C. K. R. T. Jones U. Kirchgraber H. O. Walther
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With Contributions of
R. Bielawski, H. W. Broer, R. H. Cushman, L. Gorniewicz,
G. Iooss, A. F. Ivanov, M. Levi, S. Plaskacz, A. N. Sharkovsky,
A. Vanderbauwhede, G. Vegter

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Oscillations in Singularly Perturbed Delay Equations

A.F. Ivanov, A.N. Sharkovsky

Introduction

This paper presents some recent results on the scalar singularly perturbed differentila delay equation

$$\nu \dot{x}(t) + x(t) = f(x(t-1)) \,. \tag{1}$$

The equation is of significant interest for both mathematicians dealing with qualitative theory of differential equations and for scientists applying the theory to real world problems in different areas.

Equations of form (1) have recently found a variety of applications in several fields of natural science. For instance, they model processes in radiophysics and optics (oscillations in linear arrays of tunnel diodes, high frequency generators, electro-optical bistable devices). In mathematical biology and physiology they describe systems with delayed response, particularly, the regulation of red blood cell populations, respiratory control circuits, neural interactions, etc. They were suggested to model commodity cycles in economics. Interested readers may find descriptions of applications or further references, for example in [6, 10–11, 14, 16–17, 19–25, 36, 40–42, 45, 48, 59]. Certain non-linear boundary value problems for hyperbolic equations are reducible to functional-differential equations which include equation (1). Such a procedure of reduction was proposed in 1936 by A.A. Vitt. Studies of nonlinear boundary value problems based on the mentioned reduction can be found in the monograph [55].

In spite of the simple form of equation (1) its dynamics is very rich and varifold. Apparently, the equation is one the most studied in the theory of differential functional equations, as recent results of many authors show (see list of references). These results show as well that we are rather far from a complete understanding of the dynamics given by equation (1). There is a series of natural and simple stated problems, which are still unsolved.

One efficient approach to equation (1) is to begin with step functions $f(x)$. Indeed, with $f(x)$ being finite valued any initial condition generates a piecewise exponential solution. For certain families of initial conditions the problem of the asymptotic behavior of solutions is reducible to the study of maps on finite dimensional sets, in particular, to one-dimensional maps in the simplest cases. This approach was used in a series of papers, including [1–3, 18–23, 36, 40, 48, 61] with step functions or smoothed step functions $f(x)$. Some of these papers deal with equations of the form $\dot{x}(t) = f(x(t-1))$. The idea is to determine a set Φ of initial conditions consisting of functions of special type characterized by a real parameter $z \in \Lambda$. For any $z \in \Lambda$ and $\varphi \in \Phi$ given by z there exists $t_0 = t_0(\varphi)$ such that the segment of the corresponding solution $x(t_0+t)$, $t \in [-1, 0]$, is again an element of Φ which belongs to some $\tilde{z} \in \Lambda$. Thus, one has an induced one-dimensional map $F : z \to \tilde{z}$ on the parameter set Λ. It turns out that the map F is a continuous and piecewise Moebius transformation. Therefore the dynamics given by equation (1) (on a subset of solutions) is as complicated as the dynamics of the corresponding piecewise Moebius map. In general, interval maps given by piecewise Moebius transformations may define complicated dynamical systems (see, e.g. [34]). In particular, for the solutions of the differential-difference equations of the form (1) and its modifications the authors of the papers [22, 23, 61] obtained an induced map $F_0(z)$ of the interval $[a, b]$ as shown in Fig. 1.

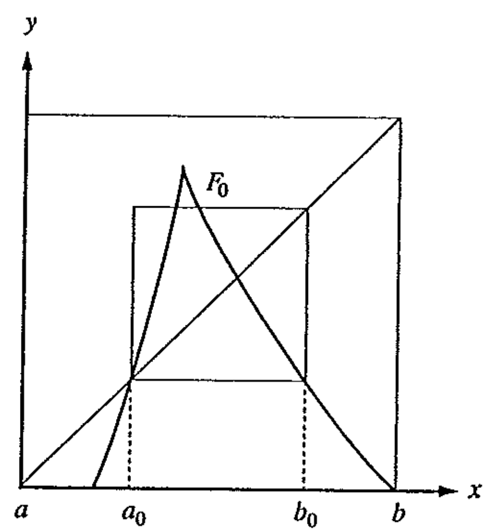

Fig. 1.

The map F_0 has a closed invariant repelling set $K_0 \subset [a_0, b_0]$ which is homeomorphic to a Cantor set. The dynamics of F_0 on the set K_0 is rather complicated (the description of the standard example is found in [55]). If $f(x)$ is smoothed in a small neighborhood of the discontinuity points the new induced map F_0' is proved to coincide with F_0 on the set K_0. Hence the dynamics of equation (1) is still complicated. As shown in [18] the described dynamics persists under small perturbations of step functions or smoothed step functions $f(x)$. In some cases the map F has an invariant measure which is absolutely continuous with respect to Lebesgue measure [3, 19].

The existence of such a measure implies in particular a probability distribution for successive maxima (or zeros) of the solutions generated by initial functions from the parameterized set Φ.

A natural approach to study equation (1) is to consider it as a singular perturbation of the difference equation with continuous argument

$$x(t) = f(x(t-1)), \quad t \in \mathbb{R}_+ . \tag{2}$$

To have a continuous solution of equation (2) for $t \geq -1$ one requires, together with continuity of f and φ, a so-called consistency condition $\lim_{t\to -0}\varphi(t) = f(\varphi(-1))$.

The asymptotic behavior of the solutions to equation (2) is determined by the dynamics of the one-dimensional map

$$f : x \to f(x) . \tag{3}$$

There exists a formal correspondence between trajectories $\{x_k = f^k(x_0),\ k = 1, 2, \cdots\}$ of the dynamical system given by equation (3) and step function solutions $x(t) = f^k(x_0)$, $t \in [k-1, k)]$, of equation (2), generated by initial functions $\varphi(t) = x_0$, $t \in [-1, 0)$. The asymptotic behavior of the solutions of equation (2) depends entirely on the properties of the iteration sequence $\{x_k\}$, $x_{k+1} = f(x_k)$, $x_0 \in \mathbb{R}$ where $\varphi(t_0) \stackrel{\text{def}}{=} x_0$ and therefore $x(k + t_0) = f^k(x_0)$, $k \in \mathbb{Z}_+$. Consequently, to see the dynamics of solutions of equation (2) one should follow the continuum of trajectories of the map (3) $\{x_k : x_k = f(x_{k-1})\}$ with $x_0 \in \{x = \varphi(t) : t \in [-1, 0)\}$.

At present there is a sufficiently complete theory of the continuous argument difference equation (2) based on the properties of the one-dimensional dynamical system (3) [55]. Generically, equation (2) has continuous solutions of three types: asymptotically constant solutions, relaxation type solutions and turbulent type solutions. Asymptotically constant solutions have a finite limit as $t \to +\infty$. They are determined by the attracting fixed points of the map (3). Relaxation type solutions are undamped solutions with constant oscillation frequency on each unit time interval $[k-1, k)$. The Lipschitz constants of such a solution on the intervals $[k-1, k)$ are unbounded as $k \to \infty$. Their existence is tied to a repelling fixed point of the map (3), separating domains of attraction of attracting fixed points or splitting the domain of attraction of an attracting 2-cycle. Turbulent type solutions are undamped solutions with unlimitedly increasing oscillation frequency and unbounded Lipschitz constants on $[k-1, k)$ as $k \to \infty$. Equation (2) has turbulent type solutions if the map (3) has periodic points with periods greater than 2. The existence of relaxation and turbulent type solutions indicates the complexity of the dynamics given by equation (2).

The natural question is to what extent the dynamics of the map (3) determines properties of the solutions for equation (1)?

Solutions of equation (1) appear to have some simple properties which are intrinsic to the dynamical system defined by the map (3). An example is the following property of invariance. If the map f has an invariant interval $I(f(I) \subseteq I)$ then any initial function $\varphi(t)$ with $\varphi([-1,0]) \subset I$ gives rise to a solution with values in I for all $t \geq 0$. This allows us to define a semiflow $\mathcal{F}^t, t \geq 0$ on $X = C^0([-1,0], I)$, as usual, by setting $(\mathcal{F}^t\varphi)(s) = x\varphi^\nu(t+s)$, $s \in [-1,0]$. Moreover, if all points $x \in I$ are attracted under f by a fixed point $x_* \in I$ then any initial function φ satisfying $\varphi([-1,0]) \subseteq I$ generates a solution x for which $\lim_{t\to\infty} x(t) = x_*$. In other words, there is a correspondence between attracting fixed points of the map (3) and constant solutions of equation (1) which are asymptotically stable. This shows that asymptotically constant solutions persist under singular perturbations of equation (2).

What happens to relaxation and turbulent solutions when equation (2) is singularly perturbed? Clearly, the Lipschitz constant for the solutions of equation (1) can not grow infinitely. Indeed, if I is an f-invariant interval then for any $\nu > 0$ and $\varphi(t) \in I$, $t \in [-1,0]$ the derivative of the corresponding solution is bounded: $|\dot{x}(t)| \leq (1/\nu)|-x(t) + f(x(t-1))| \leq (1/\nu)\,\mathrm{diam}\, I$. Therefore, the oscillation frequency of solutions with amplitudes bounded away from zero can not increase infinitely.

With ν small one naturally expects a closeness (in a sense to be made precise) between solutions of equations (1) and (2). This happens to be true within finite time intervals provided the nonlinearity $f(x)$ and the initial conditions considered are continuous. In particular, if $f(x)$ is continuously differentiable and the (same) initial condition for equations (1) and (2) is also continuously differentiable then the corresponding solutions remain close within a time interval of length $O(1/\nu), \nu \to +0$. This means that the solutions of equations (1) are as complicated as the solutions of equation (2) in this time interval.

The question how the asymptotic properties for equations (1) and (2) are related when ν is small is much more difficult.

Suppose the interval map f has an attracting cycle $x_1 \to x_2 \to \cdots \to x_n \to x_1$ with components U_i, $i = 1, 2, \cdots, n$ of its domain of immediate attraction. Clearly, an initial condition with values in say U_1, for $t \in [-1,0)$ defines a solution of equation (2) which is necessarily discontinuous at $t = i$, $i \in \mathbb{Z}_+$. This solution converges as $t \to +\infty$ to a step function solution of equation (2) defined by the initial condition $\varphi(t) = x_1$, $t \in [-1,0)$, and the convergence is uniform on time intervals $[k,\infty)$ as $k \to \infty$.

Because of the closeness results mentioned above an important problem is to find conditions for the map f which guarantee that there is a correspondence between the attracting cycle of the map f and an asymptotically stable periodic solution of equation (1), with period close to n, for small $\nu > 0$.

It appears that, in general, there is no such correspondence. This is shown by the following simple example. Let $f_0(x)$ be a smooth function mapping the interval $[-1,1]$ into itself such that $f_0(x) = 0$ for $|x| \le h, \frac{1}{2} < h < 1$, and f_0 has no fixed points outside the set $\{x \in \mathbb{R} : |x| \le h\}$. In addition $f_0(x)$ may be chosen in such a way that it has an attracting 2-cycle generating relaxation type solutions, and a cycle of period > 2 generating turbulent type solutions for equation (2). With $f_0(x)$ chosen in such a fashion, the difference equation (2) has continuous solutions of three possible types: asymptotically constant, relaxation, and turbulent ones. However, the asymptotic behavior of the solutions for equation (1) (with $f = f_0$) is very simple for any $\nu > 0$: all solutions satisfy $\lim_{t\to\infty} x(t) = 0$. This phenomenon of simplification for the asymptotic behavior of the solutions of equation (1) is caused by the existence of an attracting fixed point $x = 0$ of the map with "large" domain of immediate attraction.

Similar phenomena arise in more general cases. If an invariant interval I_0 attracts an interval $J_0 : I_0 \bigcap_{n\ge 0} f^n(J_0)$, and the part $I \backslash J_0$ which is not attracted is small enough compared to $J_0 \backslash I_0$ and does not contain fixed points then every solution x of equation (1) satisfies $x(t) \in J_0$, $t > t_0$, for some t_0 depending on the particular solution. Thus, an attractor with sufficiently large domain of immediate attraction "attracts" all solutions of equation (1).

Another aspect of the problem is an increase in complexity for the asymptotic behavior of solutions of equation (1) compared to the dynamics given by the map (3) (or equation (2)). Consider equation (1) with $f(x) = -\mathrm{sign}(x)$ for $|x| > h$, $0 < h < 1$, $f(x) = -a\,\mathrm{sign}\,(x)$ for $|x| \le h$, $a > 1$. Then there exists an open set of parameters (a, h) so that the corresponding equation (1) has a set of solutions which are governed by a quasi-random variable as $t \to \infty$. This means for instance that successive maxima of the solutions (or distances between zeros) have a probability distribution with a density which is absolutely continuous with respect to the Lebesgue measure. Note that in this case the map (3) has a cycle of period two given by -1 and 1 and this cycle is globally attracting. This implies that the increase in complexity occures in a small Hausdorff neighbourhood of the generalized 2-periodic solution of equation (2) defined by $p_0(t) = 1$, $t \in (0,1)$, $p_0(t) = -1$, $t \in (1,2)\, p_0(0) = p_0(1) = [-a, a]$. Moreover, the size of this neighbourhood goes to zero as $\nu \to +0$.

Very important are bifurcation problems for equation (1) when the nonlinearity $f(x)$ is parameter dependent. Bifurcation problems for interval maps are rather well understood (see, e.g. [54]). For some families of interval maps (including the well-known family $f_\lambda(x) = \lambda x(1-x)$, $0 \le \lambda \le 4$) the most important changes arise through period doubling bifurcations. The first bifurcation gives rise to a globally stable 2-cycle with a repelling fixed point in between. The previous example suggests that solutions of equation (1) which remain close to this 2-cycle may be very complex.

For equation (1) the problem of how the dynamics of its solutions mimics the dynamics of the interval map (3), when a parameter λ changes, was not in fact under systematic investigation. Computer simulations with equation (1) for different choices of f_λ indicate a correspondence between bifurcations for one-parameter families of equations (1) (with small ν) and one-parameter families of maps (3) [6]. In particular, for the family $f_\lambda = \lambda x(1-x)$ there is evidence that, for ν sufficiently small, one has a correspondence between attracting cycles of period 2^n, $n \leq n_0(\nu)$ and attracting periodic solutions of equation (1) with period approximately 2^n. Unfortunately, there is no rigorous result available so far.

The most natural approach to prove existence of periodic solutions for equation (1) is by the use of interval cycles for the map (3). In the simplest case, when the map f has two intervals interchanged by f and a repelling fixed point in between, the existence of a periodic solution was shown in [16].

The existence of an attracting cycle of intervals does not necessarily produce the existence of periodic solutions for equation (1) as the previous example with $f = f_0$ shows. This implies that a cycle of intervals should be subjected to additional assumptions. Some sufficient conditions are given below which guarantee the existence of a periodic solution for equation (1) when the map f has an attracting cycle of intervals.

The paper is organized in the following way.

In Chapter 1 difference equations with continuous argument are considered. Simple properties of the solutions needed later are discussed. Preliminary basic notions on interval maps are given.

Chapter 2 deals with the simplest properties of equation (1) defined by the interval map (3).

Continuous dependence results showing closeness within finite time intervals between solutions of equations (1) and (2) for small positive ν are proved in Chap. 3.

In Chapter 4 a series of examples is considered showing specific features of solutions of equation (1) which are caused by the singular perturbation term.

The role of attracting periodic intervals with relatively large immediate attraction domains is studied in Chap. 5.

Chapter 6 deals with the existence of periodic solutions for equation (1).

Concluding discussions including some naturally posed unsolved problems are presented in Chap. 7.

We do not survey all the results available for differential-delay equations of the form (1) but rather make an attempt to present studies done at the Institute of Mathematics of the Ukrainian Academy of Sciences. Therefore some intersections may be found with other results, in particular, with those obtained in [40–42].

1. Difference Equations

Basic Notions for Interval Maps

We recall briefly some basic notions from the theory of one-dimensional maps which we use throughout the paper.

Every continuous map: $f : x \to f(x)$ of an interval $I \subset \mathbb{R}$ into itself generates a dynamical system on I. Every $x_0 \in I$ defines a trajectory $\{x_k, k \in \mathbb{Z}_+\}$ by $x_{k+1} = f(x_k)$. A point x_0 is called *periodic* with period n if the points $x_0, x_1, \cdots, x_{n-1}$ are pairwise distinct and $f(x_{n-1}) = x_0$. The points $x_0, x_1, \cdots, x_{n-1}$ are said to form a *cycle* of period n. Clearly, any point from the cycle is periodic with period n. Fixed points are periodic with period 1. Points from a cycle of period n are fixed points for the map $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$.

A fixed point $x_0 \in I$ is called *attracting* if $\lim_{k\to\infty} f^k(x) = x_0$ for all x in some neighbourhood of x_0. If the function $f(x)$ is differentiable at $x = x_0$ and $|f'(x_0)| < 1$ then the fixed point x_0 is attracting. A maximal connected open (with respect to I) set which is attracted by the fixed point is called a *domain of immediate attraction* of the fixed point. The domain of immediate attraction sometimes is called an *immediate basin.*

A fixed point $x_0 \in I$ is called *repelling* if it has a neighbourhood $U(x_0)$ such that for every $x' \in U(x_0)\backslash\{x_0\}$ there exists a positive integer $k = k(x')$ with $f^k(x') \notin U(x_0)$. This means that all points from some neighbourhood leave the neighbourhood (though, they may get back afterwards). If $f(x)$ is differentiable at $x = x_0$ and $|f'(x_0)| > 1$ then the fixed point x_0 is repelling.

A periodic point $x_0 \in I$ of period n is called *attracting* if it is an attracting fixed point of the map f^n. Then the corresponding cycle $\{x_0, x_1, \cdots, x_{n-1}\}$ is also called attracting. If $f^n(x)$ is differentiable at $x = x_0$ and $|(f^n)'(x_0)| < 1$ then the periodic point x_0 is attracting. Since $(f^n)'(x_0) = f'(x_0)f'(x_1)\cdots f'(x_{n-1})$ for smooth maps the inequality mentioned holds at any point of the cycle. Repelling periodic points and cycles are defined similarly. Sometimes attracting periodic points and cycles are called *sinks,* and repelling ones are called *sources.*

A set M is called *invariant* if $f(M) \subseteq M$. Sometimes the stronger condition $f(M) = M$ is meant by invariance. If it is not specified we use the broader first notion.

A closed set $A \subset I$ will be called an *attractor* if it is invariant and there exists a neighbourhood $U(A)$ with $\bigcap_{k\geq 0} f^k(U(A)) = A$. Attracting fixed points and cycles are attractors. A maximal open set U which has a non-empty intersection with A for each connected component and such that $f^k(x) \to A$, $x \in U$ as $k \to \infty$ will be called the domain of immediate attraction for the attractor A.

The domain of immediate attraction for an attracting cycle $\{x_0, x_1, \cdots, x_{n-1}\}$ consists of n open intervals. Each of them contains a point x_i from the cycle for some i and is the domain of immediate attraction of the fixed point x_i of the map f^n, $i = 1, 2, \cdots, n$.

An interval $I_0 \subset I$ is called *periodic* with period n if $f^n(I_0) \subset I_0$ and the intervals $I_k = f^k(I_0)$, $k = 0, 1, \cdots, n-1$ satisfy $\operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset$, $i \neq j$. Here we will deal only with closed periodic intervals. The set of intervals $\{I_0, I_1, \cdots, I_{n-1}\}$ is said to form an interval cycle of period n. If the map has an interval cycle of period n then it has a periodic point with period n or $n/2$ at least (the latter may happen if $n = 2q$ and if the intervals I_i and I_{i+q}, $i = 0, 1, \cdots, q-1$, have common points on the boundary, which constitute the cycle of period $q = n/2$). A cycle of intervals $\{I_0, I_1, \cdots, I_{n-1}\}$ is called attracting if the set $\cup_i I_i$ is an attractor.

The domain of immediate attraction of an attracting cycle of intervals consists of n open (with respect to I) intervals each of them containing an element of the cycle.

Difference Equations with Continuous Argument

In this part we briefly describe properties of solutions of difference equations with continuous argument

$$x(t) = f(x(t-1)), \quad t \in \mathbb{R}_+ \tag{2}$$

which are needed later.

Throughout we assume that the one-dimensional map

$$f : x \to f(x) \tag{3}$$

has an invariant interval I and is continuous on I.

To have solutions of equation (2) for $t \geq 0$ it is necessary to define some functions $\varphi(t)$ on the initial set $[-1, 0)$. Let $\varphi \in X = C([-1,0), I)$. Then for $t \in [0, 1)$, $x(t) = f(\varphi(t-1)) \in I$ because of the invariance property $f(I) \subseteq I$. By repeating the procedure we construct the solution for $t \in [1, 2)$ and so on. Thus, to any $\varphi \in X$ there corresponds an unique solution $x_\varphi(t)$ of the equation (2) for all $t \geq 0$.

Requiring $\lim_{t \to -0} \varphi(t) = f(\varphi(-1))$ for a given $\varphi \in X$ the solution $x_\varphi(t)$ is continuous for all $t \geq 0$ since $f(x)$ is continuous. We restrict ourselves here to continuous solutions by supposing $\varphi \in X^0 = \{\varphi \in X \,|\, \lim_{t \to -0} \varphi(t) = f(\varphi(-1))\}$.

Remark. Unlike differential equations, difference equations in form (2) do not require any assumptions of continuity or smoothness of $f(x)$ or $\varphi(t)$. In particular we shall consider difference equations with discontinuous (e.g. steplike) functions. Clearly, the restriction $\varphi \in X = C([-1,0), I)$ is also not necessary.

Since the solution $x_\varphi(t)$ is defined by means of iterates of f the asymptotic behaviour of the solutions depends on properties of sequences $\{f^k(x_0),\, k \geq 0\}$ (i.e. on trajectories of the dynamical system $\{f^k,\, k \geq 0\}$ for distinct $x_0 \in \{\varphi(t),\, t \in [-1, 0)\}$. Generically, equation (2) has solutions which are asymptotic to upper semicontinuous functions (generally speaking, to discontinuous ones), see details in [55]. The complexity of the limit functions, determining the asymptotic behaviour of the solutions of equation (2), is characterized by the discontinuity set and by the jump spectrum which consists of the accumulation points of the function on the discontinuity set. Generically, the jump spectrum is finite though it may consist of countably many points. The discontinuity set usually has a complicated structure; in particular it may be homeomorphic to a Cantor set.

We shall not get into the detailed theory of equation (2) which is presented completely in the monograph [55]. Here we only consider the simplest properties of solutions which are needed in the sequel.

We distinguish three basic types of continuous solutions of equation (2): asymptotically constant solutions, relaxation type solutions and turbulent type solutions. We present simple examples which show the mechanisms of how the different types of solutions do appear.

a) Asymptotically Constant Solutions

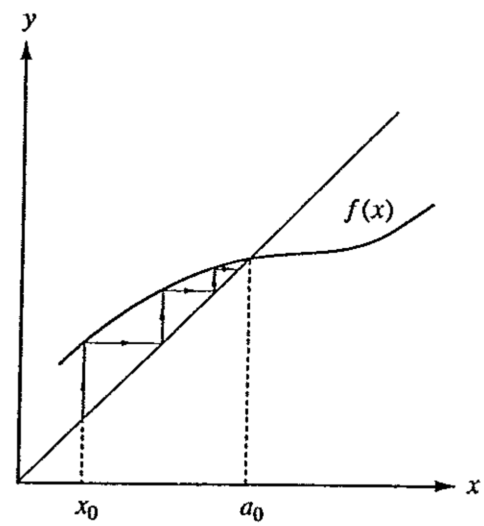

Fig. 2.

