$ is called a cycle of period $m$ if $f(\beta_k) = \beta_{k+1}$, $k = 0, 1, \ldots, m-1$, where $\beta_m$ is defined to be $\beta_0$. A cycle $\beta$ is called attracting (repelling) if $\beta_0$ is an attracting (repelling) fixed point of the map $f^m$.

Let $\beta = \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$ be an attracting cycle of the map $f$. Then its domain of immediate attraction is made up of $m$ disjoint open intervals $U_0, U_1, \ldots, U_{m-1}$ and such that $f(U_k) = U_{k+1}$, $k = 0, 1, \ldots, m-1$, where $U_m := U_0$. Here $U_0$ is the domain of immediate attraction of the fixed point $\beta_0$ for the map $f^m$.

A solution $\{x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0, x_1 \ldots, x_N, x_{N+1}, \ldots\}$ of either Eq. (1.2) (or (1.1)) or Eq. (3.1) will be called *slowly oscillating with respect to cycle* $\beta = \{\beta_0, \ldots, \beta_{m-1}\}$ if $\bar{x}_k \in U_{k+p(modulo\ m)}$ for some nonnegative integer $p$ and all $k \geq 0$. Here $\bar{x}_k$ is the $k$th string of the corresponding solution.

An important class of equations, for both Eq. (1.3) and Eqs. (1.1)/(1.2), is that with the nonlinearity $f$ of the negative feedback type: $x \cdot f(x) < 0$ for all $x \in \mathbf{R}$, $x \neq 0$. The negative feedback condition implies that $x_* = 0$ is a fixed point of the map $f$, and that all of the Eqs. (1.1)–(1.3) have the trivial zero solution. A solution of any one of the equations will be called *slowly oscillating* if its consecutive strings alternate sign. For example, all elements of $\bar{x}_0$ are positive, all elements of $\bar{x}_1$ are negative, etc., or the other way around.

Let $\beta = \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$ be a cycle of the map $f$. By $\bar{\beta}_k$ we denote a string each element of which is $\beta_k$, that is, $\bar{\beta}_k = \{\beta_k, \ldots, \beta_k\}$, $k = 0, 1, \ldots, m-1$.

Finally we use the notions of stability of a constant solution and stability with the asymptotic phase of a periodic solution of Eq. (1.2) in a standard way.

A constant solution $\{x_*, x_*, \ldots\}$ of Eqs. (1.1), (1.2), or (1.3) will be called stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for every initial string $\bar{x}_0$ with $\|\bar{x}_0 - \bar{x}_*\| \leq \delta$ one has $\|\bar{x}_k - \bar{x}_*\| \leq \varepsilon$ for all $k > 0$. The constant solution will be called asymptotically stable if it is stable and $\lim_{k \to \infty} \|\bar{x}_k - \bar{x}_*\| = 0$.

A periodic solution $\bar{p}_M = \{p_1, p_2, \ldots, p_M, p_1, p_2, \ldots\}$, $M \geq N+1$, will be called asymptotically stable with the asymptotic phase if there exists $\delta > 0$ such that for every initial string $\bar{x}_0 = \{x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0\}$ with $\|(p_1, \ldots, p_{N+1}) - (x_{-N}, \ldots, x_0)\| \leq \delta$ the corresponding solution $\bar{x} := \{x_{-N}, \ldots, x_0, x_1, x_2, \ldots, x_n, \ldots\}$ has the property: $\lim_{k \to \infty} \|\bar{x}_k^M - \bar{p}_{M+j}\| = 0$ for some $j \in \{0, \ldots, M-1\}$, where $\bar{x}_k^M := \{x_{M(k-1)+1}, \ldots, x_{Mk}\}$, and $\bar{p}_{M+j} := \{p_{j+1}, \ldots, p_M, p_1, \ldots, p_j\}$.

## 4 MAIN RESULTS

### 4.1 Continuous Dependence on the Parameter

THEOREM 4.1 (Continuous dependence on $\mu$) *Let $I := [a, b]$ be a closed interval and let $f(I) := [c, d]$. For every $\sigma > 0$ there exists $\mu_0 = \mu_0(\sigma)$ such that for every initial string $\bar{x}_0 := \{x_{-N}, \ldots, x_{-1}, x_0\} \in I$ the first string $\bar{x}_1 := \{x_1, x_2, \ldots, x_{N+1}\}$ has the property $\bar{x}_1 \in [c - \sigma, d + \sigma]$ for all $0 < \mu \leq \mu_0$.*

