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On a High Order Differential Delay Equation

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1. INTRODUCTION

Consider the differential delay equation

$$\left(\varepsilon_m \frac{d}{dt}+1\right)\cdots\left(\varepsilon_0 \frac{d}{dt}+1\right) y(t)=f(y(t-1)), \tag{1.1}$$

where $f\in C(\mathbb{R}, \mathbb{R})$ and $\varepsilon=(\varepsilon_0, ..., \varepsilon_m)\in \mathbb{R}_+^{m+1}=(0, \infty)^{m+1}$. Equation (1.1) is equivalent to the system

$$\begin{aligned}\varepsilon_0\dot{x}_0(t)+x_0(t)&=x_1(t)\\ &\cdots\\ \varepsilon_{m-1}\dot{x}_{m-1}(t)+x_{m-1}(t)&=x_m(t)\\ \varepsilon_m\dot{x}_m(t)+x_m(t)&=f(x_0(t-1)).\end{aligned} \tag{1.2}$$

Our objective is to give some conditions on the nonlinear function f which will either ensure the stability of an equilibrium solution of (1.1) or the existence of a slowly oscillating periodic solution of (1.1).

To describe the results, we need some notation. Let $x=\mathrm{col}(x_0, x_1, ..., x_m)=\mathrm{col}(y, z)\in\mathbb{R}^{m+1}$, where $z=\mathrm{col}(x_1, ..., x_m)\in\mathbb{R}^m$. If we define $X=C([-1,0], \mathbb{R})\times\mathbb{R}^m$, then, for any $\psi=(\varphi, \xi)\in X$, there is a unique solution $x(t)=x(t, \varepsilon, \psi)=\mathrm{col}(y(t), z(t))$ which exists for all $t\geq 0$ and satisfies the initial condition $y(\theta)=\varphi(\theta)$, $\theta\in[-1,0]$, $z(0)=z_0$. If we define $y_t(\theta)=y(t+\theta)$, $\theta\in[-1,0]$, and

$$T_\varepsilon(t)\psi=\mathrm{col}(y_t, z(t)), \tag{1.3}$$

then $T_\varepsilon(t)\colon X\to X$, $t\geq 0$, is a C^0-semigroup.

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For any interval $I \in \mathbb{R}$ (closed or open), let $X_I = C([-1, 0], I) \times I^m$. Our first result is

THEOREM 1.1 (Positive Invariance). *If I is an interval such that $f(I) \subset I$, then $T_\varepsilon(t) X_I \subset X_I$ for $t \geq 0$.*

If x_0 is a fixed point of f, then $x_0^* = \mathrm{col}(x_0, ..., x_0) \in X$ is an equilibrium point of (1.2) and conversely. If x_0 is an attracting fixed point of f, we say that an interval J is the *maximal interval of attraction* of x_0 if $x_0 \in J$, $f(J) \subset J$, $f^n(x) \to x_0$ as $n \to \infty$ for each $x \in J$ and there is no interval $J' \supset J$ with this property. We remark that the maximal interval of attraction is open.

THEOREM 1.2 (Stability). *If x_0 is an attracting fixed point of f with maximal interval of attraction J, then the equilibrium solution x_0^* of (1.2) is asymptotically stable and, for each $\psi \in X_J$ and every $\varepsilon \in \mathbb{R}_+^{m+1}$, we have*

$$\lim_{t \to \infty} T_\varepsilon(t)\psi = x_0^*.$$

Theorems 1.1 and 1.2 are extensions of results of Ivanov and Sharkovsky [4] for the scalar case $m = 0$.

We say that a continuous scalar function u: $[t_0, \infty) \to \mathbb{R}$ *oscillates* if it has arbitrarily large zeros. We say that u *oscillates with respect to a constant function* u_0 if $u - u_0$ oscillates. A continuous vector function x: $[t_0, \infty) :\to \mathbb{R}^{m+1}$ is said to be *slowly oscillating* if each component of x oscillates and the distance between zeros is greater than 1.

THEOREM 1.3 (Existence of a Slowly Oscillating Periodic Solution). *Suppose that I is a bounded interval such that $f(I) \subset I$, $x_0 \in I$ is a fixed point of f with $f'(x_0) < -1$, and $(y - x_0)[f(y) - x_0] < 0$ for $y \neq x_0$ (negative feedback). Then there exists $\delta > 0$ such that, for each $\varepsilon \in (o, \delta)^{m+1}$, system (1.2) has a slowly oscillating periodic solution.*

For $m = 0$, Theorem 1.3 has been given by Hadeler and Tomiuk [2] and, for $m = 1$, by an der Heiden [1] without restrictions on the parameters ε except those, of course, which imply that the origin is unstable. We prove the result for arbitrary m, but require that $|\varepsilon|$ is small.