Suppose the interval map (3) has an attracting fixed point a_0 (Fig. 2) with domain of immediate attraction J_0, implying $\lim_{k\to\infty} f^k(x_0) = a_0$ for every $x_0 \in J_0$. Then, for every φ with $\varphi(t) \in J_0$ for all $t \in [-1, 0)$, the corresponding solution $x_\varphi(t)$ satisfies $\lim_{t\to+\infty} x_\varphi(t) = a_0$. We call such solutions asymptotically constant. Clearly their existence and stability is determined by the existence of the attracting fixed points of the map f. If a fixed point x_0 is repelling for the map then the solution $x(t) = a_0$ of equation (2) is Liapunov unstable.

b) Relaxation Type solutions

Relaxation type solutions appear in the following typical cases. (i) The map f has several (> 1) attracting fixed points and their attraction domains are separated by repelling fixed points. (ii) The map f has an attracting cycle of period two with a two component domain of attraction, separated by a repelling fixed point. The two cases can be realized by monotone maps. We show them in Fig. 3 a) and b), respectively.

(a)

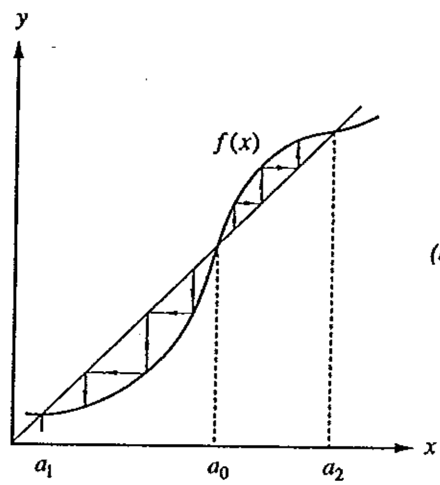

Fig. 3a.

(b)

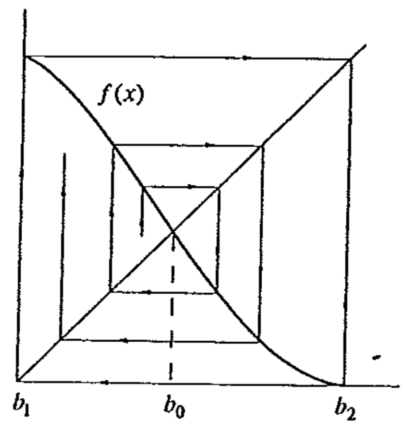

Fig. 3b.

Consider case (i). If the initial function $\varphi \in X$ satisfies $a_0 < \varphi(t) \leq a_2$ for all $t \in [-1, 0)$ then $\lim x_\varphi(t) = a_2$ since $f^k(x_0) \to a_2$ for all $x_0 \in (a_0, a_2]$, $k \to \infty$. The corresponding solution $x_\varphi(t)$ is asymptotically constant. Similarly, $\lim_{t\to\infty} x_\varphi(t) = a_1$ provided $a_1 \leq \varphi(t) < a_0$ for all $t \in [-1, 0)$. The situation changes when $\varphi(t)$ has intersections with the graph $x(t) = a_0$. Consider the simplest case of two intersections: $\varphi(t) > a_0$ for $t \in [-1, t_1) \cup (t_2, 0)$ and $\varphi(t) < a_0$ for $t \in (t_1, t_2)$ (Fig. 4a). Since $f^k(x_0) \to a_2$ for every $x_0 \in (a_0, a_2]$ and $f^k(y_0) \to a_1$ for every $y_0 \in [a_1, a_0)$, the solution $x_\varphi(t)$ approaches a generalized 1-periodic function on $[k-1, k)$ as $k \to \infty$ (see Fig. 4b). Clearly, the convergence is uniform on each compact set not containing t_1 and t_2. This means that the family $\{f^k(\varphi(t)),\ k \in \mathbb{Z}_+\}$ converges uniformly on $[-1, 0 \backslash U_\delta(t_1, t_2)$ where $U_\delta(t_1, t_2)$ is a δ neighbourhood of the points t_1 and t_2.

Case (ii) differs from (i) in that we have convergence to a 2-periodic limit function. For the initial function shown in Fig. 5a one obtains the limit function shown in Fig. 5b.

In both cases (i) and (ii), the oscillation frequency (number of zeros of $x_\varphi(t) - a_0$) of the solution $x_\varphi(t)$ on each unit segment $[k-1, k)$ is constant for all $k \in \mathbb{N}$ and is defined by the number of crossings of the initial function with

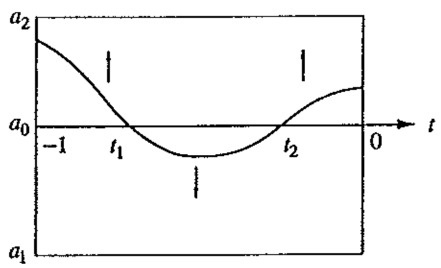

Fig. 4a.

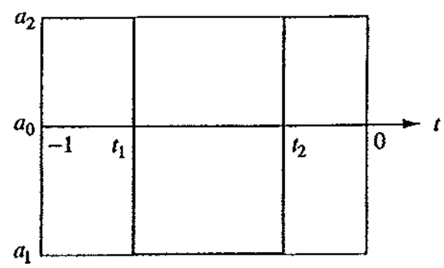

Fig. 4b.

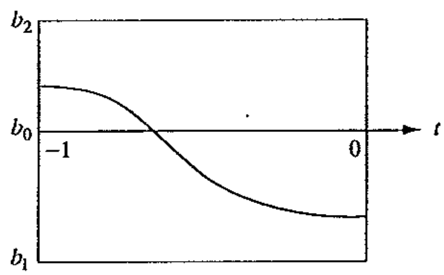

Fig. 5a.

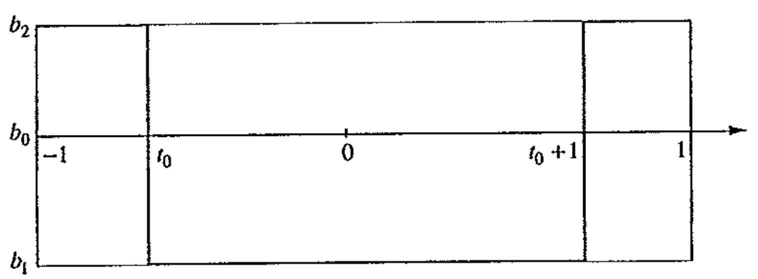

Fig. 5b.

the level of the repelling fixed point. The Lipschitz constant of the solution $x_\varphi(t)$ at the crossing points with $x = a_0$ (or $x = b_0$) increases infinitely as $t \to \infty$ and does not depend on the smoothness of $f(x)$ or $\varphi(t)$.

c) Turbulent Type Solutions

Turbulent type solutions are characterized by exponential (or polynomial) growth of oscillation frequency while their amplitudes do not damp out as $t \to \infty$. Smoothness of the solutions is predetermined by the smoothness of the map (3) and of the initial function φ, and does not change with t. Nevertheless, the Lipschitz constant for any solution on the unit interval $[t-1,t)$ grows infinitely as $t \to \infty$ (with exponential rate generically).

The generation of oscillations for turbulent type solutions can be illustrated as follows. Suppose the map f of the interval $I = [a,b]$ onto itself has the form as in Fig. 6.

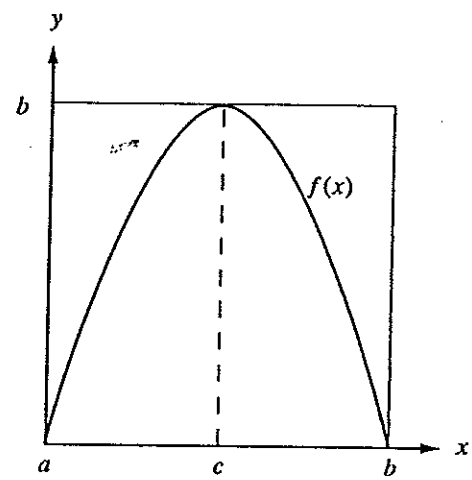

Fig. 6.

In this case there exist two intervals I_1 and I_2 such that each of them is mapped onto their union $I_1 \cup I_2$ ($I_1 = [a,c]$, $I_2 = [c,b]$ for the map in Fig. 6). Therefore, if the initial condition $\varphi(t)$ attains all the values from $I_1 = [a,c]$ on some time subinterval $[t_1,t_2]$ of $[-1,0)$, then the solution $x_\varphi(t)$ will attain all the values from $I = I_1 \cup I_2$ for the time interval $[1+t_1, 1+t_2]$. Similar, initial conditions ranging within $I_2 = [c,b]$ give rise to a solution ranging through all the interval I after one time unit. One oscillation of the initial function covering the interval I generates two similar oscillations within the same time interval but one time unit later. An initial condition φ with two oscillations (Fig. 7a) generates a solution $x_\varphi(t)$ with four oscillations on $[0,1)$ (Fig. 7b). Clearly, the number of oscillations on the initial interval $[-1,0)$ is increased by the factor 2^n for the time interval $[n-1,n)$.

The existence of turbulent type solutions with exponential growth of the frequency may be concluded in the case when the interval map f has periodic

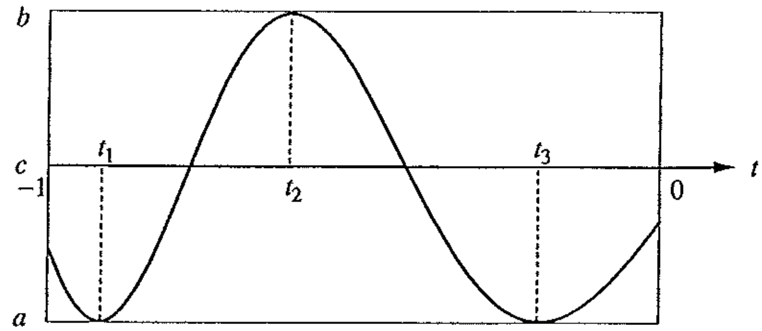

Fig. 7a.

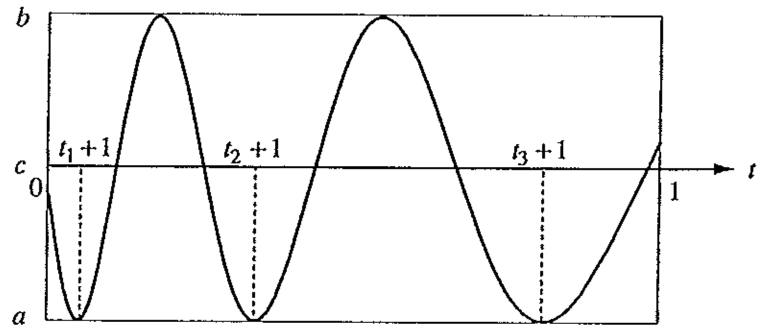

Fig. 7b.

points with periods which are not a power of two. Indeed, as it was shown in [49] in this case there exist two intervals I_1 and I_2 and a positive integer N such that $g(I_1) \supset I_1 \cup I_2$ and $g(I_2) \supset I_1 \cup I_2$ where $g = f^N$. Therefore, an oscillation of the initial function covering the union $I_1 \cup I_2$ gives rise to at least two oscillations covering $I_1 \cup I_2$ after a time interval of length N.

Turbulent type solutions with polynomial growth of the frequency (sometimes called preturbulent ones) always exist when the map f has periodic points of periods 2^i with $i > 1$ and no other periodic points. But they may exist also for some maps f which have only fixed points.

Summarizing, the existence of certain types of continuous solutions for equation (2) is defined by the dynamics of the map f. In particular, if the map f has attracting fixed points, equation (2) has asymptotically constant solutions: if there are no attracting fixed points, there are no asymptotically constant solutions (except trivial ones $x(t) = \text{const}$); if the map has several attracting fixed points on some subinterval of I such that their immediate attraction domains are separated by repelling fixed points or if it has an attracting 2-cycle with two component domain of immediate attraction separated by a fixed point of by a cycle of period two, then equation (2) has relaxation type solutions; if the map f has a periodic point with period $\neq 2^i$, $i = 0, 1, 2, \cdots$ then equation (2) has turbulent type solutions. Cleary, equation (2) may have solutions of several types simultaneously. Examples of asymptotically constant, relaxation type, and turbulent type solutions are shown on the a), b), c) parts of Fig. 8 respectively.

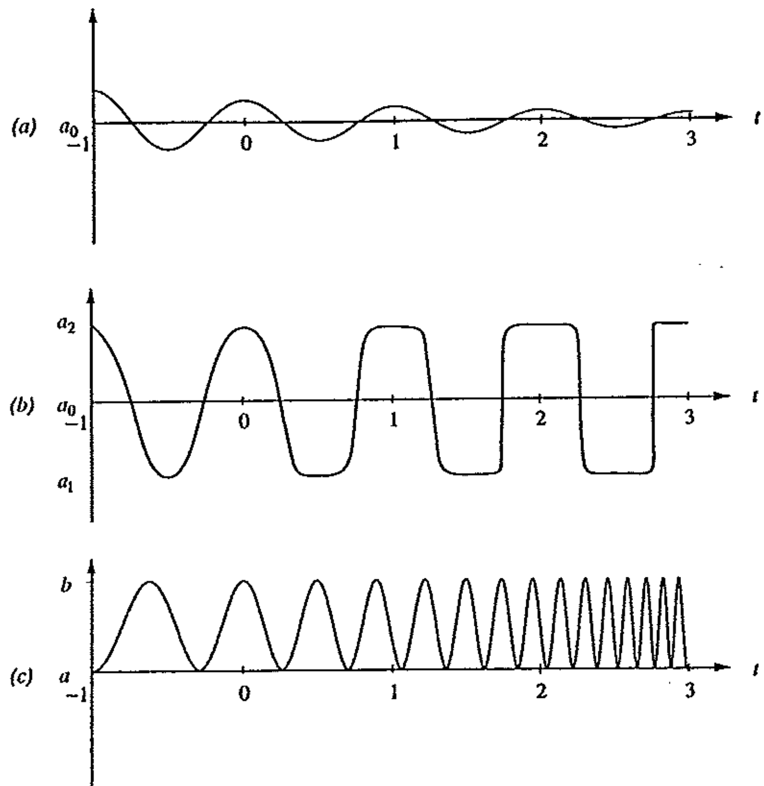

Fig. 8.

2. Singular Perturbations of Difference Equations with Continuous Argument: Simplest Properties

It is natural to consider the differential difference equation

$$\nu \dot{x}(t) + x(t) = f(x(t-1)) \tag{1}$$

as a singular perturbation ($\nu \ll 1$) of the difference equation with continuous argument

$$x(t) = f(x(t-1)) \,. \tag{2}$$

Properties of solutions for equation (2) can be understood rather well. The analysis is based on properties of the one-dimensional dynamical system generated by the corresponding map

$$f : x \to f(x) . \tag{3}$$

For a theory of equation (2) with scalar $f(x)$ see monograph [55]. Necessary preliminaries have been introduced in the previous chapter.

The map f is assumed to have an invariant interval $I = [a, b]$ and to be continuous on I. Let $x_\varphi^\nu(t)$ denote the solution of equation (1) with initial condition $\varphi \in X_I = C([-1,0], I)$. It is natural to ask what properties of solutions of equation (1) are inherited from those of equation (2) (or from the map (3))? The following statements contain some answers to this question which can be obtained immediately from the dynamics given by the map f. They are found in earlier publications by the authors [2, 55], and also in [40–42].

Theorem 2.1 (Invariance property). *If $\varphi \in X_I$ then $x_\varphi^\nu(t) \in I$ for all $t \geq 0$ and any fixed $\nu > 0$.*

Theorem 2.1 states that the solution $x_\varphi^\nu(t)$ ranges within the invariant interval I of the map f provided the initial condition $\varphi(t)$ does so for all $t \in [-1, 0]$.

Theorem 2.1 allows to define in a well-known way a semiflow $\mathcal{F}^t$, $t \geq 0$ on the state space $X_I = C([-1,0], I)$ by

$$(\mathcal{F}^t\varphi)(s) = x_\varphi^\nu(t+s),\ s \in [-1,0] . \tag{4}$$

Remark. The invariance property given by Theorem 2.1 holds true for the case of a nonstrong invariance of the map f i.e., $f(I) \subseteq I$. The invariance property also holds true in the case of discontinuous maps f .

Proof. Suppose $\varphi \in X_I = C([-1,0], I)$ and t_0 is the first time where the corresponding solution $x(t)$ leaves the interval I. To be definite we may assume that $t_0 = 0$ (due to the autonomy), $x(0) = b$, and that every right-sided neighborhood of $t_0 = 0$ contains a point t' with $x(t') > b$. Then this neighbourhood also contains a point t'' with $x(t'') > b$ and $\dot{x}(t'') > 0$. Admitting $t'' < 1$ we have $\nu\dot{x}(t'') = -x(t'') + f(x(t'' - 1)) < 0$ a contradiction. The other case, $x(0) = a$, is treated similarly. □

Suppose next that x_* is an attracting fixed point of the map f with immediate basin J_0 : $\lim_{n\to\infty} f^n(x_0) = x_*$ for any $x_0 \in J_0$. Define $X_{J_0} = C([-1,0], J_0)$. In this situation the following holds.

Theorem 2.2. *For any $\nu > 0$ and $\varphi \in X_{J_0}$, $\lim_{t\to\infty} x_\varphi^\nu(t) = x_*$.*

The result shows that asymptotically constant solutions persist under singular perturbations of equation (2).

The proof of Theorem 2.2 is based on the following Lemma, which is also of independent interest.

Lemma 2.1. *Suppose an interval J is mapped by f into itself. If none of the endpoints of the interval $f(J)$ is a fixed point then for every $\varphi \in X_J = C([-1,0],J)$ there exists a finite time $t_0 = t_0(\varphi,\nu)$ such that $x_\varphi^\nu(t) \in f(J)$ for all $t \geq t_0$.*

Proof. Suppose first that $\varphi(0) \in \overline{f(J)}$. Then we claim that $x_\varphi^\nu(t) \in f(J)$ for all $t \leq 0$. Suppose not, and let t_0 be the first point at which the solution $x_\varphi^\nu(t)$ leaves the interval $\overline{f(J)}$. Then every right-sided neighbourhood of $t = t_0$ contains a point t' for which $x_\varphi^\nu(t') \notin \overline{f(J)}$. To be definite suppose $x_\varphi^\nu(t') > \sup\{\overline{f(J)}\}$. Then the same neighbourhood contains also a point t'' for which both $x_\varphi^\nu(t'') > \sup\{\overline{f(J)}\}$ and $d/dt[x_\varphi^\nu(t'')] > 0$ holds. Since $x_\varphi^\nu(t) \in J$ for all $t \in [t_0-1,t_0]$ and $t'' < t_0+1$ may be assumed, we then have $\nu\dot{x}_\varphi^\nu(t'') = -x_\varphi^\nu(t'') + f(x_\varphi^\nu(t''-1))$, a contradiction.

Suppose next that $\varphi(0) \notin \overline{f(J)}$. To be definite, let $\varphi(0) > \sup\{f(J)\}$ the case $\varphi(0) < \inf\{f(J)\}$ is treated similar). We claim that $x_\varphi^\nu(t)$ is decreasing for all $t \in [0,t_0]$ where $t_0 \leq \infty$ is the first point with $x_\varphi^\nu(t_0) = \sup\{f(J)\}$. Suppose $t_0 = \infty$. Then $x_\varphi^\nu(t) > \sup\{f(J)\}$ for all $t \geq 0$. According to equation (1) $\nu\dot{x}_\varphi^\nu(t) = -x_\varphi^\nu(t)+f(x_\varphi^\nu(t-1)) \leq 0$. Therefore there exists $x' = \lim_{t\to\infty} x_\varphi^\nu(t) \geq \sup\{\overline{f(J)}\}$. Since x' is not a fixed point of the map f we have $-f(x')+x' := \delta > 0$. Equation (1) then replies $\nu\dot{x}_\varphi^\nu(t) = -x_\varphi^\nu(t)+f(x_\varphi^\nu(t-1) \leq -\delta/2$ for large t. This implies $x_\varphi^\nu(t) \to -\infty$ as $t \to \infty$, a contradiction. Therefore $t_0 < \infty$. Since $x_\varphi^\nu(t)$ is decreasing on $[0,t_0]$, by the above, and $x_\varphi^\nu(t) \in \overline{f(J)}$ we have $x_\varphi^\nu(t) \in f(J)$ for all $t \in [t_0-1,t_0]$. To complete the proof we repeat its first part. □

Proof of Theorem 2.2. Let J_0 denote the immediate basin of the attracting fixed point $x = x_*$. Take an arbitrary $\varphi \in X_{J_0} = C([-1,0],J_0)$ and set $m = \inf\{\varphi(s), s \in [-1,0]\}$, $M = \sup\{\varphi(s), s \in [-1,0]\}$. Then $[m,M] \subset J_0$. Consider the smallest closed invariant interval J' containing the interval $[m,M]$, which is contained in J_0. Then one has $J' \supset f(J') \supset f^2(J') \supset \cdots$ and $\bigcap_{n\geq 0} f^n(J') = x_*$. The proof follows by repeated application of Lemma 2.1 and by using the invariance property. □

Theorem 2.2 can be extended to the case of a general attractor of the map f. Suppose an interval I_0 is invariant and an attractor for the map f with immediate basin J_0:

$$\mathrm{dist}(f^n(x),I_0) \to 0 \text{ for all } x \in J_0,\ n \to \infty.$$

Theorem 2.3. *For any $\nu > 0$ and $\varphi \in C([-1,0], J_0)$*
$\inf\{I_0\} \leq \lim_{t\to\infty}\inf x_\varphi^\nu(t) \leq \lim_{t\to\infty}\sup x_\varphi^\nu(t) \leq \sup\{I_0\}$.

The proof is the same as in the case of Theorem 2.2. with J being any invariant subinterval of J_0 containing I_0.

Let x_* be a repelling fixed point of the map f, satisfying $|f'(x_*)| > 1$. The following shows that the repelling property (in a sense) persists for the constant solution $x(t) = x_*$ of equation (1) if ν is small enough.

Theorem 2.4. *Suppose x_* is a repelling fixed point for the map f, $f'(x)$ is continuous in a neighbourhood of x_* and $|f'(x_*)| > 1$. Then there exists a positive ν_0 such that for all $0 < \nu \leq \nu_0$ the constant solution $x(t) = x_*$ of equation (1) is unstable.*

A constant solution $x(t) = x_*$ is called unstable if the stationary state $\varphi(t) = x_*$ of the semiflow $\mathcal{F}^t$ is unstable.

The proof of the theorem splits into two cases.
1. $f(x)$ is increasing in a neighbourhood of $x = x_*$. In this case the proof follows directly from the following Lemma.

Lemma 2.2. *Suppose the map f increases in some (half) neighbourhood of a repelling fixed point $x = x_*$. Then there exists a (half) neighbourhood $U(x_*)$ such that for every initial function $\varphi \in X_{U(x_*)} = C([-1,0], U(x_*))$ the corresponding solution $x_\varphi^\nu(t)$ has the property $x_\varphi^\nu(t_0 + s) \notin X_{U(x_*)}, s \in [-1,0]$, for some $t_0 = t_0(\varphi, \nu) \geq 0$.*

Proof. For definiteness we suppose $f(x_*) = x_*$ and $f(x) > x$ for all $x \in I' = (x_*, x_* + \delta]$. Take an arbitrary $\varphi \in X_{I'} = C([-1,0], I')$ and consider the corresponding solution $x_\varphi^\nu(t)$. Set $\inf\{\varphi(s), s \in [-1,0]\} = m > x_*$. We claim that there exists $t_0 \geq 0$ such that $x_\varphi^\nu(t) > f(m)$ for all $t \in [t_0, t_0 + 1]$. Indeed, in the case $\varphi(0) \geq f(m)$ one has $x_\varphi^\nu(t) \geq f(m)$ for all $t \in [0,1]$. This can be shown similar to the proof of Theorem 2.1. In the other case $\varphi(0) < f(m)$ there always exists a first point $t_0 > 0$ such that $x_\varphi^\nu(t_0) = f(m)$ and $x_\varphi^\nu(t)$ is strictly increasing for $t \in [0, t_0]$ (see the second part of the proof of Lemma 2.1). According to the first case and autonomy of equation (1) $x_\varphi^\nu(t) \geq f(m)$ for all $t \in [t_0, t_0 + 1]$. Using induction arguments one has $x_\varphi^\nu(t) > f^n(m)$ for all $t \in [t_n, t_n + 1]$ and some t_n. Since $f^n(m) \notin U(x_*) = (x_*, x_* + \delta]$ for every $m \in U(x_*)$ and some $n \in \mathbb{N}$, the proof follows. □

2. $f(x)$ is decreasing in a neighbourhood of $x = x_*$ with $f'(x_*) < -1$.