*Proof* We estimate first the difference $x_1 - f(x_{-N})$. From Eq. (1.2) one finds:

$$\begin{aligned}
&|x_1 - f(x_{-N})| \\
&= \left|\frac{\mu}{\mu+1}x_0 + \frac{1}{\mu+1}f(x_{-N}) - \frac{\alpha\mu}{\mu+1}(x_{-N+1} - x_{-N}) - f(x_{-N})\right| \\
&\leq \frac{|\alpha|\mu}{\mu+1}|x_{-N+1} - x_{-N}| + \frac{\mu}{\mu+1}|x_0 - f(x_{-N})| \leq \mu \cdot M,
\end{aligned}$$

where $M > 0$ is a constant depending on $\alpha, a, b, c, d$ only. In the same way one derives $|x_{n+1} - f(x_{n-N})| \leq \mu \cdot M$, $n = 1, 2, \ldots, N$. The last step in this demonstration (that is, for $n = N$) is slightly different from the others, because from (1.2) it is seen that $x_{N+1}$ depends on $x_1$ as well as on values in the initial string. The statement of the theorem follows.

COROLLARY 4.1 (Invariance Property) *Let $I$ be a closed interval such that $f(I) \subset \operatorname{int} I$, where $\operatorname{int} I$ is the interior part of $I$. There exists $\mu_0 > 0$ such that for every initial string $\{x_{-N}, \ldots, x_{-1}, x_0\} \in I$ the corresponding solution $x_n$ of Eq. (1.1) has the property: $x_n \in I$ for all $n \geq 1$ and any $\mu \in (0, \mu_0]$.*

*Proof* This follows from Theorem 4.1 with $I := [a, b]$, $f(I) := [c, d]$, and the fact that $[c, d] \subset (a, b)$. $\sigma$ of Theorem 4.1 can be chosen then as $\sigma := \min\{c - a, b - d\}$.

COROLLARY 4.2 *Let $I$ be an arbitrary closed interval. For every positive integer $m$ and $\sigma > 0$ there exists $\mu_0 = \mu_0(\sigma, m)$ such that for every initial string $\bar{x}_0 = \{x_{-N}, \ldots, x_{-1}, x_0\} \in I$ one has: $\bar{x}_m \in [\inf\{f^m(I)\} - \sigma, \sup\{f^m(I)\} + \sigma]$, where $\bar{x}_m$ is the $m$th string of the corresponding solution and $f^m$ is the $m$th iteration of the map $f$.*

*Proof* It follows from Theorem 4.1 and the induction argument.

### 4.2 Existence of Slowly Oscillating Periodic Solutions

Throughout this subsection we assume that the function $f(x)$ satisfies the negative feedback condition:

$$xf(x) < 0 \quad \text{for all } x \in \mathbf{R},\ x \neq 0. \tag{nf}$$

Condition (nf) implies that $x_n \equiv 0$ is the only constant solution of Eq. (1.1). Note that the negative feedback condition with respect to a non-zero constant solution can be reduced to the above form of (nf) by an appropriate change of variables.

THEOREM 4.2 *Assume that the map $f$ has an invariant closed interval $I$ such that $f(I) \subset \operatorname{int} I$ and $x=0$ is a hyperbolic repelling fixed point, that is, $f'(0) < -1$. There exists $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0]$ Eq. (1.1) has a slowly oscillating periodic solution.*

*Proof* Let $I := [a, b]$ be the invariant interval with $f([a,b]) \subset (a,b)$. Condition (nf) implies that $a < 0 < b$. Condition $f'(0) < -1$ implies that there exists $\delta > 0$ such that $f([\delta, b]) \subset (a, -\delta)$, $f([a, -\delta]) \subset (\delta, b)$. Consider the set of initial strings defined by

$$S_0 := \{\bar{x}_0 = (x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0) \mid x_i \in [\delta, b] \\ \forall i \in \{-N, \ldots, -1, 0\}\}.$$

Obviously $S_0$ is a compact convex set.

Let $\bar{x}_2 = \bar{x}_2(\bar{x}_0) := \{x_{N+2}, \ldots, x_{2N+3}\}$ be the second string of the solution corresponding to the initial string $\bar{x}_0$. Define a mapping $F$ of the set $S_0$ into itself by

$$F(\bar{x}_0) = \bar{x}_2.$$

Obviously $F$ is a continuous mapping.

Corollary 4.2 implies that there exists $\mu_0 > 0$ such that $\bar{x}_2 \in [\delta, b]$ for every $0 < \mu \leq \mu_0$. Therefore, $F$ indeed maps $S_0$ into itself. By the Schauder fixed point theorem $F$ has a fixed point $\bar{x}_0^* \in S_0$. Clearly, $\bar{x}_0^*$ generates a slowly oscillating periodic solution of Eq. (1.1).