We remark that the above results hold true if we consider the equation

$$\left(\tilde{\varepsilon}_m \frac{d}{dt} + \alpha_m\right) \cdots \left(\tilde{\varepsilon}_0 \frac{d}{dt} + \alpha_0\right) y(t) = f(y(t-1)),$$

where each $\tilde{\varepsilon}_i$, α_i is positive. It is simply a scaled version of (1.1) with $\varepsilon_i = \tilde{\varepsilon}_i / \alpha_i$ and f replaced by $f/\alpha_0 \ldots \alpha_m$.

2. Proof of Theorem 1.1

We need the following auxiliary result.

Lemma 2.1. *Suppose that I is an interval (open or closed) and a: $[0, \infty) \to \mathbb{R}$ is a continuous function with values in I. If $\sigma > 0$ is a constant and $u(t)$, $t \geqslant 0$, is a solution of the equation*

$$\sigma \dot{u}(t) + u(t) = a(t) \tag{2.1}$$

satisfying $u(0) \in I$, then $u(t) \in I$ for $t \geqslant 0$.

Proof. Let $\bar{I} = [\alpha, \beta]$ and suppose that the conclusion of the lemma is not true. Then there is a first time $t_0 \geqslant 0$ at which the solution leaves I. To be specific, suppose that $u(t_0) = \beta$ and every interval $(t_0, t_0 + \delta)$, $\delta > 0$, contains a point τ such that $u(\tau) > \beta$. This interval also must contain a point s such that $u(s) > \beta$ and $\dot{u}(s) > 0$. On the other hand, it follows from (2.1) that $\dot{u}(s) < 0$, which is a contradiction. The case $u(t_0) = \alpha$ is discussed in a similar way to complete the proof.

To prove Theorem 1.1, we use the assumptions that $f(I) \subset I$ and $\psi \in X_I$ together with Lemma 2.1 to observe that $x_m(t) \in I$ for $0 \leqslant t \leqslant 1$. Using this fact and the mth equation in (1.2), we see that $x_{m-1}(t) \in I$ for $0 \leqslant t \leqslant 1$. Proceeding in this way, we observe that $x_i(t) \in I$ for $0 \leqslant t \leqslant 1$, $i = 0, 1, ..., m$. This implies that $T_\varepsilon(t)\psi \in X_I$ for $0 \leqslant t \leqslant 1$. The proof is completed by an induction argument.

3. Proof of Theorem 1.2

We need the following auxiliary result.

Lemma 3.1. *Suppose that K, L are intervals in $\mathbb{R}$ with $K \subset L$ and consider Eq. (2.1) with $a(t) \in K$ for $t \geqslant 0$. Let L_1 be any interval satisfying $K \subset L_1 \subset L$ and $L \neq L_1 \neq K$ if such an interval exists. Otherwise, let $L_1 = K$. If $u(t)$ is the solution of (2.1) with $u(0) = u_0 \in L$, then, there is a time $t_0 = t_0(u_0, L_1)$ such that $u(t) \in L_1$ for all $t \geqslant t_0$.*

Proof. If there is a time t_0 such that $u(t_0) \in K$, then Lemma 2.1 implies that $u(t) \in K$ for $t \geqslant t_0$ and Lemma 3.1 is proved.

Therefore, we may assume that $u(t) \notin K$ for all $t \geqslant 0$. To be specific, let us assume that $u(t) > \sup K = \sup\{b \in K\}$ for $t \geqslant 0$. Then (2.1) implies that $\dot{u}(t) < 0$ for all $t \geqslant 0$ and, thus, $u(t) \to u_*$ as $t \to \infty$. If $u_* = \sup K$, the lemma is proved. If $u_* > \sup K$, then $\dot{u}(t) = [-u(t) + a(t)]/\sigma \leqslant [\sup K - u_*]/\sigma < 0$ for sufficiently large t. This implies that $u(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $u(t) > \sup K$ for $t \geq 0$. The case where $u(t) < \inf K$ is treated in a similar way.