The proof in this case follows from two facts: (i) Instability of the stationary state for the semiflow generated by the linear differential – delay equation $\nu\dot{y}(t) + y(t) = f'(x_*)y(t-1)$. The characteristic quasipolynomial $\lambda(z) = \nu z + 1 - f'(x_*)\exp(-z)$ has roots with positive real parts when $0 < \nu$ is small enough, and linear instability follows [13, 17, 43, 45]. (ii) The stationary state

$x = x_*$ of the semiflow $\mathcal{F}^t$ given by $\nu\dot{x}(t) + x(t) = f(x(t-1)$ is unstable provided $\lambda(z) = \nu z + 1 - f'(x_*)\exp(-z)$ has roots with positive real parts and $f(x)$ satisfies $[f(x) - f(x_*) - f'(x_*)(x - x_*)] = O(|x - x_*|)$ as $x \to x_*$ [12].

Another but similar proof is contained in [16] (Lemma 10).

In the case when the endpoints of the invariant interval I are not attracting fixed points the invariance property (Theorem 2.1) can be strengthened in the following way.

Theorem 2.5. *Suppose the interval $I = [a, b]$ is invariant under f and if $x = a$ (or $x = b$) is a fixed point then it is a repelling one. Then there exists a positive number δ, depending only on $f(x)$ and ν, such that if we denote $I' = [a+\delta, b-\delta]$ then the following holds: (1) for every $\varphi \in X_I$ there exists a time $t_0 = t_0(\varphi, \nu)$ such that $x^\nu_\varphi(t_0 + s) \in X_{I'}, s \in [-1, 0]$ (excepting $\varphi \equiv a$ or b when $x = a$ or $x = b$ is a fixed point); (2) for every $\psi \in X_{I'}$ $x^\nu_\psi(t) \in I'$ for all $t \geq 0$.*

Remark. It is clear that in the case that one of the endpoints of the interval is an attracting fixed point or a nonisolated fixed point Theorem 2.5 does not hold. This follows from Theorem 2.2 and from the fact that every fixed point of the map f gives rise to a corresponding stationary solution of equation (1).

Proof. We shall show the existence of a positive δ such that for every $\varphi \in X_I$ the corresponding solution $x^\nu_\varphi(t)$ satisfies $x^\nu_\psi(t) \geq a+\delta$ for sufficiently large t. In a similar way the existence of positive σ may be shown for which $x^\nu_\varphi(t) \leq b-\sigma$ for large t. This evidently implies the proof.

Since $I = [a, b]$ is invariant in every case (either $f(a) > a$ or $f(a) = a$ and $x = a$ is a repelling fixed point) there exists a positive γ such that $f(x) > x$ for all $x \in (a, a+\gamma]$. Set $(a, a+\gamma] = I_1$.

Claim 1. For every $\varphi \in X_{I_1}$ there exists a time moment $t_1 = t_1(\varphi, \nu)$ such that $x^\nu_\varphi(t_1) = a + \gamma$, $x^\nu_\varphi(t_1 + s) > a + \gamma$ for all $s \in (0, 1]$. Moreover, $x^\nu_\varphi(t)$ is increasing for $t \in [0, t_1]$.

The proof essentially repeats the proof of Lemma 2.2.

Claim 2. For every $\varphi \in X_I$, either $x^\nu_\varphi(t) - (a+\gamma) > 0$ for sufficiently large t, or $x^\nu_\varphi(t) - (a+\gamma)$ has arbitrarily large zeros.

Proof. The case $x^\nu_\varphi(t) - (a+\gamma) < 0$ for all large t is excluded by claim 1.

From claim 2 it follows that we only have to consider solutions which oscillate around $x = a+\gamma$). Consider a zero t_0 of $x^\nu_\varphi(t) - (a+\gamma)$ which contains in every right-side neighbourhood a point t' for which $x^\nu_\varphi(t') - (a+\gamma) < 0$. Then on the interval $[t_0, t_0+1]$ one has $x^\nu_\varphi(t) \geq a + \gamma\exp\{-(t-t_0)/\nu\}$, which implies $x^\nu_\varphi(t) \geq a + \gamma\exp(-1/\nu) \stackrel{\text{def}}{=} a + \delta$ for all $t \in [t_0, t_0+1]$. If $x^\nu_\varphi(t)$ has a zero t_1 on $(t_0, t_0+1]$ such that every right-sided neighbourhood contains a point t' satisfying $x^\nu_\varphi(t') - (a+\gamma) < 0$, we set $t_0 = t_1$ and repeat arguments. If $x^\nu_\varphi(t)$ does not have such a zero in $(t_0, t_0+1]$ then $x^\nu_\varphi(t) - (a+\gamma) < 0$ for all $t \in (t_0, t_0+1]$. This implies $x_\varphi^\nu(t_0+1+s) \in X_{I_1}, s \in [-1,0]$ and therefore $x_\varphi^\nu(t)$ increases for $t \geq t_0+1$ until the next zero t_1 of $x_\varphi^\nu(t) - a + \gamma)$. In every case one has $x_\varphi^\nu(t) \geq a + \delta$ for all $t \in [t_0, t_1]$. The Theorem is then proved by induction. □

The following Theorem is a combination of Theorems 2.3 and 2.5.

Theorem 2.6. *Suppose the interval $I_0 \subset I$ is invariant under f and an attractor with domain of immediate attraction $J_0 \subset I : f(I_0) = I_0$, $f^n(x) \to I_0$ as $n \to \infty$ for every $x \in J_0$. If none of the endpoints of the interval I_0 is a fixed point then the following holds: there exists a positive δ depending on f and ν only such that solutions of equation (1) satisfy $x_\varphi^\nu(t) \in [\inf I_0 + \delta, \sup I_0 - \delta]$ for every $\varphi \in C([-1,0], J_0)$ and large t.*

The proof is based essentially on the arguments given in the proofs of Theorems 2.3 and 2.5. We leave the details for the reader.

Remark. All results proved in this chapter extend for piecewise continuous nonlinearities $f(x)$ which we use in the sequel. In particular, Theorem 2.1 (invariance property) and Lemma 2.1 hold without any changes.

Example 2.1. Consider equation (1) with $f(x)$ being the logistic family $f_\lambda(x) = \lambda x(1-x), 0 \leq \lambda \leq 4$, which map the interval $[0,1]$ into itself.

We briefly recall some dynamical properties of the map f_λ, depending on particular values of the parameter λ.

If $0 \leq \lambda \leq 1$ the map f_λ has the only attracting fixed point $x = 0$ which attracts all others trajectories $(x_n)(x = 0$ is a global attractor, Fig. 9a).

For $\lambda > 1$ another fixed point $x = 1 - 1/\lambda$ bifurcates which attracts all other trajectories (x_n) for $1 < \lambda \leq 3$, (except $x = 0$). The fixed point $x = 0$ is repelling for every $\lambda > 1$ (Fig. 9b).

For $\lambda > 3$ the fixed point $x = 1 - 1/\lambda$ becomes repelling and an attracting cycle $\{a_1, a_2\}$ of period two bifurcates off. For $3 < \lambda \leq 1 + \sqrt{6}$ this cycle attracts almost all trajectories (x_n) (except the repelling fixed points $x = 0$ and $x = 1 - 1/\lambda$ and their preimages) (Fig. 9c).

The further development of dynamics for the map f_λ is well-known. There exists an increasing sequence $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots (\lambda_0 = 1, \lambda_1 = 3, \lambda_2 = 1 + \sqrt{6})$ such that for every $\lambda \in (\lambda_n, \lambda_{n+1}]$ the map f_λ has an attracting cycle of period 2^n which attracts almost all trajectories (x_n) (except repelling cycles of periods $1, 2, \cdots, 2^{n-1}$ and all their preimages). The sequence converges to $\lambda_* \approx 3,569$. For each $\lambda > \lambda_*$ the map f_λ has a cycle with a period which is not a power of two.

For every $\lambda > 3$, the interval $I_\lambda = [f_\lambda^2(1/2), f_\lambda(1/2)]$ is invariant for the map f_λ and is an attractor (Fig. 9d).

We shall see now what can be said about the asymptotic behavior as $t \to \infty$ of solutions to equation (1) when the initial conditions are taken from $X = C([-1,0],[0,1])$. To draw conclusions we use Theorems 2.1–2.6.

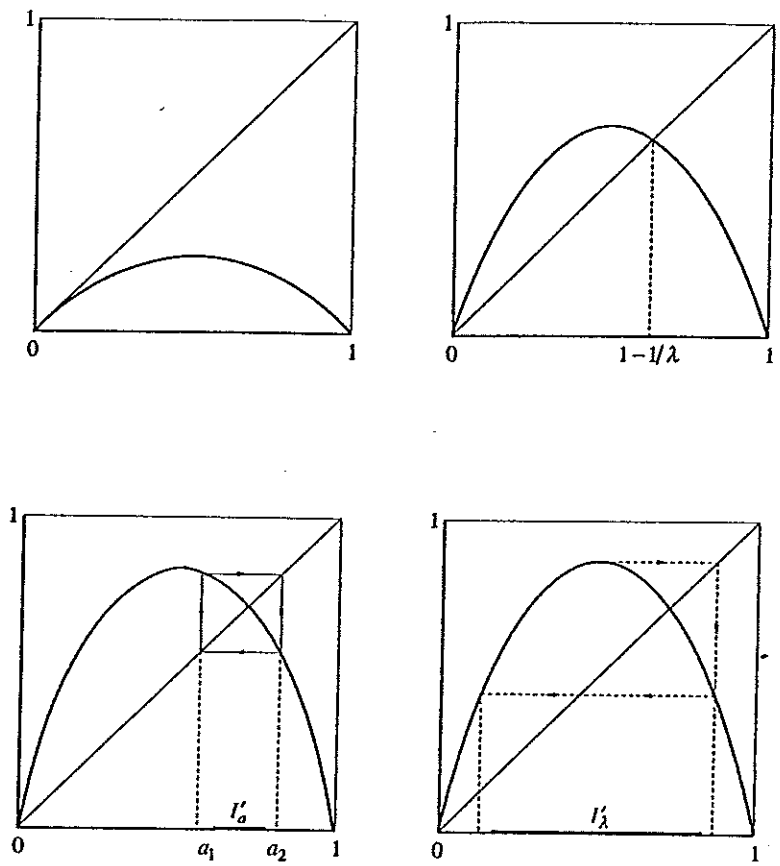

Fig. 9.

When $0 \le \lambda \le 1$ every solution of equation (1) satisfies $\lim_{t\to\infty} x(t) = 0$ (Fig. 9a).

When $1 < \lambda \le 3$ every solution of equation (1) satisfies $\lim_{t\to\infty} x(t) = x_\lambda = 1 - 1/\lambda$ (except $x \equiv 0$) (Fig. 9b).

Now consider the case $\lambda > 3$ and let $\{a_1, a_2\}$ be the cycle of period two that appears for the map f_λ. There exists a parameter value $\lambda^0 \in (\lambda_1, \lambda_2) = (3, 1 + \sqrt{6})$ such that at $\lambda = \lambda^0$ the critical point $x = 1/2$ forms a cycle of period two which is evidently attracting.

For all $\lambda \in (3, \lambda^0]$ the points a_1 and a_2 lay on the right hand side of $x = 1/2$ and the interval $I_a = [a_1, a_2]$ is invariant and a global attractor for the map f_λ. In this case one has $x(t) \in I_a$ for all sufficiently large t and every solution $x(t)$ of equation (1) (except $x \equiv 0$). Moreover, there exists a positive $\delta = \delta(\lambda, \nu)$ such that $x(t) \in I'_a$ for all large t, where $I'_a = [a_1 + \delta, a_2 - \delta]$ (Fig. 9c)(Theorem 2.6).

When $\lambda \in (\lambda^0, 1+\sqrt{6}]$ the interval $I_a = [a_1, a_2]$ is not invariant but the interval $I_\lambda = [f_\lambda^2(1/2), f_\lambda(1/2)]$ is. Similarily, every solution $x(t)(\not\equiv 0)$ of equation (1) satisfies $x(t) \in I_\lambda$ for large t. There exists $\delta = \delta(\lambda, \nu)$ such that $x(t) \in I'_\lambda$ for large t, where $I'_\lambda = [f_\lambda^2(1/2)+\delta, f_\lambda(1/2)-\delta]$ (Theorem 2.6).

The same situation holds true for every $4 \geq \lambda > \lambda^0$. The interval $I_\lambda = [f_\lambda^2(1/2), f_\lambda(1/2)]$ remains invariant. There exists a positive $\delta = \delta(\lambda, \nu)$ such that $x(t) \in I'_\lambda = [f_\lambda^2(1/2)+\delta, f_\lambda(1/2)-\delta]$ for every solution $x(t)$ of equation (1) and all large t (Fig. 9d).

3. Continuous Dependence on Parameter

Consider the differential-difference equation

$$\nu\dot{x}(t) + x(t) = f(x(t-1)) \tag{1}$$

with small positive ν and the corresponding difference equation with continuous argument

$$x(t) = f(x(t-1)) \tag{2}$$

which is obtained formally from equation (1) by letting $\nu = 0$. Assume that the one-dimensional map

$$f: \quad x \to f(x) \tag{3}$$

has a closed invariant interval $I \subseteq \mathbb{R}$ and is continuous on I.

Let $X_I = C([-1,0], I)$ denote the continuous functions from $[-1,0]$ into I. Clearly, if $\varphi \in X_I$ the corresponding solution$x_\varphi^\nu(t)$ of equation (1) is continuous (even smooth for $t > 0$), whereas the solution of equation (2) need not be continuous. It is continuous for all $t \geq -1$ if the consistency condition $\varphi(0) = f(\varphi(-1))$ holds. This motivates us to introduce the subset of initial functions $X_I^0 = \{\varphi \in X_I | \varphi(0) = f(\varphi(-1))\}$.

One naturally expects that close initial data $\varphi(t)$ and $\psi(t)$ generate solutions $x_\varphi(t)$ and $x_\psi^\nu(t)$ which are also close within (at least) a finite time interval, provided ν is small enough. In fact closeness between solutions of equations (1) and (2) holds uniformly on intervals $[0,T], T>0$ for $\varphi \in X_I^0$, and uniformly on compact subsets of $\mathbb{R}_+ = \{t | t \geq 0\}$ which do not contain discontinuity points of $x_\varphi(t)$ for $\varphi \in X_I$. Precise statements are given in Theorems 3.1 and 3.4 below.

Let $\mathcal{M}$ be a subset of $\mathbb{R}$. By the norm of $\varphi: \mathcal{M} \to \mathbb{R}$ we mean the uniform norm, that is $\|\varphi\|_{\mathcal{M}} = \sup\{|\varphi(t)|, t \in \mathcal{M}\}$.

Theorem 3.1. *For any $\varphi \in X_I^0$ and positive T, ε there exist positive δ, ν_0 depending on φ, T, ε such that $\|x_\varphi - x_\psi^\nu\|_{[0,T]} \leq \varepsilon$ for all $0 < \nu \leq \nu_0$ provided $\|\varphi - \psi\|_{[-1,0]} \leq \delta$ and $\psi \in X_I$.*

Due to induction arguments it is sufficient to prove the theorem for $T = 1$. We first prove several lemmas.

Lemma 3.1. *For any $\varphi \in X_I$ and $\varepsilon > 0$ there exists $\delta = \delta(\varphi, \varepsilon) > 0$ such that $\|x_\varphi^\nu - x_\psi^\nu\|_{[0,1]} \le \varepsilon$ for all $\nu > 0$ provided $\psi \in X_I$ and $\|\varphi - \Psi\|_{[-1,0]} \le \delta$.*

Proof. Since the differential-difference equation (1) is equivalent to the integral equation $x(t) = x(0)\exp(-t/\nu) + (1/\nu)\int_0^t \exp\{(s-t)/\nu\} f(x(s-1))ds$, we have the following relation for $t \in [0,1] |x_\varphi^\nu(t) - x_\psi^\nu(t)| \le |\varphi(0) - \psi(0)|\exp(-t/\nu) + (1/\nu) \times \int_0^t \exp\{(s-t)/\nu\}|f(\varphi(s-1)) - f(\psi(s-1))|ds$. The uniform continuity of f implies that for any $\varepsilon > 0$ there exists $\delta > 0$ with $|f(\varphi(t)) - f(\psi(t))| \le \varepsilon$ provided $|\varphi(t) - \psi(t)| \le \delta$. This yields $|x_\varphi^\nu(t) - x_\psi^\nu(t)| \le \delta \exp(-t/\nu) - \varepsilon(1 - \exp(-t/\nu) \le \max\{\delta, \varepsilon\}$ which completes the proof. □

Lemma 3.2. *For any $\varphi \in X_I^0$ and $\varepsilon > 0$ there exists $\nu = \nu_0(\varphi, \varepsilon) > 0$ such that $\|x_\varphi^\nu - x_\varphi\|_{[0,1]} \le \varepsilon$ for all $0 < \nu \le \nu_0$.*

Proof. Taking into account that $1 - \exp(-t/\nu) = (1/\nu)\int_0^t \exp\{(s-t)/\nu\}ds$ we have for $t \in [0,1] : |x_\varphi^\nu(t) - x_\varphi(t)| \le |\varphi(0) - f(\varphi(t-1))|\exp(-t/\nu) + (1/\nu)\int_0^t \exp\{(s-t)/\nu\}|f(\varphi(s-1)) - f(\varphi(t-1))|ds$. Since $f(\varphi(\cdot))$ is uniformly continuous for $t \in [-1,0]$ it follows that for any $\varepsilon' > 0$ there exists $\delta' > 0$ such that $|f(\varphi(t_1)) - f(\varphi(t_2))| \le \varepsilon'$ provided $|t_1 - t_2| \le \delta'$. Thus for $t \in [0, \delta']$. we have $|x_\varphi^\nu(t) - x_\varphi(t)| \le \varepsilon' \exp(-t/\nu) + \varepsilon'(1 - \exp(-t/\nu)) \le \varepsilon'$. Suppose next $t \in [\delta', 1]$. Then $|\varphi(0) - f(\varphi(t-1))|\exp(-t/\nu) = \operatorname{diam} I \exp(-\delta'/\nu) = \varepsilon'$ for all sufficiently small ν, say $0 < \nu \le \nu'$. On the other hand $(1/\nu)\int_0^t \exp\{(s-t)/\nu\}|f(\varphi(s-1)) - f(\varphi(t-1))|ds = (1/\nu)(\int_0^{t-\delta'} + \int_{t-\delta'}^t) \le \operatorname{diam} I \exp(-\delta'/\nu) + \varepsilon'(1 - \exp(-\delta'/\nu)) < 2\varepsilon'$ for all sufficiently small ν say $0 < \nu \le \nu''$. Therefore, setting $\nu''' = \min(\nu', \nu'')$ we have $(1/\nu)\int_0^t \exp\{(s-t)/\nu\}|f(\varphi(s-1)) - f(\varphi(t-1))|ds = 2\varepsilon'$ for all $0 < \nu \le \nu'''$. This implies $|x_\varphi^\nu(t) - x_\varphi(t)| \le 3\varepsilon' = \varepsilon$ for all $0 < \nu \le \nu'''$ and $t \in [\delta', 1]$ with $\varepsilon' = \varepsilon/3$. The proof is complete. □

The proof of Theorem 3.1. follows easily from the inequality $\|x_\psi^\nu - x_\phi\|_{[0,1]} \le \|x_\varphi - x_\varphi^\nu\|_{[0,1]} + \|x_\varphi^\nu - x_\psi^\nu\|_{[0,1]}$ and Lemmas 3.1 and 3.2.

It is not difficult to see that, similarly to the proof of Theorem 3.1, the closeness between solutions of equation (2) and solutions of equation (1) may be derived, when f in equation (1) is perturbed. For a precise statement we denote by $x_\varphi^\nu(t,f)$ and $x_\varphi(t,f)$ the solutions of equations (1) and (2), respectively, with particular $f(x)$.

Theorem 3.2. *For any $\varphi \in X_I^0$ and positive T, ε there exist positive δ, σ, ν_0 depending on φ, T, ε such that $\|x_\varphi(\cdot, f) - x_{\tilde\varphi}^0(\cdot, \tilde f)\|_{[0,T]} \le \varepsilon$ for all $0 < \nu \le \nu_0$ provided $\tilde\varphi \in X_I$, $\|\varphi - \tilde\varphi\|_{[-1,0]} \le \delta$ and $\|f - \tilde f\|_I \le \sigma$.*

With the closeness result proved the next natural question appears. How long do the solutions of equations (1) and (2) remain close, provided the initial data are close?

Suppose $\varphi \in X_I^0$ is fixed and consider arbitrary $\psi \in X_I$ with the initial deviation $\|\varphi - \psi\|_{[-1,0]} := \Delta_0$ being small. The solutions $x_\psi(t)$ and $x_\varphi^\nu(t)$ diverge, in general, as t increases. Since $\Delta_T := \|x_\psi - x_\varphi^\nu\|_{[0,T]} \leq \|x_\varphi - x_\varphi^\nu\|_{[0,T]} + \|x_\varphi - x_\psi\|_{[0,T]}$ and $\|x_\varphi - x_\varphi^\nu\|_{[0,T]} \to 0$ as $\nu \to +0$, the value $\|x_\psi - x_\varphi^\nu\|_{[0,T]}$ may be estimated by $\|x_\varphi - x_\psi\|_{[0,T]}$. The latter is determined by the initial deviation Δ_0 and equation (2). Thus, the value of Δ_T depends on both the deviation induced by difference equation (2) with Δ_0 given and the deviation caused by solutions of equations (1) and (2) through the same initial condition. The latter depends essentially on the smoothness of $f(x)$.

Theorem 3.3. *Suppose f and $\varphi \in X_T^0$ are Lispschitz. Then $\|x_\varphi - x_\varphi^\nu\|_{[0,T]} = O(\nu)$ as $\nu \to +0$ for any fixed $T > 0$.*

Proof. Suppose the Lispschitz constants are L and L' for $f(x)$ and $\varphi(t)$, respectively. First consider the interval $[0, 1]$. To estimate $\|x_\varphi^\nu - x_\varphi\|_{[0,1]}$ we use the inequality $|x_\varphi^\nu(t) - x_\varphi(t)| \leq |\varphi(0) - f(\varphi(t-1))| \exp(-t/\nu) + (1/\nu) \times \int_0^t \exp\{(s-t)/\nu\} |f(\varphi(s-1)) - f(\varphi(t-1))| ds \stackrel{\text{def}}{=} d(t)$ (see the proof of lemma 3.2). Since $\varphi(t)$ is fixed and Lipschitz and $\varphi(0) = f(\varphi(-1))$ we have $|\varphi(0) - f(\varphi(t-1)| \exp(-t/\nu) \leq LL'\nu \times (t/\nu) \exp(-t/\nu) \leq LL'\nu/e \stackrel{\text{def}}{=} c_1\nu$. On the other hand $(1/\nu) \int_0^t \exp\{(s-t)/\nu\} |f(\varphi(s-1)) - f(\varphi(t-1))| ds \leq LL' \int_0^t \exp\{(s-t)/\nu\} |s-t| ds = \nu \int_0^{t/\nu} \exp(-u)|u| du \leq c_2\nu$. Therefore $d(t) \leq c\nu$, where c is a constant depending on L and L' only. Thus $\|x_\varphi - x_\varphi^\nu\|_{[0,1]} \leq c\nu$.