## 4.3 Existence and Stability of Periodic Solutions Corresponding to Attracting Cycles of the Map $f$

THEOREM 4.3 *Assume that the map $f$ has a hyperbolic attracting cycle of period $m$: $\beta = \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$, $f(\beta_i) = \beta_{i+1}$, $i = 0, 1, \ldots, m-1$ $(m := 0)$, $\lambda := |f'(\beta_0) \cdot f'(\beta_1) \cdot \ldots \cdot f'(\beta_{m-1})| < 1$. There exists $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0]$ Eq. (1.1) has a slowly oscillating with respect to cycle $\beta$ periodic solution which is asymptotically stable with the asymptotic phase.*

*Proof* *The existence* part repeats the proof of Theorem 4.2 with the cycle of intervals of period two being replaced by the cycle of intervals of period $m$. Indeed, since $\beta$ is a hyperbolic attracting cycle, there exists a closed interval $[a,b]$ about point $\beta_0$ such that $f^m([a,b]) \subset (a,b)$. Now one defines $S_0$ in a similar way based on the interval $[a,b]$, and also defines the mapping $F$ by taking the $m$th string of the solution, and then applies the Schauder fixed point theorem. We leave details to the reader.

*Stability* of the periodic solution will be proved by using two facts: (a) the closeness of the corresponding strings of the periodic solution to the cycle $\beta$, and (b) linearization of the difference equation along the periodic solution.

(a) Let the periodic solution be determined by the initial string $\bar{x}_0 \in [a,b]$, where $[a,b]$ is the interval containing $\beta_0$ from the existence part. Then, due to the continuous dependence on $\mu$, Corollary 4.2, for every $\sigma > 0$ there exists $\mu_0 > 0$ such that $\|\bar{x}_k - \bar{\beta}_k\| \leq \sigma$ for all $0 \leq k \leq m-1$ and $0 < \mu \leq \mu_0$, where $\bar{x}_k$ is the $k$th string of the periodic solution and $\bar{\beta}_k$ is the string with all elements equal to $\beta_k$. In particular, the interval $[a,b]$ and $\sigma$ can be chosen in such a way that

$$|f'(z_0)\cdot f'(z_1)\cdot \ldots \cdot f'(z_{m-1})| \leq \delta < 1$$
$$\text{for all } z_k \in f^k([a,b]),\ k = 0,1,\ldots,m-1.$$

(b) The difference equation (1.1) is equivalent to the following $(N+1)$-dimensional map:

$$F : (y_0, y_1, \ldots, y_N) \mapsto \left(\frac{\mu}{\mu+1}y_0 + \frac{1}{\mu+1}f(y_N) \right.$$
$$\left. + \frac{\alpha\mu}{\mu+1}(y_N - y_{N-1}), y_0, y_1, \ldots, y_{N-1}\right).$$

The periodic solution of Eq. (1.1), made up of the strings

$$\begin{aligned} \bar{x}_0 &:= \{x_0^0, x_1^0, \ldots, x_N^0\},\\ \bar{x}_1 &:= \{x_0^1, x_1^1, \ldots, x_N^1\},\\ &\vdots\\ \bar{x}_{m-1} &:= \{x_0^{m-1}, x_1^{m-1}, \ldots, x_N^{m-1}\}, \end{aligned}$$

corresponds to a cycle of period $(N+1)m$ of the map $F$.

The cycle is made up of the set of points $\{\bar{y}_i := F^i(\bar{x}_0), i = 0, \ldots, (N+1)m-1\}$. Its stability is determined by the location of the eigenvalues of the corresponding linearized map along the cycle:

$$A_\mu := \sqcap_{i=0}^{(N+1)m-1} \frac{\partial F}{\partial y}(\bar{y}_i),$$

where $\partial F/\partial y$ is the Jacobian of the map $F$.

One has:

$$\frac{\partial F}{\partial y} = \begin{pmatrix} \mu/\mu{+}1 & 0 & 0 & \ldots & -\alpha\mu/(\mu{+}1) & (1/(\mu{+}1)) \cdot f'(y_N){+}\alpha\mu/(\mu{+}1) \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & \ldots & & & \ldots & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Set $A_i^\mu := \partial F/\partial y|_{y=\bar{y}_i}$. Then $A_\mu = \sqcap_{i=0}^{(N+1)m-1} A_i^\mu$.