We now turn to the proof of Theorem 1.2. Fix ε. For any $\psi \in X_J$, we have $T_\varepsilon(t)\psi \in X_J$ from Theorem 1.1.

Let $\alpha = \min\{\min_{-1 \leq s \leq 0} \varphi(s), x_1^0, ..., x_m^0\}$, $\beta = \max\{\max_{-1 \leq s \leq 0} \varphi(s), x_1^0, ..., x_m^0\}$, let L be the minimal interval containing $[\alpha, \beta]$ such that $f(L) \subset L$, and define $K = f(L)$.

Our first objective is to prove the following *Claim*: For any $\delta > 0$, there is a set $L_1 \subset [\inf K - \delta, \sup K + \delta]$ satisfying the conditions of Lemma 3.1 and a time t_0 such that $T_\varepsilon(t)\psi \in X_{L_1}$ for $t \geq t_0$.

Let $\tilde{L}_1$ be any interval satisfying the conditions of L_1 in Lemma 3.1 and consider the $(m+1)$st equation of system (1.2). From Lemma 3.1, there exists a $t_m^0 \geq 0$ such that $x_m(t) \in \tilde{L}_1$ for all $t \geq t_m^0$. Now redefine $K = \tilde{L}_1$ and choose $\tilde{L}_2$ with $L \supset \tilde{L}_2 \supset K$ satisfying the conditions of L_1 in Lemma 3.1. If we consider the mth equation of system (1.2), then Lemma 3.1 implies that there exists $t_{m-1}^0 \geq 0$ such that $x_{m-1}(t) \in \tilde{L}_2$ for all $t \geq t_{m-1}^0$. We continue in this way through the first equation of system (1.2) to obtain $x_0(t) \in \tilde{L}_{m+1}$ for all $t \geq t_0^0$. We can obviously choose the intervals $\tilde{L}_j$ so that $\tilde{L}_1 \subset \tilde{L}_2 \subset \cdots \subset \tilde{L}_{m+1} \subset [\inf K - \delta, \sup K + \delta]$ for any fixed $\delta > 0$. If we let $t_0 = \max\{t_0^0, ..., t_m^0\}$, then we have proved the claim.

With the initial L as above, the attractivity of the fixed point x_0 of f implies that $L \supset f(L) \supset f^2(L) \supset \cdots$ and $\bigcap_{n \geq 0} f^n(L) = \{x_0\}$. We may repeat the above argument with L, K replaced by $f^k(L)$, $f^{k+1}(L)$ to obtain an interval $L_k \subset f^k(L)$ and $T_\varepsilon(t)\psi \in X_{L_k}$ for all $t \geq t_k$. This obviously completes the proof.

4. Proof of Theorem 1.3

The proof of Theorem 1.3 follows the standard procedure of obtaining a mapping of a cone into itself with an ejective fixed point with some special modifications due to Hadeler and Tomiuk [2]. We recall these concepts.

Definition 4.1. Suppose that X is a Banach space, U is a subset of X, and x is a given point in U. Given a map $A: U \setminus \{x\} \to X$, the point $x \in U$ is said to be an *ejective point* of A if there is an open neighborhood $G \subset X$ of x such that, for every $y \in G \cap U$, $y \neq x$, there is an integer $m = m(y)$ such that $A^m y \notin G \cap U$.

We need also the following theorem of Nussbaum [5] (see also [3]).

Theorem 4.2. *If K is a closed, bounded, convex infinite-dimensional set in X, $A: K \setminus \{x_*\} \to K$ is completely continuous, and $x_* \in K$ is an ejective point of A, then there is a fixed point of A in $K \setminus \{x_*\}$.*

We first construct the set K. Let I be an interval such that $f(I) \subset I$, let $x_0 \in I$ be a fixed point of f with $f'(x_0) < -1$, and f satisfies the negative feedback condition: $(x - x_0)[f(x) - x_0] < 0$ for $x \neq x_0$. Without loss of generality, we may assume that $x_0 = 0$. We now define

$$K = \{\psi \in X_I : \varphi(-1) = 0, \varphi(s) > 0, \varphi(s) \in I, \varphi(s)\, e^{s/\varepsilon_0} \text{ nondecreasing for } s \in [-1, 0], x_{i0} \geqslant 0, i = 1, 2, ..., m\}.$$