Now consider the solutions $x_\varphi(t)$ and $x_\varphi^\nu(t)$ for $t \in [0,1]$ as members of the space of initial functions. Denote them by $\varphi_1 \in X_I^0$ and $\psi_1 \in X_I$ respectively. Then $\|x_\varphi - x_\varphi^\nu\|_{[1,2]} \leq \|x_{\varphi_1} - x_{\varphi_1^\nu}\|_{[0,1]} + \|x_{\varphi_1}^\nu - x_{\psi_1}^\nu\|_{[0,1]}$. But one has $\|x_{\varphi_1}^\nu - x_{\psi_1}^\nu\|_{[0,1]} \leq \sup_{t\in[0,1]}\{\varphi_1(0) - \psi_1(0)| \exp(-t/\nu) + L(1 - \exp(-t/\nu)) \sup_{(0,t)} |\varphi_1(s-1) - \psi_1(s-1)|$, which implies $\|x_{\varphi_1}^\nu - x_{\psi_1}^\nu\|_{[0,1]} \leq \max\{L, 1\} |\varphi_1 - \psi_1\|_{[-1,0]}$. Since $\varphi_1 = f(\varphi)$ and both φ and f are Lipschitz, φ_1 is Lipschitz too. Therefore $\|x_{\varphi_1} - x_{\varphi_1}^\nu\|_{[0,1]} \leq c_1\nu$ again, and we have $\|x_\varphi - x_\varphi^\nu\|_{[1,2]} \leq c_1\nu + c\max\{L, 1\}c\nu = O(\nu)$. With T fixed the above arguments can be repeated, completing the proof. □

It is worth to note that assuming $f(x)$ to be Lipschitz and $\varphi \in H^\alpha$, $0 < \alpha \leq 1$, one would obtain $\|x_\varphi - x_\varphi^\nu\|_{[0,T]} = O(\nu^\alpha)$ as $\nu \to +0$ for any fixed T. We recall that $H^\alpha = \{\varphi \in X_I | |\varphi(t') - \varphi(t'')| \leq K|t' - t''|^\alpha$ for all $t', t'' \in [-1, 0], K - \text{const}\}$.

As we have noted earlier, the consistency condition $\varphi(0) = f(\varphi(-1))$ need not hold in the case $\varphi \in X_I$. Then the solution $x_\varphi(t)$ of equation (2) is in general discontinuous at each point $t = i$, $i \in \mathbb{N}$, although the solution $x_\varphi^\nu(t)$ of equation (1) is smooth for $t > 0$. Nevertheless the solutions $x_\varphi(t)$ and $x_\varphi^\nu(t)$ are close within finite time intervals for small ν, outside the discontinuity points of $x_\varphi(t)$. To be precise we define for positive T and κ the set $J_T^\kappa = [0,T]\setminus\bigcup_{i=0}^{[T+1]}[i,i+\kappa)$. where $[\cdot]$ stands for integer part of a number. Then the following holds.

Theorem 3.4. *For any $\varphi \in X_I$ and positive T,κ,ε there exist positive δ,ν_0 depending on $\varphi,T,\kappa,\varepsilon$ such that $\|x_\varphi - x_\psi^\nu\|_{J_T^\kappa} \le \varepsilon$ for all $0<\nu\le\nu_0$ provided $\psi\in X_I$ and $\|\varphi-\psi\|_{[-1,0]}\le\delta$.*

The proof is similar to that of Theorem 3.1.

Finally we note that some continuous dependence results may be proved under weaker assumptions on f and φ than in Theorems 3.1. and 3.4. We consider here the particular case of nonlinearities $f(x)$ as needed in the next chapter. Assume: (i) $f(x)$ has a finite discontinuity set Λ, is continuous on $I\setminus\Lambda$ and $\lim_{x\to x^*\pm 0} f(x)$ exists for any $x^*\in\Lambda$; (ii) $f(I\setminus\Lambda)\bigcap\Lambda=\emptyset$.

Introduce an initial function space X_I^1 by setting $X_I^1=\{\varphi\in X_I|$ all zeros of $\varphi(t)-x^*$, $x^*\in\Lambda$, are isolated$\}$.

With $\varphi\in X_I^1$, the solution $x_\varphi(t)$ of equation (2) is defined as usual by the above iteration procedure. By a solution $x_\varphi^\nu(t)$ of equation (1) we mean a continuous and piecewise continuously differentiable function satisfying the equation for all $t>0$ except at isolated points. Clearly, with given φ the corresponding solution $x_\varphi^\nu(t)$ is constructed for $t>0$ through step by step integration.

Suppose $\varphi\in X_I^1$ is fixed. We enumerate all zeros of $\varphi(t)-x^*$, $x^*\in\Lambda$, on the initial set $[-1,0]$ by $t_1<t_2<\cdots<t_N$. The number of zeros is finite according to the definition. Take any positive κ and T and set $J_T^\kappa=J_T^\kappa(\varphi)=[0,T]\setminus\bigcup_{k=1}^N\bigcup_{i=0}^{[T+1]}U_\kappa(t_k+i)$ where $U_\kappa(z)$ is the κ-neighbourhood of z. The set J_T^κ is the interval $[0,T]$ excepting κ-neighbourhoods of points t_k+i, where $0\le i\le[T+1]$, t_k is a zero of $\varphi(t)-x^*$, $x^*\in\Lambda$, $1\le\kappa\le N$. Clearly, for any $\kappa>0$ the solution $x_\varphi(t)$ is continuous on J_T^κ.

Theorem 3.5. *Let f satisfies conditions (i) and (ii). For any $\varphi\in X_I^1$ and positive T,κ,ε there exist positive δ and ν_0 depending on $\varphi,T,\kappa,\varepsilon$ such that $\|x_\varphi - x_\psi^\nu\|_{J_T^\kappa}\le\varepsilon$ for all $0<\nu\le\nu_0$ provided $\psi\in X_I^1$ and $\|\varphi-\psi\|_{[-1,0]}\le\delta$.*

Essentially the proof is similar to that of Theorem 3.1.

4. Impact of Singular Perturbations: Examples

In this chapter we consider several simple examples showing that the asymptotic behaviour as $t\to+\infty$ of solutions for equation (1) with positive ν may differ essentially from the asymptotic behaviour of solutions for equation (2) (though, as was shown in the previous chapter, the solutions of the equations are close within any sufficiently large but fixed time interval, for small $\nu>0$).

In the examples considered the function $f(x)$ is piecewise constant (hence, the map $x \to f(x)$ is not continuous). This allows us to find an explicit form for the shift operator along solutions of equation (1) and to carry out a sufficiently complete analysis of the properties of the solutions. If it is not specified otherwise the parameter ν is assumed to be positive and small throughout the chapter.

The examples show in particular that the singular term $\nu\dot{x}(t)$ may lead to both a simplification of the limit behavior for the solutions (Examples 4.1–4.3) and a complication (Example 4.4). For the latter, the attractor of the difference equation (2) consists of generalized periodic functions, whereas the attractor of the corresponding singularly perturbed equation is described by oscillating solutions governed (in a sense) by a quasi-random quantity for which the density distribution is absolutely continuous with respect to Lebesgue measure.

In the examples which follow the map f has a maximal invariant interval I. Set $X = C([-1,0], I)$. According to Theorem 2.1 (the invariance property) and its generalization to the case of the discontinuous f, for an arbitrary $\varphi \in X$ the corresponding solution satisfies $x_\varphi^\nu(t+s) \in X$, $s \in [-1,0]$, for all $t \geq 0$ and each $\nu > 0$. This allows us to consider initial conditions from X only.

Example 4.1. Consider equation (1) with f satisfying: $f(x) = a > 0$ for $x > 0$; $f(x) = -b < 0$ for $x < 0$; $f(0) = 0$.

With $f(x)$ given, the corresponding interval map has two attracting fixed points $x_1 = a$ and $x_2 = -b$ with basins $x > 0$ and $x < 0$ respectively. Both equations (1) and (2) have three constant solutions $x_0(t) = 0$ and $x_1(t) = a$, $x_2(t) = -b$ with the latter two being attracting. Any initial function $\varphi(t) > 0$ gives rise to a solution $x(t)$ satisfying $\lim_{t\to\infty} x(t) = a$. Similarly, any initial function $\varphi(t) < 0$ generates a solution with $\lim_{t\to\infty} x(t) = -b$. This is the case for both equation (1) and equation (2) (see Chaps. 1 and 2).

Equation (2) also has relaxation type solutions, and they are the typical ones. They are generated by initial conditions having both positive and negative values. Small perturbations give rise to relaxation type solutions as well.

On the other hand for equation (1) oscillatory solutions are rare. According to the following proposition almost all solutions are asymptotically constant.

Proposition 4.1. *Almost all solutions of equation (1) satisfy one of the following* $\lim_{t\to\infty} x(t) = a$ *or* $\lim_{t\to\infty} x(t) = -b$.

"Almost all" is used in the following meaning. The set of initial conditions for which the solutions have limits a or b is an open and dense subset of $X = C([-1,0], \mathbb{R})$.

Proof. First consider a set of initial functions with at most two zeros: $\varphi_u = \{\varphi \in X | \varphi(0) = \varphi(-u) = 0, \varphi(t) < 0$ for $t \in (-u, 0)$, $\varphi(t) > 0$ for $t \in [-1, -u)\}$, depending on the real parameter $u \in [0,1]$.

Integrating (1) for $t > 0$ and u given, we have $x^u(t) = a - a\exp(-t/\nu)$ for $t \in [0, 1-u]$, $x^u(1-u) \stackrel{\text{def}}{=} x_1 = a[1 - \exp\{-(1-u)/\nu\}]$; $x^u(t) = -b + (x_1 + b)\exp\{-(t - 1 + u)/\nu\}$ for $t \in [1-u, 1]$, $x^u(1) \stackrel{\text{def}}{=} x_2 = -b + (x_1 + b)\exp(-u/\nu)$; $x^u(t) = a + (x_2 - a)\exp\{-(t-1)/\nu\}$ for $t \in [1, 2-u]$.

Claim 1. With u fixed there exists a unique solution $x^u(t)$ of equation (1) generated by φ_u. The solution does not depend on the particular $\varphi \in \Phi_u$.

Since $f(x)$ takes constant values for $x < 0$ and $x > 0$ the claim is obvious.

Claim 2. Given $u \in [0,1]$ either (a) there exists $t_1 \in (0,1]$ with $x^u(t_1) = 0$, $x^u(t) > 0$ for $t \in (0, t_1)$ and $x^u(t) < 0$ for $t \in (t_1, 1]$, or (b) $x^u(t) = a + (x^u(1) - a)\exp\{-(t-1)/\nu\}$, $t \geq 1$.

Clearly, $x^u(t)$ is increasing for $t \in (0, 1-u)$ and decreasing for $t \in (1-u, 1)$. Thus, either $x^u(1) > 0$ meaning $x^u(t) > 0$ for all $t \in (0,1]$, implying $x^u(t) = a + (x^u(1) - a)\exp\{-(t-1)/\nu\}$ (Fig. 10).

Claim 3. With ν fixed there exists $u_1 > 0$ such that $x^{u_1}(t) > 0$ for $t \in (0,1)$ and $x^{u_1}(1) = 0$. For any $u \in [0, u_1)$ $x^u(t) > 0$ for all $t > 0$ (therefore $x^u(t) = a + (x^u(1) - a)\exp\{-(t-1)/\nu\}$, $t \geq 1$).

The claim is obvious and u_1 is calculated directly $u_1 = \nu \ln[(a + b)/a \exp(-1/\nu) + b)]$. The claim states the existence of a threshold value for the parameter u: for every $u < u_1$ the solution $x^u(t)$ has no zeros for $t > 0$, while for $u \geq u_1$ it has at least one zero $t_1 > 0$.

Claim 4. In case $x^u(1) < 0$ there either (a) exists $1 < t_2 < t_1 + 1$ with $x^u(t_2) = 0$, or (b) $x^u(t) < 0$ for all $t \in [1, 2]$.

The claim is obvious.

Similarly to claim 3 the following holds.

Claim 5. With ν fixed there exists $u_2 > 0$ such that $x^{u_2}(t) < 0$ for $t \in (t_1, t_2)$, $t_2 = t_1 + 1$, $x^{u_2}(t_2) = 0$. For any $u \in (u_2, 1]$ $x^u(t) < 0$ for all $t > t_1$ (therefore $x^u(t) = -b + (x^u(1 + t_1) + b)\exp\{-(t - t_1 - 1)/\nu\}$).

The claim follows as a symmetric counterpart of claim 3. Indeed, for any $u > u_1$ there always exists $t_1 > 0$ such that $x^u(t_1) = 0$, and $x^u(t) > 0$ for all $t \in (0, t_1)$ (see claim 3). Due to the symmetry arguments to those of claim 3, there is $t_1^* = \nu \ln[(a + b)/(b\exp(-1/\nu) + a)]$ and a corresponding value u_2 of the parameter u such that $x^{u_2}(t_1^* + 1) = 0$, $x^{u_2}(t) < 0$ for all $t \in (t_1^*, t_1^* + 1)$. The parameter value u_2 is a real root of the equation $1 - u_2 + \nu \ln[1 + (a/b)(1 - \exp\{(u_2 - 1)/\nu\})] = \nu \ln[(a + b)/(b\exp(-1/\nu) + a)]$.

Claims 2–5 allow to define a map $\mathcal{F}$ on the parameterized sets Φ_u in the following way. To any Φ_u, $u \in (u_1, u_2)$, there corresponds $\Phi_{u'}$ with $u' = t_2 - t_1$. In the case $u \in [0, u_1]$ we set $u' = 0$, and in the case $u \in [u_2, 1]$ we set $u' = 1$.

The map $\mathcal{F}$ induces an one-dimensional map by $u' = F(u)$. We shall find the explicit form of F next. If both t_2 and t_1 exist then it is easy to calculate:

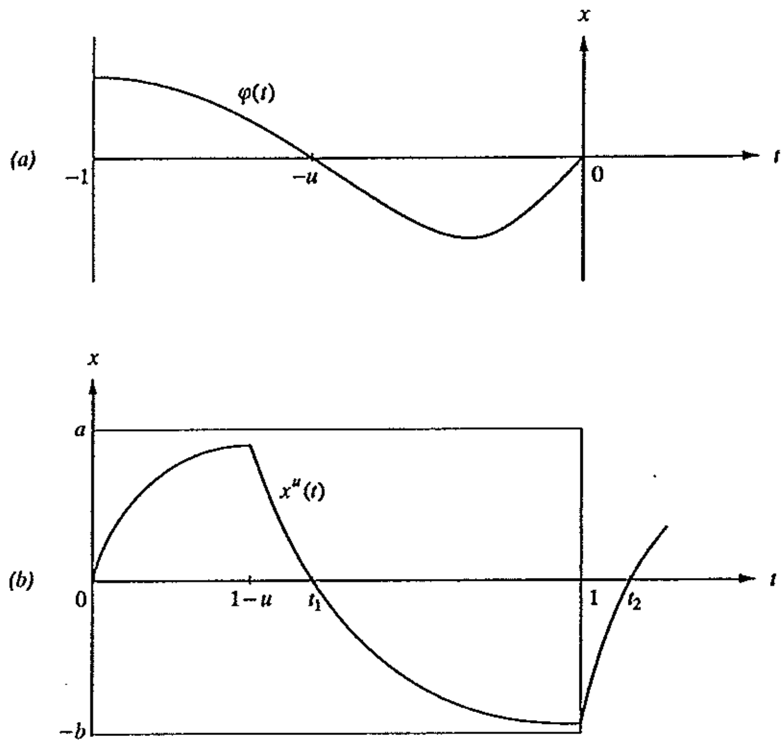

Fig. 10.

$$\Delta_1 \stackrel{\text{def}}{=} t_1 - (1-u) = \nu \ln[1 + a(1 - \exp\{(u-1)/\nu\})/b]\,,$$
$$\Delta_2 \stackrel{\text{def}}{=} t_2 - 1 = \nu \ln[1 + b/a + \exp(-1/\nu) - (b/a+1)\exp(-u/\nu)]\,.$$

In this case: $F(u) = u' = u + \Delta_2 - \Delta_1$, implying

$$F : u \to u + \nu \ln[1 + b/a + \exp(-1/\nu) - (b/a+1)\exp(-u/\nu)] - \nu \ln[1 + \frac{a}{b}(1 - \exp\frac{u-1}{\nu})]\,.$$

Changing the variable u to $z = h(u) \stackrel{\text{def}}{=} \exp(-u/\nu)$ we get the toplogically equivalent map

$$G : z \to z\frac{1 + (a/b)(1 - \exp(-1/\nu)/z)}{1 + b/a + \exp(-1/\nu) - (b/a+1)z}\,, z \in [\exp(-1/\nu), 1]\,,$$

with F and G being conjugate by $G \circ h = h \circ F$ for $u \in [0,1]$. Denoting $a/b = k$ and $a\exp(-1/\nu)/(a+b) = \varepsilon$ we obtain

$$G : z \to k\frac{z-\varepsilon}{1+\varepsilon-z}, \quad z \in [\exp(-1/\nu), 1] .$$

Note that F is defined for all $u \in [0,1]$. This implies that $z = h(u)$ varies within $[\exp(-1/\nu), 1]$ and the latter interval is the domain of G. The function $k(z-\varepsilon)/(1+\varepsilon-z)$ is defined for all real $z \neq 1+\varepsilon$ and is strictly increasing (Fig. 11). If $\nu > 0$ is small enough then there exist $z_1 < z_2$ such that $G(z_1) = \exp(-1/\nu)$, $z_1 > \exp(-1/\nu)$ and $G(z_2) = 1$, $z_2 < 1$ (by direct calculation one obtains from the formula for G : $G(1) > 1$, $G(z_1) = \exp(-1/\nu)$, $z_1 = \varepsilon[1-(k+1)/\{k^2+\varepsilon(k+1)\}] > \exp(-1/\nu)$, $0 < \nu \ll 1$. It is clear that the values z_1 and z_2 are related to u_1 and u_2 in the following way: $z_1 = \exp(-u_1/\nu)$, $z_2 = \exp(-u_2/\nu)$. According to the definition of the map F and claims 3, 5 we have to set $G(z) = \exp(-1/\nu)$, $z \leq z_1$, and $G(z) = 1$, $z \geq z_2$. Then the map G has a form as shown in Fig. 11 by continuous curve.

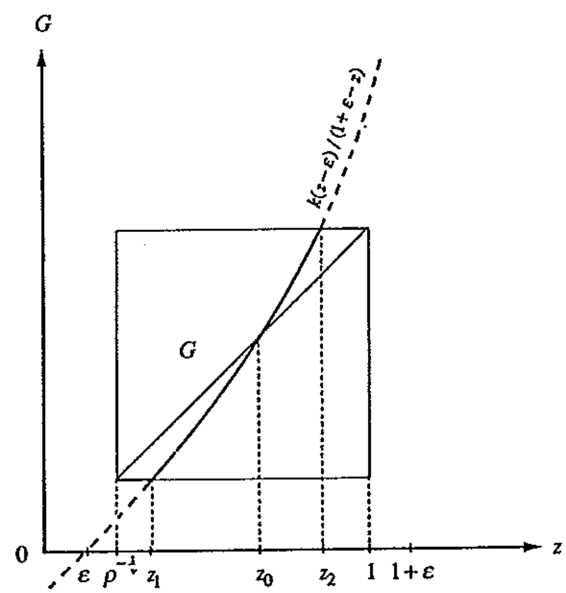

Fig. 11.

The map G is related to the dynamics given by (1) in the following way. For any particular $u \in [0,1]$ equation (1) has a unique solution $x^u(t)$, $t \geq 0$ generated by Φ_u. The distance u' between its successive zeros t_1, and t_2 (Fig. 10) is given for u by $u' = F(u)$, where $F = h^{-1} \circ G \circ h$, $h = \exp(-u/\nu)$. Therefore, if the solution $x^u(t)$ oscillates with successive zeros $t_1 < t_2 < t_3 < t_4 < \cdots$ then the distance u_n between zeros t_{2n-1} and t_{2n} is governed by G as follows: $u_n = h^{-1} \circ G^n \circ h$, where G^n is n-th iterate of G.

The map G has one repelling fixed point z_0 (Fig. 11). (The value z_0 is found as a larger (real) root of the equation $z = k(z-\varepsilon)/(1+\varepsilon - z)$). Clearly, the corresponding value $u_0 = \nu \ln(1/z_0)$ gives rise to an unstable periodic solution of the equation (1). For any $z \neq z_0$ the sequence $z_n = G^n(z)$ is monotone. There always exists a first $n_0 \in \mathbb{N}$ with either $z_{n_0} = \exp(-1/\nu)$ or $z_{n_0} = 1$. This implies monotonicity of the corresponding $u_n = \nu \ln(1/z_n)$ with $u_{n_0} = 1$ or $u_{n_0} = 0$, respectively. Recall that we set $u' = F(u) = 0$ for $u \in [0, u_1]$ and $u' = F(u) = 1$ for $u \in [u_2, 1]$. If the initial value satisfies $u \in [0, u_1] \cup [u_2, 1]$ then $x^u(t)$ has no zeros for $t > 0$, and either $x^u(t) \to a$ or $x^u(t) \to -b$ as $t \to \infty$ (see Claims 3, 5). Therefore, the solution $x^u(t)$ for which $u_{n_0} = 0$ (or $u_{n_0} = 1$) monotonically tends to a (or to $-b$) for large t.

Similar (but more complicated) calculations show that for any even $m \geq 4$ there exists precisely one unstable periodic solution with m zeros per period. Almost all other solutions are asymptotically constant. There exist infinitely many solutions which merge into the mentioned periodic solutions. Almost all small perturbations within the initial set give rise to asymptotically constant solutions. Details are found in [2].

Since the solutions depend continuously on initial conditions for the nonlinearity f of Example 4.1, the set of asymptotically constant solutions (which have limits a or b as $t \to \infty$) is open. For every initial function $\varphi \in C([-1,0], \mathbb{R})$ and arbitrary $\varepsilon > 0$ there exists an initial function $\varphi_m \in C([-1,0], \mathbb{R})$ which has an even number m of zeros on the inital set $[-1, 0]$ and is in an ε-neighbourhood of φ. Together with the previous arguments this gives density and completes the proof of Proposition 4.1.

Next, we briefly describe the second

Example 4.2. Consider equation (1) with $f(x)$ satisfying: $f(x) = a > 0$ for $x > 0$ and $f(x) = -b < 0$ for $x \geq 0$.

The corresponding interval map has a globally attracting cycle of period 2: $a \to -b \to a$ (Fig. 12). All solutions of equation (2) oscillate. Initial functions having m zeros on $[-1, 0)$ produce solutions having m zeros on each interval $[k, k+1)$, $k \in \mathbb{N}$. This follows from the fact that each solution of equation (2) is obtained by successive iterations of the corresponding initial function (see Chap. 2).

For equation (1) the situation is different as the following proposition shows.

Proposition 4.2. *Equation (1) has an asymptotically stable periodic solution generated by any initial function* $\varphi(t)$ *satisfying* $\varphi(t) > 0$ *for all* $t \in [-1, 0]$ *(or* $\varphi(t) < 0$*). Almost all other inital functions generate solutions which merge into this periodic solution.*

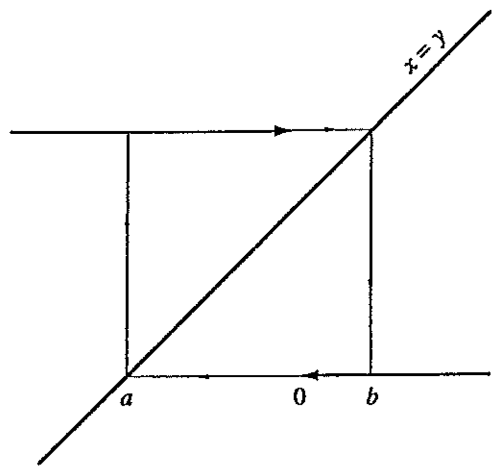

Fig. 12.

"Almost all" means that the corresponding set is open and dense in $X = C([-1,0],\mathbb{R})$. The details of the proof can be found in [2].