We shall use next the fact that eigenvalues of $A_\mu$ depend continuously on $\mu$ at $\mu = 0$. Let $\mu = 0$. Then

$$\left.\frac{\partial F}{\partial y}\right|_{\mu=0} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & f'(y_N) \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & \ldots & & & \ldots & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

From the matrix's characteristic equation $\lambda^n - f'(y_n) = 0$ it follows that any eigenvalue $\lambda$ satisfies the inequality $|\lambda| \leq |f'(y_N)|^{1/(N+1)}$. Therefore, since $\lambda_0$ in the hypothesis of Theorem 4.3 is less than 1, any eigenvalue $\lambda$ of matrix $A_0 = \sqcap_{i=0}^{(N+1)m-1} A_i^0$ satisfies the inequality

$$\begin{aligned} &|\lambda| \\ &\leq |f'(x_0^0) \cdot f'(x_1^0) \cdot \ldots \cdot f'(x_N^0) \cdot f'(x_0^1) \cdot \ldots \cdot f'(x_N^1) \cdot f'(x_0^2) \cdot \ldots \cdot f'(x_N^{m-1})| \\ &\leq \delta^{N+1} < 1. \end{aligned}$$

This completes the proof of the stability part.

### 4.4 Further Remarks, Conclusion, and Future Directions

1. We first note that the periodic solution of Theorem 4.3 is not a unique periodic solution of Eq. (1.1) that is close to a particular cycle $\beta = \{\beta_0, \ldots, \beta_{m-1}\}$. The nonuniqueness follows from a simple observation that in the proof of the existence part the initial string $\bar{x}_0 = \{x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0\}$ in general does not have to belong to a sufficiently small neighborhood of a single point of the cycle $\beta$. If, for example, parts of the initial string $\bar{x}_0$ are chosen to belong to sufficiently small neighborhoods of two different points of the cycle $\beta$ the corresponding solution will converge as $n \to \infty$ to a different periodic solution. To obtain the latter periodic solution one has to construct an appropriate set of initial strings, and to apply the Schauder fixed point theorem in essentially the same way as it was done in Theorem 4.2/Theorem 4.3. The periodic solution obtained is asymptotically stable with the asymptotic phase – the proof of this fact repeats the main points of the stability part in the proof of Theorem 4.3.

Now it is rather obvious that any combination of the elements of the initial string $\bar{x}_0$ belonging to the appropriate neighborhoods of two or more points of the cycle $\beta$ will result in, generally speaking, a different periodic solution. The periodic solution can be proved to be asymptotically stable with the asymptotic phase.

2. Our main results, Theorems 4.2 and 4.3, also remain valid for the difference equation (2.5). This fact follows from the analysis of the corresponding proofs – their basic steps remain the same.

We note that the same results as described by Theorems 4.2 and 4.3 can be obtained for a difference equation resulting from the backward Euler discretization of (1.3) at $\dot{x}(t)$, and *any* Euler discretization at $\dot{x}(t-1)$ (backward, forward, or symmetric) with the same discretization step.

3. Our third remark is about the asymptotic shape as $\mu \to 0+$ of the periodic solution in Theorem 4.3. As it can be easily seen from the construction of the mapping $F$ (see proof of Theorems 4.2 and 4.3), and the continuous dependence on the parameter $\mu$ (Theorem 4.1) the periodic solution of Theorem 4.3 converges as $\mu \to 0+$ to a periodic function $\bar{\beta}$ defined by the consecutive sequence of the strings $\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_{m-1}$, that is, $\bar{\beta} := \{\underbrace{\beta_0, \ldots, \beta_0}_{N+1}, \underbrace{\beta_1, \ldots, \beta_1}_{N+1}, \ldots, \underbrace{\beta_{m-1}, \ldots, \beta_{m-1}}_{N+1}\}$. The function

$\bar{\beta}$ is a periodic solution of the limiting ($\mu = 0$) Eq. (3.1). Likewise, a corresponding convergence takes place for any other periodic solution described in Remark 1 of this subsection.

A principal implication of the results of this paper is that the dynamics of singularly perturbed delay differential equations of the form (1.3) and (1.4) are generally speaking essentially different from those determined by their Euler discretizations, the corresponding difference equations (1.1). This in turn implies that in some cases the results of numerical simulations of Eqs. (1.3) and (1.4) may be irrelevant to the actual dynamics in the differential equations. This is true at least for the Euler numerical schemes. An important question of interest here is whether this phenomenon persists for other standard numerical schemes of solving delay differential equations.

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