LEMMA 4.3. *Let $\psi \in K$ be arbitrary and suppose that the solution $T_\varepsilon(t)\psi$ of* (1.2) *has the property that its first component $x_0(t)$ oscillates. Then*

(i) *all zeros of $x_0(t)$ are simple and distances between successive zeros are larger than* 1;

(ii) *between each two successive zeros α and β, $\alpha < \beta$, of $x_0(t)$, there exists only one zero γ^i of $x_i(t)$, $i = 1, 2, ..., m$, and $\gamma^i - \alpha > 1$ and $\gamma^m < \gamma^{m-1} < \cdots < \gamma^1$;*

(iii) *if α is a zero of $x_0(t)$, then $|x_0(t) \exp(t/\varepsilon_0)|$ is nondecreasing for $t \in [\alpha, \alpha + 1]$.*

Proof. Let $T > 0$ be fixed and consider Eq. (2.1). If $u(0) > 0$ and $a(t) \geqslant 0$ for $t \in [0, T]$, then it is obvious that $u(t) > 0$ for all $t \in [0, T]$. From the form of system (1.2) and the assumption that $x_0(t)$ oscillates, it follows that each $x_i(t)$, $i = 1, ..., m$, must have a zero. Furthermore, if $t_1^0 > 0$ is the first zero of $x_0(t)$, then it is easy to see that there exist $0 \leqslant t_1^m < \cdots < t_1^1 < t_1^0$ such that $x_i(t_1^i) = 0$, $i = 1, 0, 1, ..., m$, and these are the first such zeros.

Since $f(x_0(t-1)) < 0$ for $t \in (-1, t_1^0)$, it follows that t_1^m is a simple zero of $x_m(t)$ and $x_m(t) < 0$ for $t \in (t_1^m, t_1^0)$. Using the same type of reasoning, we see that $x_i(t) < 0$ for $t \in (t_1^i, t_1^0 + 1)$ for each $i = 0, 1, ..., m$ and each of the zeros is simple. Since $x_1(t) < 0$ for $t \in (t_1^0, t_1^0 + 1)$, the first equation of system (1.2) implies that $(d/dt)[x_0(t) \exp(t/\varepsilon_0)] < 0$, which implies that the function $|x_0(t) \exp(t/\varepsilon_0)|$ is nondecreasing on $[t_1^0, t_1^0 + 1]$. This completes the proof for the first two successive zeros of $x_0(t)$. It is clear that the other situations can be treated in the same way.

LEMMA 4.4. *If $f'(0) < 0$ and $yf(y) < 0$ for $y \neq 0$, then there exists a $\delta > 0$ such that, for every ε satisfying $0 < \varepsilon_i < \delta$, $i = 0, 1, ..., m$, all solutions of Eq.* (1.1) *oscillate.*

Proof. Suppose that $x(t) = \text{col}(x_0(t), x_1(t), ..., x_m(t))$ is a nonoscillatory solution of system (1.2). Then there must be one component of $x(t)$, say $x_k(t)$, that is nonoscillatory.

Claim 1. There is a t_0 such that every component $x_i(t)$ of $x(t)$ has a fixed sign for $t \geqslant t_0$.

To prove this, we make the following observation: If $v(t)$ is a given function of constant sign for large t and $u(t)$ is a nonzero solution of $\delta \dot{u}(t) + u(t) = v(t)$, $\delta > 0$, then there is a t_0 such that $u(t)$ is of constant sign for $t > t_0$. In fact, if $v(t)$ is of constant sign for $t > t_0$ and either $u(t_0)\, v(t_0) > 0$ or $u(t_0) = 0$, then the variation of constants formula implies that $u(t)\, v(t) > 0$ for $t > t_0$. If $u(t_0)\, v(t_0) < 0$, then $u(t)$ is strictly monotone near t_0 and, therefore, either $u(t)\, v(t) < 0$ for $t > t_0$ or $u(t_0)\, v(t_0)$ eventually becomes positive and remains so.

We can now apply this remark to (1.2) to see that $x_{k-1}(t)$ is nonoscillatory. Proceeding in this way, we observe that $x_0(t)$ is nonoscillatory. From the $(m+1)$st equation in (1.2), it follows that $x_m(t)$ is nonoscillatory. The proof that the other components of $x(t)$ are nonoscillatory is completed using the same argument as before. This completes the proof of the claim.