Thus both Examples 4.1 and 4.2 show that the term $\nu\dot{x}(t)$ added to equation (2) leads to the disappearance of solutions having more than one zero on intervals of length 1.

A similar phenomenon also takes place for the equation $\dot{x}(t) = g(x(t-1))$ if the nonlinearity g satisfies the negative feedback condition $xg(x) < 0$, $x \neq 0$. Strong results for the case of a general nonlinearity $g(x)$ are found in [60].

Example 4.3. Fix a constant $h \in (0,1)$ and consider equation (1) with $f(x) = f_h(x)$ where nonlinearity $f_h(x)$ is defined by $f_h(x) \equiv 0$ for $|x| \leq h$ and $f_h(x) = -\text{sign}(x)$ for $|x| > h$ (Fig. 13).

Proposition 4.3 *Any solution of equation (1) satisfies* $\lim_{t\to\infty} x^\nu_\varphi(t) = 0$ *provided* $h \geq 1/2$.

Note that $x = 0$ is an attracting fixed point of the map f_h in the interval $[-1,1]$ with basin $|x| < h$ and $1 \to -1 \to 1$ is an attracting cycle with basin $|x| > h$. This means that equation (2) has both asymptotically constant solutions and relaxation type solutions. At the same time, if $h \geq 1/2$ all solutions of equation (1) are asymptotically constant.

Proof of the proposition.. It is clear that any initial function $\varphi(t)$ satisfying $|\varphi(t) \leq h$ for $t \in [-1,0]$ generates the solution $x^\nu_\varphi(t) = \varphi(0)\exp(-t/\nu)$ of equation (1) for which $\lim_{t\to\infty} x^\nu_\varphi(t) = 0$. □

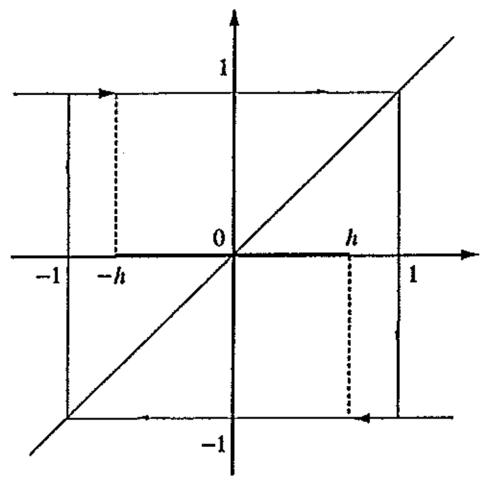

Fig. 13.

Introduce a family of sets of initial functions depending on the real parameter u, $0 \le u \le 1$, by $\varphi_u = \{\varphi \in X | \varphi(0) = \varphi(-u) = h,\ \varphi(t) > h$ for $t \in (-u, 0)$, $|\varphi(t)| \le h$ for $t \in [-1, -u)\}$.

Claim 1. With u fixed there exists a unique solution $x^u(t)$ of equation (1) generated by φ_u. The solution does not depend on the particular choice of $\varphi \in \varphi_u$.

Claim 1 is obvious.

Claim 2. Given $u \in [0,1]$ there either (a) exists a $t_1 \in (0,1)$ with $x^u(t_1) = -h$, or (b) $x^u(t) = c\exp\{-(t-1)/\nu\}$, $t \ge 1$.

Integrating (1) on $[0,1]$ we have: $x^u(t) = h\exp(-t/\nu)$ for $t \in [0, 1-u]$, $x^u(t) = -1 + (x_1 + 1)\exp\{-(t - 1 + u)/\nu\}$ for $t \in [1-u, 1]$ with $x_1 = h\exp\{(u-1)/\nu\}$. Since $x^u(t)$ is monotone on $[0,1]$ the claim follows.

Claim 3. In the case (a) of claim 2 there exists $t_2 > 1$ with $x^u(t_2) = -h$, $x^u(t) < -h$ for all $t \in (t_1, t_2)$.

Since $x^u(t) \in [-h, h]$ for $t \in [0, t_1]$ and $x^u(t) < -h$ for $t \in (t_1, 1]$, $x^u(t)$ is monotonically increasing for $t \ge 1$ with $x^u(t) \ge x^u(1)\exp\{-(t-1)/\nu\}$. This implies the existence of t_2 (Fig. 14).

It is natural to introduce a map $\mathcal{F}$ on the family φ_u in the following way. To given $u \in [0,1]$ and φ_u there corresponds $v \in [0,1]$ and φ_v such that $v = t_2 - t_1$. If the second zero t_2 of $x^u(t) + h$ does not exist we put $v = 0$ (the latter case means that the solution $x^u(t)$ has the form $x^u(t) = x^u(1)\exp\{-(t-1)/\nu\}, t \ge 1$ and goes to zero as $t \to \infty$. So do the solutions generated by φ_0).

The map $\mathcal{F}$ induces an one-dimensional map $F : u \to v$. We shall calculate the explicit form of F.

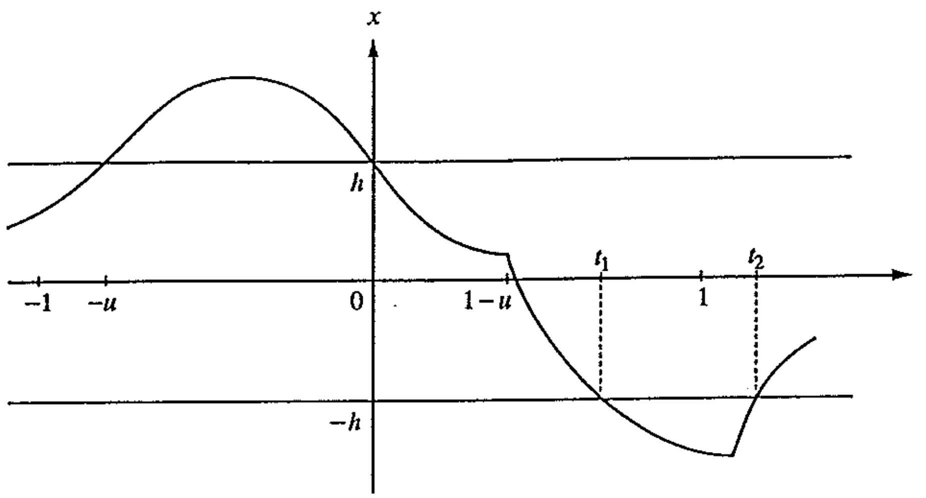

Fig. 14.

Denote $\Delta_1 = t_1 - (1-u)$. Then $-h = -1 + (1+x_1)\exp(-\Delta_1/\nu)$. This implies $\Delta_1 = \nu\ln[(1+h\exp\{(u-1)/\nu\})/(1-h)]$. Denoting $\Delta_2 = t_2 - 1$ we have $x_2\exp(-\Delta_2/\nu) = -h$ in the case $t_2 \le t_1+1$, with $x_2 = -1+(x_1+1)\exp(-u/\nu)$ implying $\Delta_2 = \nu\ln[\{1-h\exp(-1/\nu)-\exp(-u/\nu)\}/h]$. In the case $t_2 > t_1+1$ we set $v = F(u) = 1$. Let $F_1(u) = u+\nu\ln[1-\beta-\exp(-u/\nu)]/[\alpha(1+\beta\exp(u/\nu))]$ for all u where $F(u)$ exists. Here $\alpha = h/(1-h)$, $\beta = h\exp(-1/\nu)$. Then we have for $u \in [0,1]$:

$$F : u \to \begin{cases} 0 & , \text{ if } \quad F_1(u) < 0 \\ u + \nu\ln\frac{1-\beta-\exp(-u/\nu)}{\alpha(1+\beta\exp(u/\nu))} & , \text{ if } \quad F_1(u) \in [0,1] \\ 1 & , \text{ if } \quad F_1(u) > 1\,. \end{cases}$$

Introducing a new variable by $z = h(z) \overset{\text{def}}{=} \exp(-u/\nu)$ and denoting $G_1(z) = \alpha(z+\beta)/[1-(z+\beta)]$ we get the equivalent map

$$G : z \to \begin{cases} 1 & , \text{ if } \quad G_1(z) > 1 \\ \alpha\frac{z+\beta}{1-(z+\beta} & , \text{ if } \quad G_1(z) \in [\exp(-1/\nu), 1]\,, \\ \exp(-1/\nu) & , \text{ if } \quad G_1(z) < \exp(-1/\nu)\,, \end{cases}$$

where F and G are conjugate by $G \circ h = h \circ F$. The map G from the interval $[\exp(-1/\nu), 1]$ into itself and the dynamics of solutions of equation (1) for initial functions from φ_u, $0 \le u \le 1$, are related in the following way. Fix $u \in [0,1]$, φ_u, and consider the corresponding solution $x^u(t)$ of equation (1). $x^u(t)$ either oscillates around $x = 0$ or tends to zero as $t \to +\infty$. Enumerate successive zeros of the function $z(t) = x^u(t) - h$ by $0 < t_1 < t_2 < t_3 < t_4 < \cdots$

(which are finite or infinite in number) and let $u_n = t_{2n} - t_{2n-1}$. Then u_n and u_{n-1} are related by $u_n = F^2(u_{n-1})$. Therefore $u_n = F^{2n}(u)$. Since $F = h^{-1} \circ G \circ h$ we have $u_n = h^{-1} \circ G^{2n} \circ h(u)$. Here f^{2n} and G^{2n} are the $2n$-th iterates of the maps. Thus, the dynamics of solutions for initial functions from φ_u is completely determined by the dynamics given by the map G.

The graph of $G(z)$ and small $\nu > 0$ is shown in Fig. 15a for $\alpha < 1$ and in Fig. 15b for $\alpha > 1$.

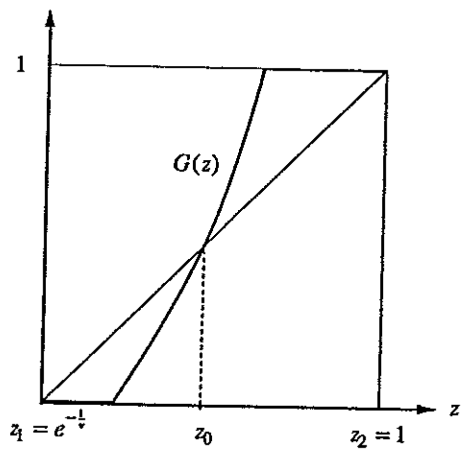

Fig. 15a.

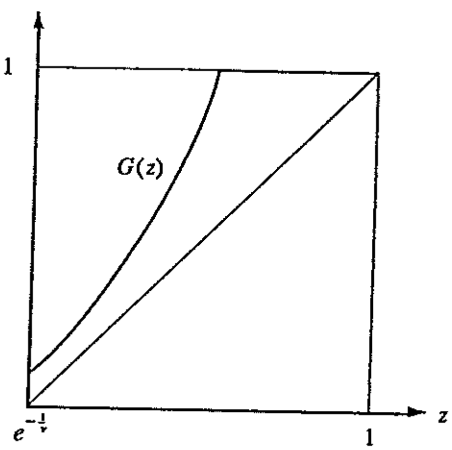

Fig. 15b.

The case $\alpha < 1$ corresponds to $h < 1/2$. There exists a positive ν_0 such that for all $0 < \nu \leq \nu_0$ the map G has three fixed points $z_1 = \exp(-1/\nu)$, $z_2 = 1$, z_0 where z_0 is the root of the equation $G(z) = z$ which belongs to the interval (z_1, z_2). The fixed points z_1 and z_2 are attracting with domains of attraction $[z_1, z_0)$ and $(z_0, z_2]$ respectively, while the fixed point z_0 is repelling. Therefore, in this case equation (1) has two periodic solutions corresponding to $u = 1$ and $u = \nu \ln(1/z_0)$ (the value $u = 0$ corresponds to the solution $x(t) = h\exp(-t/\nu)$ which is attracted by the trivial periodic solution $x(t) \equiv 0$). The first one attracts all solutions from φ_u if $u > \nu\ln(1/z_0)$. If $u < \nu\ln(1/z_0)$ the corresponding solution $x^u(t)$ tends to zero exponentially as $t \to +\infty$.

The case $\alpha \geq 1$ corresponds to $h \geq 1/2$. For any $\nu > 0$, the map G has the only attracting fixed point $z = 1$ which corresponds to $u = 0$. Clearly, for a given $u \in (0, 1]$ we have $F(u) < u$ and there exists an integer $n_0 = n_0(u)$ with $F^n(u) = 0$ for all $n \geq n_0$ (Fig. 15b). Thus for any $\varphi \in \varphi_u$ there exists $t_0 = t_0(u)$ such that the solution $x_\varphi^\nu(t)$ is of the form $x_\varphi^\nu(t) = x_\varphi^\nu(t_0)\exp\{-(t-t_0)/\nu\}$ for $t \geq t_0$ and hence tends to zero.

To show that $\lim_{t\to\infty} x_\varphi^\nu(t) = 0$ for any $\varphi \in X$ in the case $h \geq 1/2$ we make use of the following observation. Take an arbitrary $\varphi \in X$ and consider the corresponding solution $x^{\nu}_{\varphi}(t)$. If $|x^{\nu}_{\varphi}(t)| \le h$ then $x^{\nu}_{\varphi}(t) \to 0$ as $t \to \infty$. Therefore, we have to consider initial conditions for which the corresponding solutions oscillate with respect to both $x = h$ and $x = -h$. In this latter case there always exists a sequence (possibly finite) $0 < s_1 < s_2 < s_3 < \cdots$ such that $x^{\nu}_{\varphi}(s_{4k-3}) = x^{\nu}_{\varphi}(s_{4k}) = h$, $x^{\nu}_{\varphi}(s_{4k-2}) = x^{\nu}_{\varphi}(s_{4k-1}) = -h$, $x^{\nu}_{\varphi}(t) \in [-h, h]$ for all $s \in [s_{2k-1}, s_{2k}]$, and $x^{\nu}_{\varphi}(t) \le h$ for all $t \in [s_{4k-3}, s_{4k}]$, $x^{\nu}_{\varphi}(t) \ge -h$ for all $t \in [s_{4k-2}, s_{4k-1}]$, $k \in \mathbb{N}$. Consider $\psi(s) = x^{\nu}_{\varphi}(t_1 + s)$, $s \in [-1, 0]$ as an element of X. Let $u = \sup\{v \in [0,1] : |\psi(s)| \le h$ for all $s \in [-1, -v]\}$. The value $-u$ is the largest point on the interval $[-1, 0]$ such that $|\psi(s)| \le h$ for all $s \in [-1, -u]$. Compare now the solution $x^{\nu}_{\psi}(t)$, $t \ge 0$, and the solution $x^{u}(t)$ constructed by φ_u with given u. Define $w_k = s_{2k+1} - s_{2k}$ if s_{2k+1} exists and $w_k = 0$ otherwise, $k \in \mathbb{N}$ (in the latter case the sequence (s_k) is finite). Direct calculation shows that $s_3 - s_2 = w_1 \le t_2 - t_1 = F(u)$, where t_1, t_2 are the first and second zeros of $x^u(t)$ constructed above. Induction arguments show that $w_k = s_{2k+1} - s_{2k} \le F^k(u)$, $k \in \mathbb{N}$. Since $F^k(u) = 0\, \forall k \ge k_0$ for some $k_0 \in \mathbb{N}$ we have $w_k = 0\, \forall k \ge n_0$ for some positive integer $n_0 \le k_0$. This implies that the sequence (s_k) is finite and therefore $x^{\nu}_{\varphi}(t) = c \exp\{-(t - t_0)/\nu\}$ for some $c, t_0 > 0$.

Example 4.4. Consider equation (1) with $f(x)$ given by $f(x) = 1$ for $x > h$, $0 < h < 1$, $f(x) = a > 1$ for $0 < x \le h$, $f(0) = 0$, $f(x) = -f(-x)$ for $x < 0$ (Fig. 16).

Proposition 4.4. *For any positive integer n there exists an open subset of the parameter space $\{(a, h),\ a > 1,\ 0 < h < 1\}$ such that for any particular choice of a and h from this subset and any sufficiently small $0 < \nu < \nu_0 = \nu_0(h, a)$ the corresponding equation (1) has an asymptotically stable periodic solution with period $2n + O(\nu)$, $\nu \to +0$.*

Proposition 4.5. *There exists an open subset of the parameter space such that for any particular choice of (a, h) from this subset and any sufficiently small $\nu (0 < \nu \le \nu_0(h, a))$ the corresponding equation (1) has a set of solutions such that subsequent maxima (or distances between zeros) behave quasi-randomly (the associated probability density is absolutely continuous with respect to Lebesgue measure).*

Given $f(x)$ for any $0 < h < 1$ and $a > 1$ the interval map f has a globally attracting cycle of period two formed by the points -1 and 1. The corresponding difference equation (2) has relaxation type solutions which are two-periodic for $t > 0$. Nevertheless, the asymptotic behavior of the solution for equation (1) is much more complicated as the propositions indicate.

We briefly sketch an approach to the proofs of the two propositions, as given in [1, 3] (see also [40]). Consider the set Φ of initial functions defined by

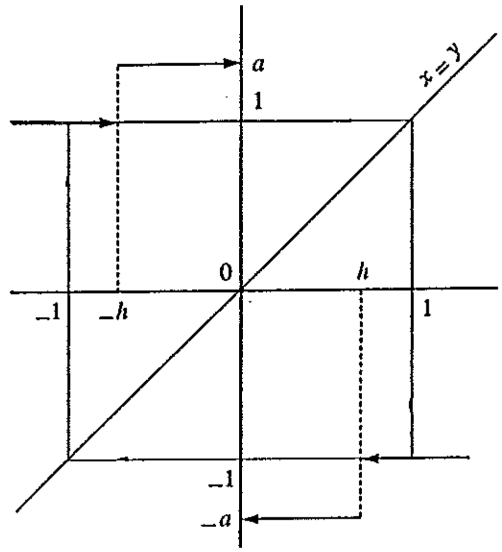

Fig. 16.

$\Phi = \{\varphi \in X | \varphi(-1) = h$, $\varphi(t)$ is unimodal on $[-1,0]$, that is, there exists a $t' \in (-1,0)$ such that $\varphi(t)$ is increasing on $(-1,t')$ and decrasing on $(t',0)\}$.

Straightforward calculations show that for ν sufficiently small and any $\varphi \in \Phi$, the corresponding solution $x^{\nu}_{\varphi}(t)$ has the property: there exists $t_1 > 0$ such that $x^{\nu}_{\varphi}(t_1+t)$, $t \in [-1,0]$ belongs to φ. This allows us to define a map $\mathcal{F}$ on Φ by $\mathcal{F}(\varphi) = \tilde{\varphi}$, $\tilde{\varphi}(t) = x^{\nu}_{\varphi}(t_1+t)$, $t \in [-1,0]$. Next consider the set $\Psi = \mathcal{F}(\Phi)$. Functions in Ψ depend on the real parameter $z = \varphi(0)$. Again, to any $\psi \in \Psi$ there corresponds an unique $\tilde{\psi} \in \Psi$ given by $\tilde{\varphi} = \mathcal{F}(\psi)$. Now $\mathcal{F}$ on Ψ induces an one-dimensional map of the parameter set $G :\mapsto z \to \tilde{z}$, which turns out to be piecewise Moebius and continuous. Simple, but technically rather complicated, calculations allow us to find the explicit form of G. All details can be found in the mentioned paper [3]. In particular, there exists a parameter subset such that G has a slope greater than 1 at each point of an invariant interval (Fig. 17). According to the theory of interval maps, this implies the existence of an invariant measure μ which is absolutely continuous with respect to Lebesque measure λ (that is $\mu(A) = \mu(G^{-1}(A))$ for every measurable $A \subset I$).

Similarly, for any n there exists an (a,h)-parameter subset for which equation (1) has an asymptotically stable periodic solution of period $2n + O(\nu)$ (ν is small) [1]. The results are based on the analysis of parametrized families of the obtained piecewise Moebius maps G [34].

We note that the asymptotic behaviors described by Propositions 4.4 and 4.5 occur in a δ-neigbourhood (in the Hausdorff metric) of the generalized peri-

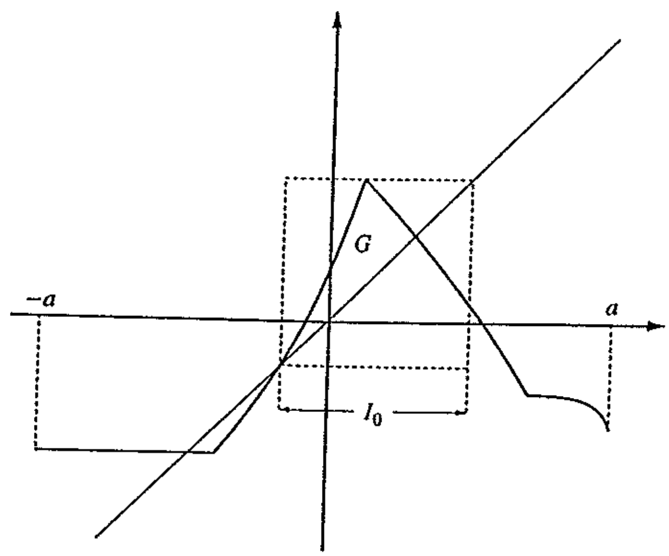

Fig. 17.

odic solution $p_0(t)$ of equation (2) defined by $p_0(t) = 1$ for $t \in (0,1)$, $p_0(t) = -1$ for $t \in (1,2)$, $p_0(0) = p_0(1) = [-a, a]$, $p_0(t)$ continued periodically outside $[0,2)$.

The following should be also noted. The induced interval map for the shift operator along solutions in the cases considered in Example 4.4 (and in many others examples in recent publications; see, e.g. [18–23, 40, 48, 61]) has Cantor sets or (as in the case of Proposition 4.4) closed intervals as invariant sets. The map on these sets is transitive (there exists a dense trajectory). Moreover it is mixing (a map G is said to be mixing on an invariant set F, if for any open (with respect to F) subset U there exist positive integers m and k such that $G^j(\cup_{i=1}^{m-1} G^i(U)) = F$ for $j \geq k$). The above sets contain trajectories (Liapunov unstable ones) with diverse asymptotic behavior: periodic trajectories with arbitrarily large periods; recurrent trajectories (for which the ω-limit sets are minimal sets different form cycles; there exists a continuum of such sets and they are pairwise disjoint Cantor-like sets); and simply Poisson stable trajectories (they are almost all trajectories), etc. Each such trajectory, as we know, gives rise to an unstable solution of equation (1) with a corresponding asymptotic behvior.

Since a set with the mixing property is invariant, every point of it has at least one preimage which belongs to the set. Therefore, any trajectory on the set may be prolonged for negative n (usually in several ways) to obtain a two-sided trajectory. Such two-sided trajectories may have the same asymptotic behaviors for both $n \to \infty$ and $n \to -\infty$, or different ones. In particular, in this way we obtain homoclinic trajectories, and each of them is attracted by a periodic trajectory or by a fixed point as $n \to \infty$ and $n \to -\infty$. To every two-sided trajectory there corresponds a solution of equation (1) defined for all $t \in \mathbb{R}$ (in fact a family of solutions differing by a shift in time). In particular, there exist solutions (homoclinic ones) modeling solitons (the corresponding trajectory for the interval map is homoclinic to a fixed point).

Every homoclinic trajectory on an invariant set with the mixing property is known to attract a continuum of trajectories as $n \to \infty$ [54, 55]. Hence, there exists a continuum of different solutions of equation (1), defined for $t \geq 0$ only (and differing not only by a shift along t). Every such solution will reproduce the original homoclinic solution, as $t \to \infty$, with increasing accuracy on time intervals of increasing length. In particular each such solution of equation (1), corresponding to a one-dimensional trajectory which is homoclinic to a fixed point, simulates a sequence of single waves (solitons) scattering away as $t \to +\infty$.

In the present paper we do not deal with bifurcation problems for equation (1) when the nonlinearity $f(x)$ is parameter dependent. Bifurcations for periodic solutions and possible paths of transition to chaos are studies extensively now in different classes of dynamical systems. The problem seems to be difficult for equation (1) and is not studied widely (some numerical results may be found in [6]).

In this situation the study of bifurcation problems for relatively simple examples seems to be worthwhile. (In particular, it is of interest what happens with the dynamics of solutions for example 4.4 as the parameters a, h vary, and how chaos may appear there). We shall not get into more detail but would like to conclude with a remark. If the shift operator $\mathcal{F}^t$ on a subset of solutions is reducible to an one-dimensional map G which is continuous and piecewise Moebius then its Schwartz derivative $S(G) = G'''/G' - 3/2(G''/G')^2$ equals zero (for all points where it exists). This implies [34] that period doubling bifurcations for the map G may occur only finitely many times. The scenary of the bifurcation itself is the following (see Fig. 18). An attracting cycle of period n (a fixed point on Fig. 18a) while losing stability is replaced by a parameterized family of cycles of period n (Fig. 18 (b)). After this an attracting cycle of period $2n$ (Fig. 18 (b)), or repelling cycle of period $2n$ (Fig. 18 (c) may appear. In the second case a local chaos in the vicinity of the $2n$ cycle appears. The cycle of period n itself becomes unstable in every case.

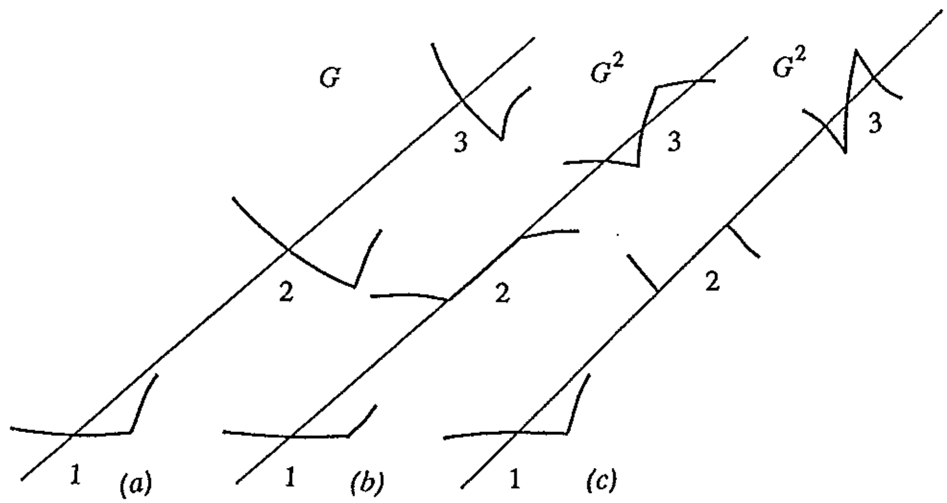

Fig. 18.

5. Attractors of Interval Maps and Asymptotic Behavior of Solutions

In this chapter we try to show the role which is played by the attractors of interval maps with large domains of immediate attraction for the asymptotic behavior of the solutions of equation (1).

Consider first a simple but important generalization of the previous Example 4.3.

Example 5.1. Fix $h \in (0,1)$ and consider equation (1) with $f(x) = f_0(x)$, where $f_0(x)$ is smooth, $f_0(x) = 0$ for $|x| \leq h$, $|f_0(x)| < 1$ for $|x| > h$ and $x = 0$ is the only fixed point of the map f_0 (a particular $f_0(x)$ is shown in Fig. 19).

Proposition 5.1. *In the case $h \geq 1/2$ every solution of equation (1) satisfies* $\lim_{t\to\infty} x^{\nu}_{\varphi}(t) = 0\, (\forall \nu > 0)$.

Proof. As in Example 4.2 we introduce a family of sets of initial functions by $\varphi_u = \{\varphi \in X | \varphi(0) = \varphi(-u) = h,\, \varphi(t) > h,\, t \in (-u, 0),\, |\varphi(t)| \leq h,\, t \in [-1, -u]\}$ depending on a real parameter $u \in [0,1]$. Given u and a particular $\varphi \in \varphi_u$ there exists an unique solution $x^u(t)$ of equation (1) defined for all $t \geq 0$.

Claim. For any u there either, (a) exist $t_1 \in (0,1)$ and $t_2 > 1$ with $x^u(t_1) = x^u(t_2) = -h$, $x^u(t) \in [-h, h]$ for $t \in [0, t_1]$ and $x^u(t) < -h$ for $t \in (t_1, t_2)$, or (b) $x^u(t) = c\exp\{-(t-1)/\nu\}$, $t \geq 1$.

The proof is quite similar as in the case of claims 2 and 3 of Example 4.2.

Denoting $w = t_2 - t_1$ we have an induced interval map F_1 on the parameter set $[0,1]$ with $F_1(u) = w$. In the case that t_1 does not exist or $t_2 = t_1$ we set $w =$

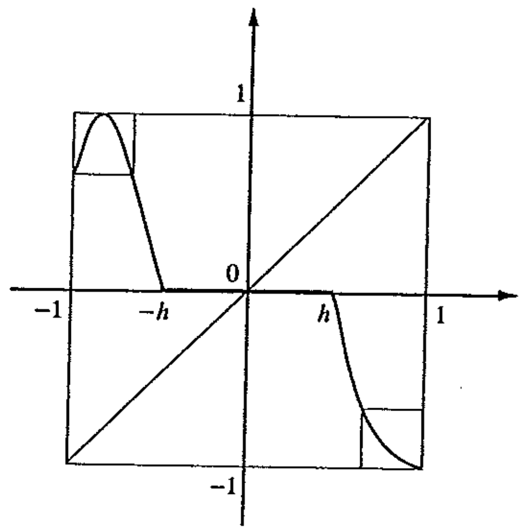

Fig. 19.

$F_1(u) = 0$. Compare now w and v obtained in Example 4.3. Since $|f_0(x)| \leq 1$ for $|x| > h$ and $f_0(x) \equiv f_h(x)$ for $|x| \leq h$ we have $w \leq v = F(u)$, where $F(u)$ is the same as in Example 4.3 (Fig. 20). Because of the monotonicity of F we conclude $F_1^n(u) \leq F^n(u)$. The exact form of $F(u)$ (see Fig. 15b) shows us that for any u there exists a positive integer $n_0 = n_0(h, u)$ with $F_1^n(u) = 0$ for all $n \geq n_0$. This implies the existence of t_0 such that $|x^u(t)| \leq h$ for all $t \geq t_0$. The proposition is proved. □

Now we observe that generally $f_0(x)$ may be defined on $\{x | h < |x| < 1\}$, $h > \frac{1}{2}$ in an arbitrary way. In particular, f_0 can have an invariant subinterval with an attracting cycle of period two, a repelling cycle of period two and no other cycles on it (and the already existing fixed point $x = 0$). This guarantees that relaxation type solutions are typical for equation (1). In addition we can require f_0 to have periodic points with periods $(2k+1)2^m$, k, m positive integers. The latter will ensure the existence of turbulent type solutions for equation (1) (see Chap. 1 for the details). An example of f_0 is shown in Fig. 19).

The proposition says, that for $h \geq 1/2$ all solutions of equation (1) are asymptotically constant, notwithstanding the particular form of $f_0(x)$ outside $[-h, h]$. For small positive ν the continuous dependence results of Chap. 3 imply that the solutions of equation (1) follow the solutions of equation (2) within finite time interval (which is the larger the smaller ν is). Then, after some transient time interval, they begin to decrease to zero exponentially. The duration of the transient state my be estimated generically as $O(1/\nu)$).

Thus the example suggests that the main factor to the determination of the asymptotic behaviour of solutions of equation (1) may be an attractor having

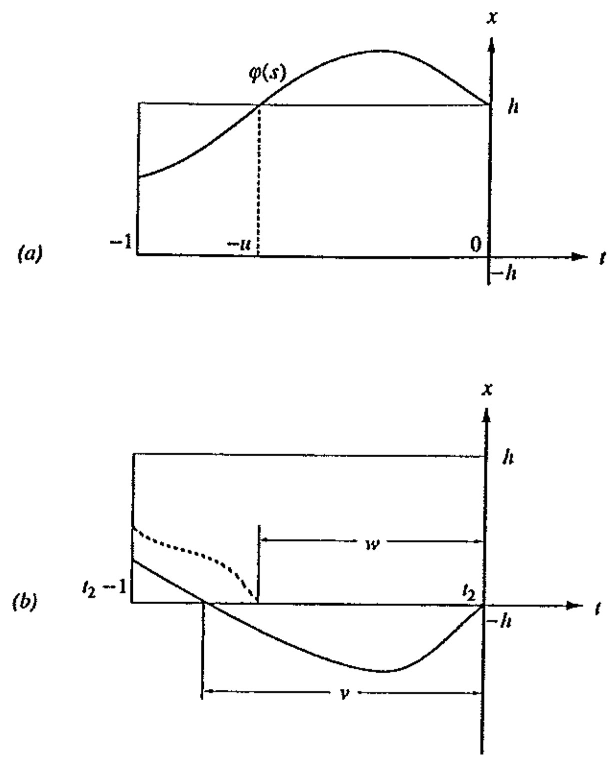

Fig. 20.

a large domain of immediate attraction compared with the remaining part of the invariant interval. The following theorem shows that this is the typical situation.

Theorem 5.1. *Suppose the map f of the compact interval I into itself has a compact invariant subinterval I_0 with immediate attraction domain $J_0 : I_0 \subset J_0 \subset I$, $f^k(x) \to I_0$ as $k \to \infty$ for any $x \in J_0$. If the set $I \backslash J_0$ does not contain fixed points and the value $meas(I \backslash J_0)/meas(J_0 \backslash I_0)$ is small enough then for every solution $x(t)$ of equation (1) there exists a time $t_0 = t_0(x, \nu)$ such that $x(t) \in J_0$ for all $t \geq t_0$. Moreover,* $\inf I_0 \leq \lim_{t\to\infty} \inf x(t) \leq \lim_{t\to\infty} \sup x(t) \leq \sup I_0$.

Remark. If $x(t) \in J_0$ for a unit time interval then one immediately obtains the required conclusion $\lim_{t\to\infty} \inf x(t) \geq \inf I_0$ and $\lim_{t\to\infty} \sup x(t) \leq \sup I_0$. This follows from Theorem 2.3.

Therefore, we only have to show that every solution of equation (1) satisfies $x(t) \in J_0$ for $t \in [s-1, s]$ and some (sufficiently large) s.

Remark. Suppose that no endpoint of the interval I_0 is a fixed point. Then there exists a positive number $\delta = \delta(f, \nu)$ such that $\inf I_0 + \delta \leq \lim_{t\to\infty} \inf x(t) \leq \lim_{t\to\infty} \sup x(t) \leq \sup I_0 - \delta$ for every solution $x(t)$ of equation (1). This follows from Theorem 2.5.

We give a proof of Theorem 5.2 using several auxilary propositions. They are listed below as Lemmas 5.1–5.3. The proofs of these Lemmas are technical and only outlined her. The proof of Lemma 5.2 is similar to the considerations of example 5.1, and the proof of Lemma 5.3 is similar to the arguments given in example 4.3 (case $h > 1/2$). For more details see [27, 28].

Let $x_0 \in I$ be arbitrary and consider a solution $x(t)$ of equation (1). We say that $x(t)$ oscillates with respect to x_0 if $x(t) - x_0$ has zeros for $t \geq 0$.

Under the assumptions of Theorem 5.1 there exist three intervals $I \supset J_0 \supset I_0$ which are invariant under f. Denote $I = [a, b]$, $\bar{J}_0 = [a_0, b_0]$, $I_0 = [a_1, b_1]$.

Lemma 5.1. *Every solution $x(t)$ of equation (1) either oscillates with respect to both a_0 and b_0 or satisfies $x(t) \in J_0$, $t \geq 0$.*

The statement of the Lemma is evident. If $x(t)$ does not oscillate then $x(t) \in \bar{J}_0$, $t \in [0, 1]$. Due to the invariance property (Theorem 2.1) we have $x(t) \in \bar{J}_0$ for all $t \geq 0$.

Lemma 5.1 allows us to consider only those solutions which oscillate with respect to both a_0 and b_0.

Now introduce a set Φ_u of initial functions depending on the real parameter $u \in [0, 1]$ by $\Phi_u = \{\varphi \in X_I | \varphi(0) = \varphi(-u) = a_0,\ \varphi(s) \in J_0,\ s \in [-1, -u)\}$. Let $x^u(t)$, $t \geq 0$ be a solution of equation (1) constructed by a particular $\varphi \in \Phi_u$ for a given u. Since we consider solutions which oscillate around a_0 and b_0, there always exist $t_1 < 1$ and $t_2 > 1$ such that $x^u(t_1) = x^u(t_2) = b_0$, $x^u(t) < b_0$ for $t \in [0, t_1)$ (a particular case is shown in Fig. 21). Consider now $x^u(t_2 + s) = \psi(s)$, $s \in [-1, 0]$ as an element of X_I. Then, similarly, we may assume that there exist $t_3 > t_2$ and $t_4 > t_2 + 1$ such that $x^u(t_3) = x^u(t_4) = a_0$, $x^u(t) > a_0$ for $t \in [t_2, t_3,)]$ (Fig. 21). Consider again $x^u(t_4 + u) = \tilde{\varphi}(s)$ as an element of X_I. It is clear that $\tilde{\varphi}(s)$ belongs to some $\Phi_{u'}$ with u' given by $u' = t_4 - t_3$. Define now a one-dimensional map $\mathcal{F}$ on the parameter set $\{u : 0 \leq u \leq 1\}$ by $\mathcal{F} : u \to u'$. In the case that zeros t_3, t_4 do not exist for some u and $\varphi \in \Phi_u$ we set $\mathcal{F}(u) = 0$. Due to the continuous dependence of solutions of equation (1) on initial conditions it is clear that for every fixed ν there exists $u_0 = u_0(f, \nu) \in (0, 1)$ such that $\mathcal{F}(u) = 0$ for every $0 \leq u \leq u_0$ and arbitrary $\varphi \in \Phi_u$.

Note that in general for every fixed u the value u' depends on the particular choice of $\varphi \in \Phi_u$. The following lemma shows that there exists an one-dimensional map to which the mapping $\mathcal{F}$ is subjected.

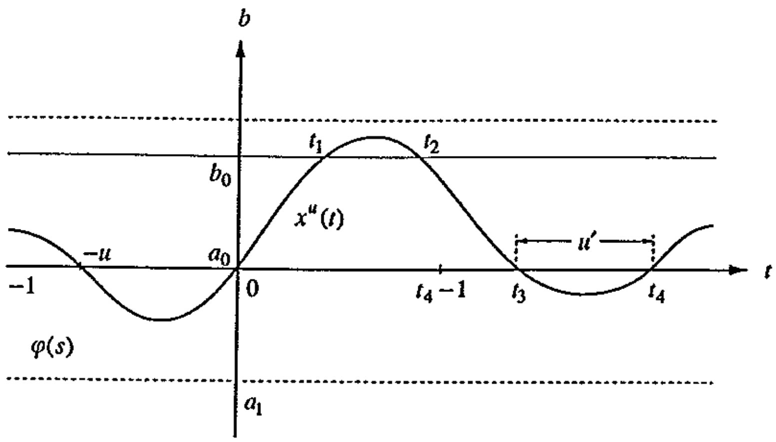

Fig. 21.

Lemma 5.2. *For $f(x)$ given and $meas(I\backslash J_0)/meas(J_0\backslash I_0)$ small, there exists an one-dimensional map F of the interval $[0,1]$ into itself which majorizes the map $\mathcal{F}$ in the following sense: $F(u) > u'$ for every particular $\varphi \in \Phi_u$ and arbitrary $u \in [0,1]$.*

The proof of the lemma is based on the properties of solutions of equation (1) with a step function $f^*(x)$ which is constructed on the basis of a given nonlinearity $f(x)$. It is similar to the considerations of Example 5.1. We suppose that $f(x)$ is fixed an continue with the construction of the function $f^*(x)$ which we will need.

We have the three invariant intervals $I = [a,b]$, $\bar{J_0} = [a_0, b_0]$ and $I_0 = [a_1, b_1]$ which satisfy $I \supset J_0 \supset I_0$. Now $f^*(x)$ is defined as follows

$$f^* = \begin{cases} b,\ x \in [a, a_0] \\ a,\ x \in [b_0, b] \\ b_1,\ x \in [a_1, c) \\ a_1,\ x \in (c, b_1] \qquad \text{for some} \quad c \in (a_1, b_1) \\ \text{an arbitrary step functions satisfying:} \\ x < f^*(x) \leq b_0,\ f^*(x) \not\equiv b_0,\ x \in (a_0, a_1) \\ x > f^*(x) \geq a_0,\ f^*(x) \not\equiv a_0,\ x \in (b_1, b_0) \end{cases} .$$

For a $f(x)$ and the three intervals $I \supset J_0 \supset I_0$ given, $f^*(x)$ is chosen to satisfy $f^*(x) \geq f(x)$ for $x \in (a_0, a_1)$ and $f^*(x) \leq f(x)$ for $x \in (b_1, b_0)$. Then, due to the construction one has $f^*(x) \geq f(x)$ for $x \leq c$ and $f^*(x) \leq f(x)$ for $x > c$. A particular $f(x)$ and its $f^*(x)$ are shown in Fig. 22.

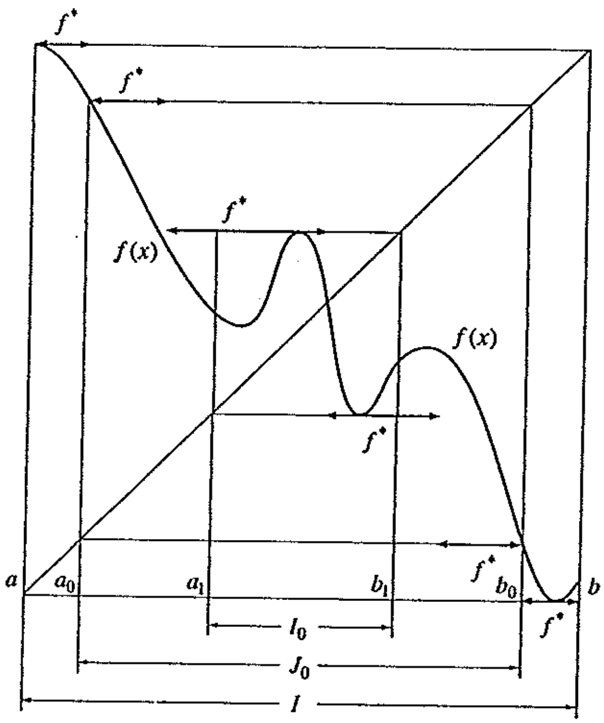

Fig. 22.

A technical calculation shows that, for every $u \in [0, 1]$ and particular $\varphi \in \Phi_u$ the map $\mathcal{F}$ depending on f and f^* has the property $u'(f^*) > u'(f)$, where $u' = t_4 - t_3$ is defined above. This is proved by step by step comparison of the corresponding solutions depending on the particular f and f^* constructed above. Details are found in [27].

The map $\mathcal{F}$ associated to f^* induces an one-dimensional map F given by $u' = F(u)$. Indeed u', constructed by u and φ, does not depend on the particular choice of $\varphi \in \Phi_u$ when f is replaced by a step function f^* (cf. in Examples 4.1–4.4).

The following statement is verified by a direct calculation of the map F. They are similar to Example 4.3, case $h > 1/2$ (see Chap. 4).

Lemma 5.3. *The one-dimensional map* $F : [0, 1] \to [0, 1]$ *is conjugate by* $z = \exp(-u/\nu)$ *to a map* $G : [\exp(-1/\nu), 1] \to [\exp(-1/\nu), 1]$. *If* $b - b_0$ *and* $a_0 - a$ *are small enough then the map* G *has the only fixed point* $z = 1$, *which is globally attracting:* $\lim_{k\to\infty} G^k(z) = 1 \, \forall z \in [\exp(-1/\nu), 1]$. *The map* G *has the form shown in Fig. 15b.*

Remark. Since $G(z) \equiv 1$ for $z \in [z_0, 1]$ and $G(z) > z$ for all $z \in [\exp(-1/\nu), 1]$ it follows that for every fixed $z \in [\exp(-1/\nu), 1]$ there exists a positive integer n_0 such that $G^k(z) = 1 \, \forall k \geq n_0$.

With Lemmas 5.1–5.3 at hand, the proof of Theorem 5.1 is straightforward. Take an arbitrary $\varphi \in X$. If the function $x^\nu_\varphi(t) - a_0$ (or $x^\nu_\varphi(t) - b_0$) has no zeros for $t > 0$ then the theorem holds (Lemma 5.1). If it has a zero $t_0 > 0$ we consider $x^\nu_\varphi(t_0 + s)$, $s \in [1, 0]$ as an element of Φ_u for some $u \in [0, 1]$. Then the map $\mathcal{F}$ is defined, for this initial condition, as well as the induced one-dimensional map F majorizing $\mathcal{F}$ (Lemma 5.2). If meas $(I \backslash J_0)/\text{meas}(J_0 \backslash I_0)$ is small enough, then the map $G = h^{-1} \circ F \circ h$, $z = h(u) = \exp(-u/\nu)$ has the property: $\lim_{k \to \infty} G^k(z) = 1$ (Lemma 5.3). Moreover, with $z = \nu \ln(1/u)$ given there exists $k_0 \in \mathbb{N}$ such that $G^{k_0}(z) < 1$, $G^k(z) = 1$, $k > k_0$ (Remark 5.3). Therefore, the corresponding $u_k = F^k(u)$ satisfy $u_{k_0} > 0$, $u_k = 0 \, \forall k > k_0$. This implies that for every $u \in [0, 1]$ and every $\varphi \in \Phi_u$ there exists $k_1 \leq k_0$ such that $\mathcal{F}^{k_1}(u) = 0$. But for $u = 0$ one has $x^\nu_\varphi(t) \in J_0 \, \forall t > 0$. This completes the proof.

One naturally expects to extend (in a sense) Theorem 5.1. to cycles of intervals. What conditions should a cycle of intervals $\{I_k\}_{k=1}^n$ be subjected to, in order to guarantee existence of solutions for equation (1) which range cyclically in intervals $I_1, I_2, \cdots, I_n$? An answer is given by Theorem 5.2 below.

We suppose that a set of intervals $I_k = [a_k, b_k]$, $k = 1, 2, \cdots, n$, forms a cycle of period n for the map $f : I_1 \to I_2 \to \cdots \to I_n \to I_1$. For every k, $1 \leq k \leq n$, denote by J_k the interval which connects I_k and $I_{k+1 (\mathrm{mod}\, n)}$ and has joint endpoints with them: $J_k = [b_k, a_{k+1}]$ for $b_k \leq a_{k+1}$, and $J_k = [b_{k+1}, a_k]$ for $b_{k+1} \leq a_k$ (from now on we identify the indices $n+1$ and 1). J_k may consist of one point. For any k, $1 \leq k \leq n$, intervals I_k, I_{k+1}, J_k given we define the following numbers:

$$
\begin{aligned}
& l_1(I_k) = [\sup f(J_k) - \sup I_{k+1}]/[\sup f(J_k) - \sup f(I_k)] \,, \\
& \qquad \text{if } \sup f(J_k) > \sup f(I_k) \,, \text{ and} \\
& l_1(I_k) = 0 \,, \quad \text{if } \sup f(J_k) \leq \sup f(I_k) \,, \\
& l_2(I_k) = [\inf I_{k+1} - \inf f(J_k)]/[\inf f(I_k) - \inf f(J_k)] \,, \text{ if} \\
& \inf f(I_k) > \inf f(J_k) \,, l_2(I_k) = 0 \text{ if } \inf f(I_k) \leq \inf f(J_k) \,, \\
& l(I_k) = \max\{l_1(I_k) \,, l_2(I_k)\} \,, \\
& m(I_k) = [\inf f(I_k) - \inf I_{k+1}]/[\inf f(I_k) - \sup I_k] \,, \\
& \qquad \text{if } \sup I_k < \inf f(I_k) \,, \\
& m(I_k) = [\sup I_{k+1} - \sup f(I_k)]/[inf\, I_k - \sup f(I_k)] \,, \\
& \qquad \text{if } \sup f(I_k) < \inf I_k \,.
\end{aligned}
$$

Introduce subsets X_k, $k = 1, 2, \cdots, n$, of X by setting $X_k = C([-1, 0], I_k)$.