From Claim 1, we know that every component of $x(t)$ is nonoscillatory. For definiteness, suppose that $x_0(t) > 0$ for $t \in (t_0, \infty)$ (the case where $x_0(t) < 0$ is analogous). Let $x_k(t)$, $k \geqslant 0$, be the last component of $x(t)$ which is positive for large t. Therefore, $x_j(t) < 0$ for $j > k$.

Claim 2. Every component $x_j(t)$ of $x(t)$ approaches 0 as $t \to \infty$ and there is a t_0 such that each $x_j(t)$ is strictly monotone for $t \geqslant t_0$.

To prove this, let us first suppose that $k = m$. From (1.2) and the fact that $f(x_0(t-1)) < 0$ for $t > t_0 + 1$, it follows that $x_m(t)$ is a strictly monotone decreasing function. This implies that there is a constant ξ such that $x_m(t) \to \xi$ as $t \to \infty$. From (1.2), this implies that all $x_j(t) \to \xi$ as $t \to \infty$ for all $0 \leqslant j \leqslant m$. On the other hand, using the last equation in (1.2), we see that $x_m(t) \to f(\xi)$ as $t \to \infty$. Thus, $f(\xi) = \xi$, which implies that $\xi = 0$.

Since $x_m(t)$ is strictly monotone for large t, we see that $x_{m-1}(t)$ is strictly monotone from the following fact: if the function $v(t)$ has fixed sign and is strictly monotone, then every solution of the equation $\delta \dot{u}(t) + u(t) = v(t)$ is eventually strictly monotone. If this statement were not true, then we would have two distinct points $t_1 > t_0$ such that $u(t_1) = v(t_1)$, $u(t_0) = v(t_0)$. For definiteness, suppose that $v(t) > 0$ and $v(t) < v(s)$ for all $t > s$. Using the variation of constants formula, we have

$$
\begin{aligned}
e^{\delta^{-1}(t_1 - t_0)} v(t_1) &= v(t_0) + \frac{1}{\delta} \int_{t_0}^{t_1} e^{\delta^{-1}(s - t_0)} v(s)\, ds \\
&> v(t_0) + v(t_1)\left[e^{\delta^{-1}(t_1 - t_0)} - 1\right],
\end{aligned}
$$

and this last relation leads to the contradiction $v(t_1) > v(t_0)$.

We can now use the same argument to see that each component of $x(t)$ is eventually strictly monotone.

If $k < m$, then $x_k(t)$ is eventually strictly monotone since $x_k(t)\, x_{k+1}(t) < 0$ for large t. Now we can use the same argument as in the previous case to see that $x_j(t)$ is eventually strictly monotone for $0 \leqslant j \leqslant k$. Since $x_0(t) \to 0$ as $t \to \infty$ and $f'(0) < 0$, it follows that the function $f(x_0(t-1))$ is eventually strictly monotone. Therefore, we will have that $x_m(t)$ is eventually strictly monotone. Now we can continue the argument to complete the proof of Claim 2.

Claim 3. If $x_i(t)$ and $x_{i+1}(t)$ eventually have the same sign, then there is a t_0 such that $|x_i(t)| > |x_{i+1}(t)|$ for $t \geqslant t_0$.

To be specific, suppose that $x_i(t) > 0$, $x_{i+1}(t) > 0$, and that both are strictly monotone decreasing for $t \geqslant t_1$. Then the variation of constants formula for $x_i(t)$ implies that

$$\begin{aligned} x_i(t) &= x_i(t_1)\, e^{-\varepsilon_i^{-1}(t-t_1)} + \frac{1}{\varepsilon_i} \int_{t_1}^{t} e^{-\varepsilon_i^{-1}(t-s)} x_{i+1}(s)\, ds \\ &\geqslant x_i(t_1)\, e^{-\varepsilon_i^{-1}(t-t_1)} + \frac{x_{i+1}(t)}{\varepsilon_i} \int_{t_1}^{t} e^{-\varepsilon_i^{-1}(t-s)}\, ds \\ &= x_{i+1}(t) + [x_i(t_1) - x_{i+1}(t)]\, e^{-\varepsilon_i^{-1}(t-t_1)} \\ &> x_{i+1}(t) \end{aligned}$$

for large t since $x_{i+1}(t) \to 0$ as $t \to \infty$ and $x_i(t_1) > 0$. This completes the proof of Claim 3.