Theorem 5.2. *Suppose the map f has an interval cycle $I_1 \to I_2 \to \cdots \to I_n \to I_1$, satisfying $m(I_k) > l(I_k)$ for $k = 1, 2, \cdots, n$. Then there exists a positive*

ν_0 such that for any $0 < \nu \leq \nu_0$ and arbitrary $\varphi \in X_1$, the corresponding solution $x_\varphi^\nu(t)$ has the following property: there exists a sequence $0 < t_1 < t_2 < t_3 < \cdots \to \infty$ with $t_{k+1} - t_k > 1$ for all $k \in \mathbb{N}$ and $x_\varphi^\nu(t_k + t) \in I_{k(\mathrm{mod}\, n)}$ for $t \in [0,1]$.

Remark. Let $\mathcal{F}^t, t \geq 0$ be the semiflow on $X_I = C([-1,0], I)$ given by $\mathcal{F}^t\varphi(s) = x_\varphi^\nu(t+s)$, $s \in [0,1]$. The teorem says that if ν is sufficiently small for every $\varphi \in X_1$ there exists a sequence $(t_k) \to \infty$ such that $(\mathcal{F}^{t_k} \in X_{I_{k(\mathrm{mod}\, n)}}$. Every initial function $\varphi \in X_1$ generates a solution $x_\varphi^\nu(t)$ which ranges cyclically in the intervals $I_1, I_2, \cdots, I_n$ within time segments of length at least 1.

Remark. Since $l(I_k) \geq 0$ the conditions of Theorem 5.2 imply that $m(I_k) > 0$ for all $1 \leq k \leq n$. The inequality $m(I_k) > 0$ implies in turn (see the definition) of $m(I_k)$) that the set $\{f(I_k)\}$ is a proper subset of $\{I_{k+1}\}$. This means that each of the intervals I_k is mapped strictly inside the interval I_{k+1}, $k = 1, 2, \cdots, n$. Then it is not difficult to see that under the conditions of Theorem 5.2 there exists a cycle of intervals $\{I_1', I_2', \cdots, I_n'\}$ satisfying $I_k' \subset I_k\, I_k' \neq I_k$, $f(I_k') = I_{k+1}'$. Indeed, it is enough to set $I_k' = \cap_{i \geq 0} f^{ni}(I_k)$. The interval cycle $\{I_1', I_2', \cdots, I_n'\}$ (which may coincide with the trajectory of a periodic point) is an attractor and the set $I_1 \cup I_2 \cup \cdots \cup I_n$ is a proper subset of its domain of immediate attraction. The conditions $m(I_k) > l(I_k)$, $k = 1, 2, \cdots, n$, can be considered (in a sense) as reflecting the fact that the set $I_1 \cup I_2 \cup \cdots \cup I_n$ is much larger than the remaining part $I \backslash (\cup_{k=1}^n I_k)$ of the invariant interval I.

A particular map of an interval into itself having a cycle of intervals of period two is depicted in Fig. 23. A cycle of intervals is formed by $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ with $f(I_1) = [a_2', b_2]$, $f(I_2) = [a_1, b_1']$ where $a_2' > a_2$, $b_1' < b_1$. The set J_1 coincides with the set J_2 and is the interval $[b_1, a_2]$. Define $\xi = \inf\{f(x), x \in J_1\}$, $\eta = \sup\{f(x), x \in J_1\}$. Directly from the definitions we have:

$$
\begin{aligned}
&l_1(I_1) = 0,\ l_2(I_2) = (a_2) - \xi)/(a_2' - \xi) \Rightarrow l(I_1) = (a_2 - \xi)/(a_2' - \xi);\\
&m(I_1) = (a_2' - a_2)/(a_2' - b_1)\ ;\\
&l_1(I_2) = (\eta - b_1)/(\eta - b_1'),\ l_2(I_2) = 0 \Rightarrow l(I_2) = (\eta - b_1)/(\eta - b_1');\\
&m(I_2) = (b_1 - b_1')/(a_2 - b_1')\ .
\end{aligned}
$$

The conditions of Theorem 5.2 in the case considered take the form $(a_2' - a_2)/(a_2' - b_1) > (a_2 - \xi)/(a_2' - \xi)$ and $(b_1 - b_1')/(a_2 - b_1') > (\eta - b_1)/(\eta - b_1')$. They hold always if the inequalities $(a_2' - a_2)/(a_2' - b_1) > (a_2 - a_1)/(a_2' - a_1)$ and $(b_1 - b_1')/(a_2 - b_1') > (b_2 - b_1)/(b_2 - b_1')$ are satisfied. This follows from the fact $a_1 \leq \xi \leq \eta \leq b_2$. In particular, the inequalities are justified when a_2' and b_1' are fixed and $a_2 - b_1$ is small enough.

The proof of Theorem 5.2 is based on the following lemma.

Lemma 5.4. *Suppose that the inequality $m(I_k) > l(I_k)$ holds. Then there exists a positive ν_0 such that for all $0 < \nu \leq \nu_0$ and any $\varphi \in X_k$ the corresponding*

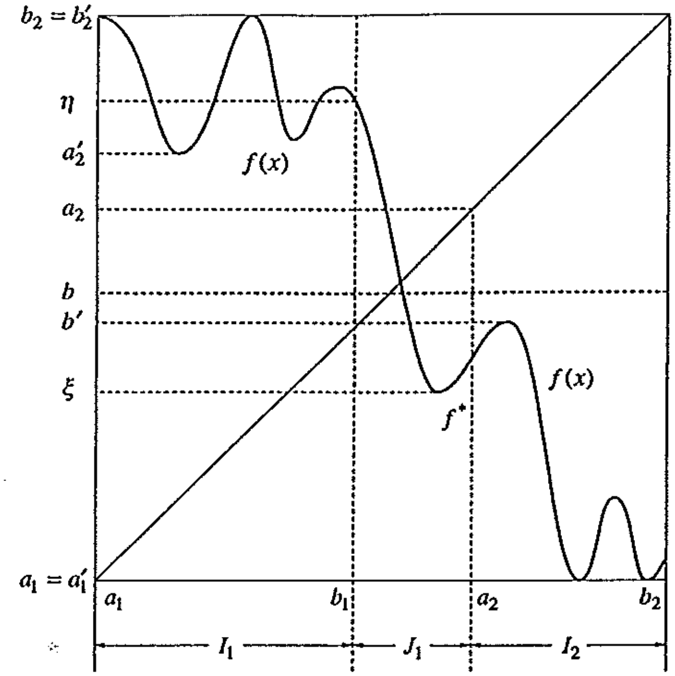

Fig. 23.

solution $x_\varphi^\nu(t)$ has the following property: there exists a time $t_* = t_*(\varphi,\nu) > 1$ with $x_\varphi^\nu(t_* + t) \in I_{k+1}$ for all $t \in [0,1]$.

The proof is divided into several parts. To be definite we suppose $k = 1$, and set $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $a_1 < b_1 < a_2 < b_2$, $f(I_1) = [a_2', b_2'] \subset I_2$, $J_1 = [b_1, a_2]$, $\xi_1 = \inf f(J_1)$, $\eta_1 = \sup f(J_1)$; $X_1^0 = \{\varphi \in X_1 | \varphi(0) = b_1\}$.

Claim 1. For any $\varphi \in X_1$, there exists $t_1 = t_1(\varphi,\nu) > 0$ with $x_\varphi^\nu(t_1) = b_1$.

Since $f(I_1) \subset I_2$ and $b_1 < a_2'$, according to equation (1) the solution $x_\varphi^\nu(t)$ increases (for any $\nu > 0$ and $\varphi \in X_1$) in some righthand neighbourhood of $t = 0$. Clearly, $d/dt(x_\varphi^\nu(t)) > 0$ for all $t > 0$ where $x_\varphi^\nu(t) < a_2'$. Moreover $x_\varphi^\nu(t) \geq \alpha(t) \stackrel{\text{def}}{=} a_2' + [\varphi(0) - a_2']\exp(-t/\nu)$. The latter implies $t_1 \leq t_1^0 = \nu \ln[(a_2' - a_1)/(a_2' - b_1)]$.

Claim 1 allows to restrict the considerations to X_1^0 rather than X_1.

Claim 2. For any $\varphi \in X_1^0$ there exists $t_2 = t_2(\varphi,\nu)$ with $x_\varphi^\nu(t_2) = a_2$. Moreover, $x_\varphi^\nu(t)$ increases on $[0, t_2]$ and $t_2 \leq \nu \ln[(a_2' - b_1)/(a_2' - a_2)]$.

Let $\varphi \in X_1^0$. Then for $t \in [0,1]$ we have $\varphi) \geq \alpha(t) \stackrel{\text{def}}{=} a_2' + (b_1 - a_2')\exp(-t/\nu)$. Clearly, $x_\varphi^\nu(t)$ is monotone for all those t from a righthand neighbourhood of $t = 0$ where $x_\varphi^\nu(t) < a_2'$.

Hence, $t_2 \leq t_2^0 = \nu \ln[(a_2' - b_1)/(a_2' - a_2)]$.

Claim 3. If the inequality $m(I_1) > l_2(I_1)$ holds, then there exists a positive ν_0^1 such that for all $0 < \nu \leq \nu_0^1$ and for any $\varphi \in X_1^0$ the corresponding solution $x_\varphi^\nu(t)$ satisfies the inequality $x_\varphi^\nu(t) \geq a_2$ for all $t \in [t_2, t_2 + 1]$.

According to claim 2, for any $\varphi \in X_1^0$ there exists $t_2 = t_2(\varphi, \nu)$ such that $x_\varphi^\nu(t)$ is increasing on $[0, t_2]$ and $x_\varphi^\nu(t_2) = a_2$. Moreover, $t_2 \leq t_2^0 = v\ln[(a_2' - b_1)/(a_2' - a_2)]$. We may choose ν_0^1 guaranteeing $t_2^0 < 1$ for all $0 < \nu \leq \nu_0^1$. Then $x_\varphi^\nu(t) \geq \alpha(t) \overset{\text{def}}{=} a_2' + (b_1 - a_2')\exp(-t)/\nu)$ and hence $x_\varphi^\nu(t) \geq b_1$ for all $t \in [t_2, 1]$. Put $\alpha_1 \overset{\text{def}}{=} \alpha(1) = a_2' + (b_1 - a_2')\exp(-1/\nu)$, and consider the solution $x_\varphi^\nu(t)$ within the time interval $[1, 1 + t_2]$. We have: $x_\varphi^\nu(t) \geq \xi_1 + (\alpha_1 - \xi_1)\exp\{-(t - 1)/\nu\} \geq \xi_1 + [a_2' + (b_1 - \alpha_1)\exp(-1/\nu) - \xi_1](a_2' - a_2)/(a_2' - b_1)$. From the definition we have $m(I_1) = (a_2' - a_2)/(a_2' - b_1)$ and $l_2(I_1) = (a_2 - \xi_1)/(a_2' - \xi_1)$. Since $m(I_1) > l_2(I_1)$, there exists a positive $\nu_0' < \nu_0^1$ with $\xi_1 + [a_2' + (b_1 - a_2')\exp(-1/\nu) - \xi_1](a_2' - a_2)/(a_2' - b_1) \geq a_2$ for all $0 < \nu < \nu_0'$. This implies $x_\varphi^\nu(t) \geq a_2$ for $t \in [1, 1 + t_2]$.

Claim 4. If the inequality $m(I_1) > l_1(I_1)$ holds, then for all $0 < \nu < \nu_0'$ and any $\varphi \in X_1^0$ the solution $x_\varphi^\nu(t)$ satisfies $x_\varphi^\nu(t) \leq b_2$ for $t \in [t_2, t_2 + 1]$.

Clearly $x_\varphi^\nu(t) \leq \beta(t) \overset{\text{def}}{=} b_2' + (b_1 - b_2')\exp(-t/\nu) \leq b_2'$ for all $t \in [0, 1]$. Suppose $\nu \leq \nu_0'$. Since $m(I_1) = (a_2' - a_2)/(a_2' - b_1)$ and $l_1(I_1) = (\eta_1 - b_2)/(\eta_1 - b_2')$ we have $x_\varphi^\nu(t) \leq \eta_1 + (b_2' - \eta_1) \times \exp\{-(t-1)/\nu\} \leq \eta_1 + (b_2' - \eta_1)(a_2' - a_2)/(a_2' - b_1) = b_2$ for $t \in [1, 1 + t_2]$.

To complete the proof of the lemma we set $\nu_0 = \nu_0'$ and $t_* = t_2 + 1$. The case $b_2 < a_1$ is treated similarly.

With the proved Lemma, the proof of Theorem 5.2 is straight-forward.

The conditions involved in Theorem 5.2 are explained in the following example.

Example 5.2. Suppose δ is a small positive number. Define $f_\delta(x)$ by $f_\delta(x) = 0$ for $x < -1 - \delta$, $f_\delta(x) = 2$ for $x \in (-1 + \delta, 1 - \delta)$, $f_\delta(x) = a < -1 - \delta$ for $x > 1 + \delta$ and let $f_\delta(x)$ be an arbitrary monotone function for $x \in (1 - \delta, -1 + \delta) \cup (1 - \delta, 1 + \delta)$ such that $f_\delta(x) \in C^0(\mathbb{R})$ (Fig. 24). The map f_δ has an interval cycle of period three: $I_1 \to I_2 \to I_3 \to I_1$ with $I_1 = [a, -1 - \delta]$, $I_2 = [-1 + \delta, 1 - \delta]$, $I_3 = [1 + \delta, 2]$. Note that there exists an attracting cycle of period three $a \to 0 \to 2 \to a$ whose immediate attraction domain contains $I_1 \cup I_2 \cup I_3$ at least. For the interval cycle $\{I, I_2, I_3\}$ we have $J_1 = [-1 - \delta, -1 + \delta]$, $J_2 = [1 - \delta, 1 + \delta]$, $J_3 = [-1 - \delta, 1 + \delta]$. One easily obtains that $l_1(I_1) = 0$, $l_1(I_2) = 0$, $l_2(I_1) = 0$, $l_2(I_2) = (1 + \delta - a)/(2 - a)$, $l_1(I_3) = (3 + \delta)/(2 - a)$, $l_2(I_3) = 0$, , $m(I_1) = m(I_2) = (1 - \delta)/(1 + \delta)$, $m(I_3) = (-1 - \delta - a)/(1 + \delta - a)$. Then the conditions of Theorem 5.2 become $(1 - \delta)/(1 + \delta) > 0$, $(1 - \delta)/(1 + \delta) > (1 + \delta - a)/(2 - a)$, $(-1 - \delta - a)/(1 + \delta - a) > (3 + \delta)/(2 - a)$.

This means that for any $a < -1 - \sqrt{6}$ there exists a positive $\delta_0 = \delta_0(a)$ such that the assumptions of Theorem 5.2 hold for f_δ provided $0 < \delta \leq \delta_0$. For the particular choice $a = -4$ a sufficient condition is $0 < \delta < \sqrt{52} - 7 \approx 0,21$.

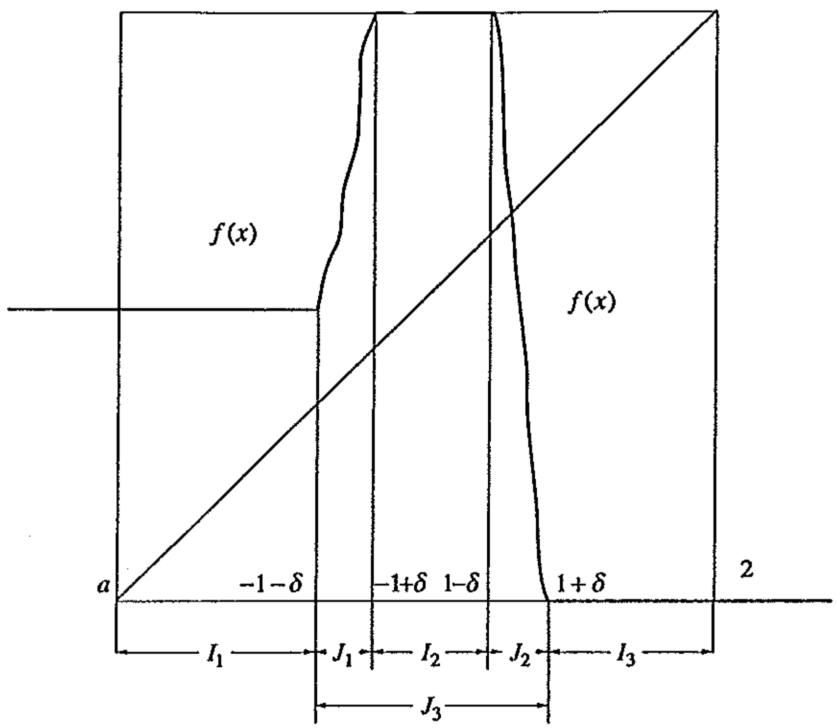

Fig. 24.

Note that if the conditions of Theorem 5.2 hold for a particular continuous function $f(x)$, then they are also satisfied for all sufficiently small C^0 perturbations of $f(x)$.

6. Existence of Periodic Solutions

The existence of nonconstant periodic solutions for funtional-differential equations has been studied in many papers. We refer, for example, to [5, 9, 15, 17, 30, 31, 33, 39, 42–45, 58, 59, 62] for autonomous differential-difference equations related to the equation considered here. Several methods were developed to prove the existence of periodic solutions including technique based on recent results of functional analysis. Some of them are applicable to the singularly perturbed differential-difference equation

$$\nu\dot{x}(t) + x(t) = f(x(t-1)) \,. \tag{1}$$

If the interval map f has only one attracting point and no other cycles, then all solutions of equation (1) are asymptotically constant. This means that the rest point of the semiflow $\mathcal{F}^t$ corresponding to the fixed point of the map f is a global attractor on $X_I = C([-1,0], I$ as $t \to +\infty$. This was shown in Chap. 2. Therefore, the dynamics of the map f on an invariant interval has to be more complex to produce nonconstant periodic solution. The next situation is when the map f has only one repelling fixed point and a globally attracting cycle of period two. This means, in particular, that the fixed point divides the invariant interval into two subintervals which are permuted by f and which cover the whole invariant interval. In this situation, equation (1) has a nonconstant periodic solution if $\nu > 0$ is small enough. This result was proved in [16].

Theorem 6.1. *Suppose the map f has an invariant interval I with exactly one repelling fixed point $x_* \in I$, $|f'(x_*)| > 1$, and f satisfies the negative feedback conditions $(x - x_*)[f(x) - x_*] < 0$, $x \neq x_*$. Then there exists a positive ν_0 such that for every $\nu \in (0, \nu_0]$ equation (1) has a slowly oscillating periodic solution.*

We recall that a solution $x(t)$ is called slowly oscillating if successive zeros of $x(t) - x_*$ for large t are spaced apart by distances more than the time delay 1.

Since the complete proof of this theorem is given in [16] for the slightly different equation $\dot{x}(t) + \nu x(t) + f(x(t-1)) = 0$ we briefly sketch here the main ideas of the proof refering to the original paper for details.

It is well known fact that for $0 < \nu < |f'(x_*)|$ all solutions of equation (1) oscillate with respect to $x = x_*$. This means that the function $x(t) - x_*$ has an unbounded set of zeros for every solution $x(t)$ of equation (1).

Consider a set of initial functions defined by $K = \{\varphi \in X_I | \varphi(-1) = x_*, \, \varphi(s) - x_* < 0 \text{ for all } s \in (-1, 0]\}$.

For any $\varphi \in K$ there exists a sequence $\{t_k\}_{k=1}^{\infty}$ of zeros of $x_\varphi^\nu(t) - x_*$ with $t_{k+1} - t_k > 1$ for all $k \in \mathbb{N}$, $x_\varphi^\nu(t) - x_* < 0$ for $t \in (t_{2i-1}, t_{2i})$, $x_\varphi^\nu(t) - x_* > 0$ for $t \in (t_{2i}, t_{2i+1})$, $i \in \mathbb{N}$. In other words, any $\varphi \in K$ gives rise to a slowly oscillating solution. This makes it possible to define a map G on K in the following way. If $\varphi \in K$ and t_2 is the second zero of the function $x_\varphi^\nu(t) - x_*$ then $(G\varphi)(t) = x_\varphi^\nu(t_2 + 1 + t)$, $t \in [-1, 0]$. Clearly, G maps K into itself. It is convenient to consider the constant solution $x_\varphi^\nu(t) \equiv x_*$ as a fixed point of G. Other fixed points of G (if any) generate nontrivial periodic solutions of equation (1). The main result of [16] is to show the existence of a fixed point of G different from x_*.

It is possible to find a subset $K_0 = \{\varphi \in K | \varphi(t)\exp(t/\nu) \text{ does not decrease for } t \in [-1, 0]\}$ of K which is invariant under G.

Suppose next that $f'(x_*) < -1$. Then for all sufficiently small $\nu > 0$ the trivial solution $x(t) \equiv x_*$ is unstable. Using ideas of [33, 44, 64] this allows us to show that the fixed point $\varphi \equiv x_*$ is repelling under G on K_0. More precisely, it is possible to find a neighbourhood U_* in K_0 of the fixed point $\varphi = x_*$ such that for every $\varphi \in U_*$ there exist a positive integer $N = N(\varphi)$ such that $G^N(\varphi) \notin U_*$. Here G^N is the N-th iterate of the map G. This implies, according to [44], the existence of a fixed point of G different from the trivial one $\varphi = x_*$. As we have remarked before every nontrivial fixed point of G gives rise to a slowly oscillating periodic solution of equation (1).

Theorem 6.1 is existence theorem which does not say anything about the particular form of the periodic solutions when ν is small. In general, little can be said about it even when the structure of the map f is known in great detail. This is one of the unsolved problems (see Chap. 7).

However, when the map f has a globally attracting cycle of period two, the structure of periodic solutions and their asymptotics as $\nu \to +0$ can be studied. This is done in [42]. Here we cite only the following particular result.

Suppose the map f has a globally attracting cycle $\{a_1, a_2\}$ of period two on I and $f'(x_*) < -1$ for the repelling fixed point $x = x_*(H)$. Let $p_0(t) = a_1$, $t \in [0,1)$, $p_0(t) = a_2$, $t \in [1,2)$ and continuate $p_0(t)$ periodically for all $t \in \mathbb{R}$.

Theorem 6.2. *If the map f satisfies (H), then there exists a positive ν_0 such that for every $0 < \nu \le \nu_0$ equation (1) has a periodic solution $p_\nu(t)$ with period $2 + O(\nu)$, $\nu \to +0$. The periodic solution $p_\nu(t)$ converges to $p_0(t)$ as $\nu \to +0$, uniformly on every compact interval not containing integer points $t = k$, $k \in \mathbb{Z}$.*

For a proof of this theorem see [42].

Remark. Note that neither stability nor uniqueness of the periodic solution $p_\nu(t)$ is asserted in Theorem 6.2. In fact, there may be several or even infinitely many periodic solutions (see Example 4.4 or [1, 3, 40]).

Example 6.1. Consider equation (1) with $f(x) = f_\lambda(x) = \lambda x(1-x)$.

For every $0 \le \lambda \le 3$ all solutions have finite limits (see Example 2.1).