Claim 3 implies that $x_0(t) > \ldots > x_k(t) > 0 > x_m(t) > \ldots > x_{k+1}(t)$ for all $t \geqslant t_1$ with t_1 sufficiently large. Therefore,

$$\begin{aligned} \varepsilon_k \dot{x}_k(t) + x_k(t) = x_{k+1}(t) < x_m(t) &= f(x_0(t-1)) - \varepsilon_m \dot{x}_m(t) \\ &< f(x_0(t-1)) < f(x_{k+1}(t-1)). \end{aligned}$$

Therefore, there is an $\eta > 0$ and a t_2 such that $a = f'(0) + \eta < 0$ and, for $t \geqslant t_2$, we have

$$\varepsilon_k \dot{x}_k(t) + x_k(t) \leqslant a x_k(t-1).$$

Using the variation of constants formula on the interval $[t, t+1]$, we have

$$x_k(t+1) \leqslant x_k(t)\, e^{-1/\varepsilon_k} + \frac{1}{\varepsilon_k} a \int_{t-1}^{t} e^{(1/\varepsilon_k)(s-t)} x_k(s)\, ds,$$

which clearly contradicts the fact that $x_k(t) > 0$ if ε_k is small enough. This completes the proof of Lemma 4.4.

Our next objective is to define a map $A\colon K \to K$. From Lemma 4.4, we know that $x_0(t)$ oscillates. From Lemma 4.3, we have $-T_\varepsilon(t_1^0+1)\psi \in K$. If t_2^0 is the second zero of $x_0(t)$, then $T_\varepsilon(t_2^0+1)\psi \in K$. Therefore, we define

$$A\psi = T_\varepsilon(t_2^0+1)\psi \qquad \text{for} \quad \psi \in K. \tag{4.1}$$

It is natural to extend the definition of A to the closure $\bar{K}$ of K by setting $A0=0$.

The set $\bar{K}$ is closed bounded and convex. Following the same proof as in Hadeler and Tomiuk [2], we see that A is a completely continuous map on $\bar{K}$. As a first step in showing that 0 is an ejective fixed point of A, we need specific information about the characteristic equation

$$(\varepsilon_m\lambda+1)\cdots(\varepsilon_0\lambda+1) - ae^{-\lambda} = 0, \qquad a = f'(0), \tag{4.2}$$

corresponding to the linear variational equation of (1.1) about the origin:

$$\left(\varepsilon_m \frac{d}{dt}+1\right)\cdots\left(\varepsilon_0 \frac{d}{dt}+1\right) y(t) = ay(t-1). \tag{4.3}$$

LEMMA 4.5. *There exists a* $\delta>0$ *such that, for every* ε *with* $0<\varepsilon_i\leqslant\delta$, $i=0, 1, \ldots, m$, *the characteristic equation* (4.2) *has a solution* $\lambda_0=\mu+i\nu$ *with* $\mu>0$, $o<\nu<\pi$.

Proof. For $\varepsilon=0$, there is a solution of (4.2) with $\mu=\ln|a|$, $\nu=\pi$. This solution also is simple. By Rouché's Theorem, there exists a $\delta>0$ and a neighborhood U of $\mu+i\nu$ in the complex plane such that, for $0<\varepsilon_i\leqslant\delta$, $i=0, 1, \ldots, m$, there is a unique solution $\mu(\varepsilon)+i\nu(\varepsilon)$ of (4.2) in U. To show that $\nu(\varepsilon)<\pi$ for δ sufficiently small, it is enough to show that $(\partial\nu/\partial\varepsilon_i)(0)<0$ for all i. Setting $\lambda=\mu(\varepsilon)+i\nu(\varepsilon)$ in (4.2), differentiating with respect to ε_i, and setting $\varepsilon=0$, we deduce that the imaginary part of the resulting expression satisfies

$$\nu(0) + ae^{-\mu(0)}(\cos\nu(0))\frac{\partial\nu}{\partial\varepsilon_i}(0) = 0.$$

Since $ae^{-\mu(0)}(\cos\nu(0))=1$, this implies that $(\partial\nu/\partial\varepsilon_i)(0) = -\pi<0$, $i=0, 1, \ldots, m$. This completes the proof.

LEMMA 4.6. *There exists a* $\delta>0$ *such that, for every* ε *with* $0<\varepsilon_i\leqslant\delta$, $i=0, 1, \ldots, m$, *the fixed point* 0 *of* A *is ejective.*

Proof. To prove the ejectivity of the fixed point 0 of A, we plan to use some general results from [3, Chap. 11]. To do this, we need some specific but elementary facts about the linear equation (4.3).

We may write (4.3) as a system of equations

$$\dot{x}(t) = Cx(t) + Dx(t-1), \tag{4.4}$$

where

$$C = \begin{bmatrix} -\frac{1}{\varepsilon_0} & \frac{1}{\varepsilon_0} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -\frac{1}{\varepsilon_{m-1}} & \frac{1}{\varepsilon_{m-1}} \\ 0 & 0 & \cdots & 0 & -\frac{1}{\varepsilon_m} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{a}{\varepsilon_m} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The equation adjoint to (4.4) (see [3]) is

$$\frac{d}{ds} w(s) = -w(s)\, C - w(s+1) D, \tag{4.5}$$

where $w \in (\mathbb{R}^{m+1})^*$ is an $(m+1)$-dimensional row vector. From the form of the matrix D, the initial data for (4.5) is taken in the space $X^* = (\mathbb{R}^m)^* \times C([0, 1], \mathbb{R})$.

For any $\psi^* = (\eta, \zeta) \in X^*$ and any $\psi = (\varphi, \xi) \in X$, there is an associated bilinear form

$$(\psi^*, \psi) = \psi^*(0)\, \psi(0) + \int_0^1 \zeta(\alpha)(a/\varepsilon_m)\, \varphi(\alpha - 1)\, d\alpha. \tag{4.6}$$

Let λ_0 be the solution of (4.2) guaranteed by Lemma 4.5. A few computations show that a corresponding eigenfunction of (4.5) is given by

$$\psi^*_{\lambda_0}(s) = e^{-\lambda_0 s}[\varepsilon_0(\varepsilon_1 \lambda_0 + 1) \cdots (\varepsilon_m \lambda_0 + 1), \varepsilon_1(\varepsilon_2 \lambda_0 + 1) \cdots (\varepsilon_m \lambda_0 + 1), ..., \varepsilon_m].$$

The results in [3, Lemma 4.4, Chap. 11] imply that the fixed point 0 of A is ejective if

$$\inf\{ |(\psi^*, \psi)| : \psi \in \bar{K}, |\psi| = 1 \} > 0, \tag{4.7}$$

where, for $\psi = (\varphi, \xi)$, $\xi = \operatorname{col}(x_1, ..., x_m)$, we define

$$|\psi| = \max\{ \max_{1 \leqslant s \leqslant 0} |\varphi(s)|, |x_1|, ..., |x_m| \}.$$

For any $\psi \in \bar{K}$, we let $(\psi^*_{\lambda_0}, \psi) = R(\psi) + iI(\psi)$, where $R(\psi)$ and $I(\psi)$ are real. Using (4.6) and the expression for $\psi^*_{\lambda_0}$, we see that (we have put $\varphi(0) = x_0$)

$$I(\psi) = [\varepsilon_0(\varepsilon_1 + \ldots + \varepsilon_m)\, x_0 + \varepsilon_1(\varepsilon_2 + \ldots + \varepsilon_m)\, x_1 + \ldots + \varepsilon_{m-1}\varepsilon_m x_{m-1}]\, \nu u(0)$$

$$+ a \int_0^1 e^{\mu(0)\alpha} \sin(\nu(0)\alpha)\, \varphi(\alpha - 1)\, d\alpha + O(|\varepsilon|).$$

We claim next that there exists an index $i \in \{0, 1, ..., m\}$ such that $x_i \geqslant \exp\{-1/\varepsilon_i\}$. In fact, since $|\psi| = 1$, it follows that either $x_i = 1$ for some $i \in \{1, 2, ..., m\}$ or $\max_{-1 \leqslant s \leqslant 0} |\varphi(s)| = 1$. In the first case, the claim is obvious. In the second one, since $\varphi(s) \exp\{s/\varepsilon_0\}$ is increasing, we have $x_0 = \varphi(0) \geqslant \max_{-1 \leqslant s \leqslant 0} \varphi(s) \exp\{-1/\varepsilon_0\}$. This proves the claim.

Finally, since the kernel in the integral term of $I(\psi)$ is positive, we have $I(\psi) \geqslant c = c(\varepsilon) > 0$, for $|\varepsilon| \leqslant \delta$ and $|\psi| = 1$. This obviously implies that (4.7) is satisfied and completes the proof of the lemma.

Theorem 1.3 is now a consequence of Theorem 4.2.

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