Suppose that $3 < \lambda \le \lambda^*$. Here $\lambda^* \approx 3,57$ is the value of the parameter λ for which the map f_λ has cycles of every period 2^n, $n = 0, 1, 2, \cdots$ but no other cycles (Feigenbaum point). It is well-known (see, e.g. [54]) that for every $\lambda \in (3, \lambda^*)$ the fixed point $x_* = 1 - 1/\lambda$ is repelling and $(x - x_*)[f_\lambda(x) - x_*] < 0$ for all x belonging to the invariant interval $[f_\lambda^2(1/2), f_\lambda(1/2)] \subset [0,1]$. Therefore, for every $\lambda \in (3, \lambda^*)$ and $\nu > 0$ sufficiently small, equation (1) has a periodic solution $p_\nu(t)$ which is slowly oscillating (Theorem 6.1). Moreover, if $\lambda \in (3, 1+\sqrt{6})$ (for this range of parameter values, the map f_λ has a globally attracting cycle of period 2) $p_\nu(t)$ converges to the function $p_0(t)$ on compact sets not containing integer points $t = n$, $n \in \mathbb{Z}$, as $\nu \to +0$. Here $p_0(t) = a_1$, $t \in [0,1)$, $p_0(t) = a_2$, $t \in [1,2)$ and $\{a_1, a_2\}$ is the cycle of period two of the map f_λ. The points a_1, a_2 are found as the real roots of the equation $f_\lambda^2(x) = x$, different from $x = 0$ and $x = 1 - 1/x$.

It is natural, however, to relate periodic solutions of equation (1) and cycles of the map f. This can be seen from the following heuristic arguments. Suppose $\beta = \{x_1, x_2, \cdots, x_n\}$ is an attracting cycle of the map f with components $U(x_i), i = 1, \cdots, n$, of domain of immediate attraction. Define subsets of the phase space $X = C([-1,0], I)$ by setting $Y_i = \{\varphi \in X | \varphi(t) \in U(x_i) \forall t \in [-1,0)\}$. Since β is an attracting cycle any $\varphi \in Y_i$, $i = 1, 2, \cdots, n$, gives rise to a solution $x_\varphi(t)$ of equation (2) converging uniformly on $[k, k+1)$ as $k \to \infty$ to the steplike function $x^* = x_i$ for $t \in [i-1, i)$, $i = 1, 2, \cdots, n$. By virtue of the continuous dependence results (Chap. 3) which guarantee, for small ν, the closeness between solutions of the equations (1) and (2), it is natural to expect that, in some cases, a periodic solution of equation (1) with period close to n will correspond to the attracting cycle β. Generally speaking there may not be any such correspondence at all, as Example 4.3 and Theorem 5.1 show. Therefore some additional restrictions are needed to guarantee the correspondence. The idea is to use the continuous dependence results together with the existence of an attracting cycle of intervals, subjected to additional conditions. The conditions are roughly speaking, to ensure, the existence of an interval cycle with a large domain of immediate attraction.

Suppose the map f has an interval cycle $\{I_1, I_2, \cdots, I_n\}$. Consider the numbers $m(I_i)$ and $l(I_i)$, $i = 1, 2, \cdots, n$, introduced in the previous chapter. Recall the notations $X_I = C([-1,0], I)$, $X_k = C([-1,0], I_k)$, $k = 1, 2, \cdots, n$.

Theorem 6.3. *If the conditions $m(I_i) > l(I_i)$, $i = 1, 2, \cdots, n$, hold, then there exists a $\nu_0 > 0$ such that for every $0 < \nu \leq \nu_0$ equation (1) has a periodic solution $p(t)$ with the following properties:*

(i) $p(t)$ has period $T = n + O(\nu)$, $\nu \to +0$;
(ii) there exists a sequence $0 < t_1 < t_2 < \cdots < t_n < T$ such that $p(t_k + t) \in X_k$, $t \in [-1, 0]$, $1 \leq k \leq n$.

We recall (see Remark 5.5) that the conditions $m(I_i) > l(I_i)$ imply the existence of an attracting cycle of intervals $\{I'_1, \cdots, I'_n\}$ defined by $I'_i = \cap_{k \geq 0} f^{nk}(I_i)$. The set $I_1 \cup I_2 \cup \cdots \cup I_n$ is a part of its domain of immediate attraction.

Theorem 6.3 is a straightforward corollary of Theorem 5.2 and the Schauder fixed point theorem. Indeed, according to Theorem 5.2, for arbitrary $\varphi \in X_1$ there exists a sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq \cdots \to \infty$ such that $t_{k+1} - t_k > 1$ for all $k \in \mathbb{N}$ and $x_\varphi^\nu(t_k + s) \in X_{k (\mathrm{mod}\, n)}$, $s \in [-1, 0] (0 \leq \nu \leq \nu_0$, ν_0 is small enough) (see Remark 5.4). Define a map $G : X_1 \to X_1$ by setting: $(G\varphi)(t) = x_\varphi^\nu(t_n + t)$, $t \in [-1, 0]$. More or less standard arguments show that X_1 is convex and bounded, and G is compact. Therefore, there exists a fixed point $\varphi_0 \in X_1$ of G which corresponds to a periodic solution of equation (1).

In the case when the cycle of intervals $\{I_1, I_2, \cdots, I_n\}$ contains a unique attracting cycle of the map f, the statement of Theorem 6.3 can be strengthened substantially. Let a finite set $\{A_1, A_2, \cdots, A_n\}$ of real numbers be given. We define a step function $p_0(t)$, $t \in \mathbb{R}$, by setting $p_0(t) = A_k$, $t \in [k-1, k) (\operatorname{mod} n)$.

Theorem 6.4. *Suppose the map f has a cycle of intervals $\{I_1, I_2, \cdots, I_n\}$ for which the inequalities $m(I_k) > l(I_k)$ hold for all $1 \leq k \leq n$. If the cycle of intervals contains a unique cycle $\{A_1, A_2, \cdots, A_n\}$ of the map f then the periodic solution $p(t)$, guaranteed to exist by Theorem 6.3, converges to the step function $p_0(t)$ as $\nu \to +0$. The convergence is uniform on every compact set not containing points $t = k$, $k \in \mathbb{Z}$.*

Remark. Under the conditions of Theorem 6.4 the convergence $p(t) \to p_0(t)$, $\nu \to +0$ holds but uniqueness of $p(t)$ is not claimed. In fact equation (1) may still have several or even countably many periodic solutions $p(t)$ [1, 3, 40]. All of them will be close to $p_0(t)$ (and therefore close to each other) and converge to $p_0(t)$ as $\nu \to +0$ in the following sense. Take a compact set $K \subset \mathbb{R}$ not containing points $t = i$, $i \in \mathbb{Z}$ and an arbitrary positive ε. Then there exists a positive ν_0 such that for every $0 < \nu \leq \nu_0$ one has $\sup\{|p(t) - p_0(t)|, t \in K\} \leq \varepsilon$ for any periodic solution $p(t)$ from Theorem 6.4. This result can be derived from the fine structure of periodic solution $p(t)$ obtained in [1, 3, 40] for the cases considered there. For our case it is proved below.

The proof of Theorem 6.4 is based on the properties of the shift operator $\mathcal{F}^t$ along solutions of equation (1) and on the continuous dependence results of Chap. 3 which we now adopt in Lemma 6.1.

Lemma 6.1 *Suppose $\varphi \in X_I$ and $[s_1, s_2]$ is a subinterval of $[-1, 0]$. For any positive ε, δ there exists a positive ν_0 such that* $\sup\{|x_\varphi^\nu(t) - f(\varphi(t-1))|, t \in [s_1 + 1 + \delta, s_2 + 1]\} \leq \varepsilon$ *for all $0 < \nu \leq \nu_0$.*

The lemma says that the iterate $f \circ \varphi$ of an initial function φ and the corresponding solution $x_\varphi^\nu(t)$ of equation (1) are as close on $[s_1 + 1 + \delta, s_2 + 1]$ as desired, provided ν is small enough.

Proof of the lemma. Since equation (1) is autonomous we may set $s_1 = -1$. For $t \in [0, s_2 + 1]$ the solution $x_\varphi^\nu(t)$ of equation (1) may be written in the form $x_\varphi^\nu(t) = \varphi(0)\exp(-t/\nu) + (1/\nu) \int_0^t \exp\{(s-t)/\nu\} f(\varphi(s-1)) ds$. Using the identity $\exp(-t/\nu) + (1/\nu) \int_0^t \exp\{(s-t)/\nu\} ds = 1$ we have: $|x_\varphi^\nu(t) - f(\varphi(s-1))| \leq |\varphi(0) - f(\varphi(t-1))| \exp(-t/\nu) + (1/\nu) \int_0^t \exp\{(s-t)/\nu\} |f(\varphi(s-1)) - f(\varphi(t-1))| ds$, $t \in [0, s_2 + 1]$.

Since ε and δ are fixed there always exists ν_0' such that $\sup\{|\varphi(0) - f(\varphi(t-1))| \exp(-t/\nu), t \in [\delta, s_2 + 1]\} \leq \varepsilon/3$ for all $0 < \nu \leq v_0'$. This also implies $(1/\nu) \int_0^{t-\sigma} \exp\{(s-t)/\nu\} |f(\varphi(s-1)) - f(\varphi(t-1))| ds \leq M'(1/\nu) \int_0^{t-\sigma} \exp\{(s-t)/\nu\} ds \leq \varepsilon/3$, $t \in [0, s_2 + 1]$, for every $\sigma > 0$ and all $0 < \nu \leq \nu_0 = \nu_0(\sigma)$.

The function $f(\varphi(\cdot))$ is uniformly continuous since both $f(\cdot)$ and $\varphi(\cdot)$ are continuous. Therefore, for given $\varepsilon > 0$ there exists $\sigma > 0$ such that $|f(\varphi(t')) - f(\varphi(t''))| \leq \varepsilon/3$ provided $|t' - t''| \leq \sigma$. This implies $(1/\nu)\int_{t-\sigma}^{t} \exp\{(s-t)/\nu\}|f(\varphi(s-1)) - f(\varphi(t-1))|ds \leq \sup\{|f(\varphi(s-1)) - f(\varphi(t-1))|, s : |s-t| < \sigma\} \leq \varepsilon/3$.

Therefore $\sup\{|x_\varphi^\nu(t) - f(\varphi(t-1))|, t \in [\delta, s_2+1]\} \leq |\varphi(0) - f(\varphi(t-1))|\exp(-t/\nu) + (1/\nu)[\int_0^{t-\sigma} + \int_{t-\sigma}^{t}] \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ for every $0 < \nu \leq \min\{\nu_0', \nu_0''\}$. □

Corollary 6.1. *Suppose $\varphi \in X_I$, $[s_1, s_2]$ is a subinterval of $[-1, 0]$. Then for any positive ε, δ and any positive integer N there exists a positive ν_0 such that $\sup\{|x_\varphi^\nu(t) - f^N(\varphi(t-N))|, t \in [s_1 + N + \delta, s_2 + N]\} \leq \varepsilon$ for all $0 < \nu \leq \nu_0$.*

The corollary is proved by induction using Lemma 6.1. Indeed $\varphi_1(t) = x_\varphi^\nu(t)$ and $\varphi_2(t) = f(\varphi(t-1))$ are close on $[s_1 + 1 + \delta/N,\ s_2 + 1]$ by the lemma. Consider φ_1 and φ_2 as elements of X_I and set $s_1' = s_1 + \delta/N$, $s_2' = s_2$. Then $x_{\varphi_1}^\nu(t)$ and $f(\varphi_1(t-1))$ are close on $[s_1' + 1 + \delta/N, s_2' + 1]$ by Lemma 6.1. The functions $f(\varphi_1(t-1))$ and $f(\varphi_2(t-1))$ are close on $[s_1' + 1, s_2' + 1]$ since f is uniformly continuous on I. This implies that $x_\varphi^\nu(t)$ and $f^2(\varphi(t-2))$ are close on the interval $[s_1 + 2 + 2\delta/N, s_2 + 2]$, and so on.

Proof of Theorem 6.4. Let $\{I_1, I_2, \cdots, I_n\}$ be a cycle of intervals of the map f for which the inequalities $m(I_k) > l(I_k), 1 \leq k \leq n$, hold. Let $\{A_1, \cdots, A_n\}$ be the only (point) cycle contained in the cycle of intervals $\{I_1, \cdots, I_n\}$. Then $\{A_1, \cdots, A_n\}$ is an attracting cycle of the map f and I_k is a proper subset of the component $U(A_k)$ of its domain of immediate attraction, $k = 1, 2, \cdots, n$. Therefore, for any positive ε there exists a positive integer k_0 such that $|f^{kn}(x) - A_k| \leq \varepsilon$ for every $k \geq k_0$ and all $x \in I_k$.

Let $X_k = C([-1, 0], I_k)$, $k = 1, \cdots, n$ be fixed (say $k = 1$; the case $k > 1$ is similar). Then for arbitrary $\varepsilon > 0$ and every $\varphi \in X_1$ there exists a positive integer $N_0 = n_0 \cdot n$ such that $\sup\{|f^{N_0}(\varphi(t-N_0)) - A_1|, t \in [N_0 - 1, N_0)\} \leq \varepsilon/2$. On the other hand according to Corollary 6.1 for any positive δ and given ε there exists a positive ν_0 such that $\sup\{x_\varphi^\nu(t) - f^{N_0}(\varphi(t-N_0))|, t \in [N_0 - 1 + \delta, N_0]\} \leq \varepsilon/2$ (we apply the Corollary setting $s_1 = -1, s_2 = 0$). The latter inequality holds for every $\varphi \in X_1$ including those which give rise to periodic solutions. Use next Theorem 6.3. Take the particular $\varphi_0 \in X_1$ which generates the periodic solution $p_\nu(t)$ with period $T = n + O(\nu)$, $\nu \to +0$. To be definite we may always assume that $p_\nu(-1) = \inf I_1$ while considering $p_\nu(t)$ as an element of X_1. Denote by G the translation operator of time T along the periodic solution $p_\nu(t)$. We have $G^{ni}p_\nu(t) \in X_1$ for every integer i. Then $|p_\nu(t) - f^{N_0}(\varphi_0(t-N_0))| = |G^{ni}p_\nu(t) - f^{N_0}(\varphi_0(t-N_0))| = |x_{\varphi_0}^\nu(t) - f^{N_0}(\varphi_0(t-N_0))| \leq \varepsilon/2$ for all $t \in [N_0 - 1 + \delta, N_0]$, and small $\nu > 0$. Therefore, we have $|p_\nu(t) - A_1| = |G^{n_0}p_i(t) - f^{N_0}(\varphi_0(t-N_0)) + f^{N_0}(\varphi_0(t-N_0)) - A_1|$ $\leq |G^{n_0}p_\nu(t) - f^{N_0}(\varphi_0(t-N_0))| + |f^{N_0}(\varphi_0(t-N_0)) - A_1| = |x_{\varphi_0}^\nu(t) - f^{N_0}(\varphi_0(t-$

$N_0))| + |f^{N_0}(\varphi_0(t-N)) - A_1| \le \varepsilon$ for $t \in [N_0 - 1 + \delta, N_0 - \delta]$ and for every $0 < \nu \le \nu_0$. This implies the convergence $p_\nu(t) \to p_0(t)$ as $\nu \to +0$ on the segments corresponding to A_1. For the other cases $k > 1$ arguments have to be repeated. This completes the proof. □

Finally, to illustrate Theorem 6.4 we give the following

Example 6.2. Consider equation (1) with a continuous nonlinearity $f(x)$ close to a step function which is constructed as follows.

Suppose $n \ge 2$ is fixed. Take two sets of real numbers $\{a_1, \cdots, a_{n-1}\}$ $\{A_1, \cdots, A_n\}$ satisfying $A_n < a_1 < A_1 < a_2 < A_2 < \cdots < a_{n-1} < A_{n-1}$. Let δ be positive and small (say $\delta = \delta_0$ where $\delta_0 = \min\{(a_1 - A_n)/2, (a_i - a_{i-1})/2, i = 1, 2, \cdots, n-1\}$. Define $f(x)$ by setting: $f(x) = A_1$ for $x \le a_1 - \delta$; $f(x) = A_k$ for $x \in [a_{k-1} + \delta, a_k - \delta]$, $k = 2, \cdots, n-1$, $f(x) = A_n$ for $x \ge a_{n-1} + \delta$. Let $f(x)$ be an arbitrary monotone function on $[a_k - \delta, a_k + \delta]$, $k = 1, 2, \cdots, n-1$ such that $f(x)$ is continuously differentiable everywhere.

It is easy to see that the map f has the invariant interval $I = [A_n, A_{n-1}]$ and a cycle of intervals $\{I_1, \cdots, I_n\}$ belonging to I. Here $I_1 = [A_n, a_1 - \delta]$, $I_2 = [a_1 + \delta, a_2 - \delta], \cdots, I_{n-1} = [a_{n-2} + \delta, a_{n-1} - \delta]$, $I_n = [a_{n-1} + \delta, A_{n-1}]$. Moreover, there exists the attracting cycle $\{A_1, A_2, \cdots, A_n\}$ of period n. Its domain of immediate attraction contains the cycle of intervals $\{I_1, \cdots, I_n\}$ as a proper subset.

By direct calculation one has $l_1(I_1) = (A_2 - a_2 + \delta)/(A_2 - A_1)$, $l_2(I_1) = 0, \cdots, l_1(I_{n-2}) = (A_{n-2} - a_{n-1} + \delta)/(A_{n-1} - A_{n-2})$, $l_2(_{n-2}) = 0$, $l_1(I_{n-1}) = 0$, $l_2(I_{n-1}) = (a_{n-1} + \delta - A_n)/(A_{n-1} - A_n)$, $l_1(I_n) = (A_{n-1} - a_1 + \delta)/(A_{n-1} - A_n)$, $l_2(I_n) = 0$, $m(I_k) = (A_k - a_k - \delta)/(A_k - a_k + \delta)$, $k = 1, 2, \cdots, n$.

Since $\lim_{\delta \to +0} l(I_k) < 1$ and $\lim_{\delta \to +0} m(I_k) = 1$ for all $1 \le k \le n$, there exists a positive δ_1 such that for every $0 < \delta \le \delta_1$ the conditions of Theorem 6.4 are fulfilled. Therefore, equation (1) has a periodic solution $p_\nu(t)$ with period $n + O(\nu)$, $\nu \to +0$. Using the steplike form of $f(x)$, the solution $p_\nu(t)$ can be obtained explicitly. When $\nu \to +0$, the periodic solution $p_\nu(t)$ converges to a step function $p_0(t)$ on compact sets not containing integer points $t = i$, $i \in \mathbb{Z}$. Here $p_0(t) = A_{i(\mathrm{mod}\, n)}$, $t \in [i-1, i)$, $i \in \mathbb{Z}$.

Theorem 6.4 still holds true for small C^1 perturbations of the given steplike nonlinearity $f(x)$.

7. Concluding Remarks and Open Questions

The main problem we have discussed in this paper on the differential-delay equation

$$\nu \dot{x}(t) + x(t) = f(x(t-1)) \tag{1}$$

with a small positive parameter ν concerns the relation between properties of its solutions and the dynamics given by the corrsponding one-dimensional map $x \to f(x)$. The map f completely determines the properties of solutions of the difference equation with continuous argument

$$x(t) = f(x(t-1)) \tag{2}$$

obtained formally from (1) by setting $\nu = 0$. Thus, alternatively, we are concerned with the correspondence between solutions of equations (1) and (2), when ν is small enough.

The natural question of closeness between solutions of equations (1) and (2) arises. Two particular cases may be stated as follows:

(i) how are the solutions of equations (1) and (2) related within a finite time interval?

(ii) to what extent does the one-dimensional map f define asymptotic properties of solutions for equation (1) as $t \to +\infty$?

The solution of problem (i) is natural and complete. When considered within any finite time interval $[0, T]$, the solutions of equations (1) and (2) are close provided the corresponding initial conditions are close and ν is small enough. In particular, solutions of equation (1) follow, within a finite segment of time solutions of equation (2), the behavior of which is studied in reasonable generality [55].

The question about correspondence of asymptotic properties of the map f and solutions of equation (1) is much more difficult. Except for some relatively simple properties, this correspondence is not too direct. Two phenomena can be observed from the results presented in this paper:

(iii) the dynamics given by the map f is simple while the asymptotic behavior of solutions to equation (1) is complicated:

(iv) the dynamics given by the map f is complicated while the asymptotic behavior of solutions of equation (1) is simple.

The first phenomenon was illustrated by an example for which the map f has a globally attracting cycle of period two while the asymptotic behavior of solutions to equation (1) (on a subject) is described by the induced one-dimensional map which may exhibit very complex asymptotic properties. In particular, the induced map may have an invariant measure which is absolutely continuous with respect to the Lebesgue measure. It is worth noting that the arising chaos is small, in a sense. For the example considered above all chaotic solutions are close to a particular step function generated by the cycle of period two. There are no results proved on the phenomenon for the case of a general nonlinearity $f(x)$.

The second phenomenon occurs in the general case, and may be considered as typical. Simplification occurs in that the dynamics of the map $f(x)$ on some invariant subsets does not define any corresponding asymptotic properties of the solutions to equation (1), at all. This can be explained by the existence of attractors of the map f with "large" immediate basin. These large sets make solutions to gradually damp out in amplitude in the long run, since every solution is continuous and spends a relatively large fraction of its transient time within large attractors. The phenomenon may be viewed, on the other hand, to be caused by the "damping" term $\nu\dot{x}(t)$.

Although there are many publications on equation (1) (see the List of References, for example), a series of natural and easily formulated questions about its dynamics have not been resolved in general so far. We would like to indicate some of these questions here.

1. Suppose the map f has an attracting cycle $\{a_1, a_2, \cdots, a_n\}$ of period n. Under what additional conditions does equation (1) possess an asymptotically stable periodic solution $p_\nu(t)$ with period $n+O(\nu)$, which converges to $p_0(t) = a_{k(\bmod n)}$, $t \in [k-1, k)(\bmod n)$ as $\nu \to +0$?

As was shown above (Theorem 5.1) attracting cycles of the map f need not give rise to nearby periodic solutions of equation (1). Some existence results on periodic solutions corresponding to cycles of intervals were given (Theorem 5.2.). No particular results on the stability of such solutions is known (except examples; see no. 4.2, 4.4).

A related question is the following. In what cases does a cycle $\{a_1, a_2, \cdots, a_n\}$ of the map f give rise to several nearby periodic solutions of equation (1)?

A particular variant of this first problem is the following.

2. Suppose the map f has an attracting cycle $\{a_1, a_2, \cdots, a_n\}$ which is a global attractor (this means that it attracts almost all trajectories, in a topological or measure sence). Does equation (1) have a periodic solution nearby this cycle? If so, under what additional conditions is the periodic solution asymptotically stable?

3. Supppose the map f has a so-called simple structure. That is, its topological entropy equals zero. This means that the map f has only cycles with periods given by powers of two. What additional conditions guarantee that the dynamics of equation (1) is also simple. That is, does equation (1) possess an asymptotically stable periodic solution which attracts almost all solutions? Almost all means that the attracted set is residual.

4. Suppose the map f has a complicated structure. For example, suppose there exists a cycle of period $(2k+1)2^{i-1}$ for some positive integers k and i. This guarantees (see, e.g. [54]) the existence of a homoclinic trajectory for the map f, implying its chaotic dynamics.

What additional hypotheses are needed to produce complicated behavior of the solutions of equation (1)? When is it possible to prove the existence of a homoclinic solution for equation (1) which would imply complicated behavior?

Transversal homoclinic solutions with chaotic behavior for differential-delay equations were shown to exist in several examples [19, 23, 36, 61] (see remarks at the end of Chap. 4 also) and for an equation on the circle [63] in a general case.

5. A bifurcation problem

For some families f_λ of interval maps depending on a real parameter λ, a complication in the dynamics arises through period doubling bifurcations as λ varies (increases or decreases). In what cases is this complication followed by corresponding changes for equation (1)?

In the simplest case, the situation is as follows. There exists a sequence $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ of parameter values, convergent to some $\lambda_* < \infty$, such that the map f_λ has a globally attracting cycle of period 2^n for every $\lambda \in (\lambda_n, \lambda_{n+1}]$. For what families f_λ does the corresponding equation (1) possess an attracting periodic solution nearby the cycle of period 2^n, $n = 0, 1, 2, \cdots, \lambda \in (\lambda_n, \lambda_{n+1}]$?

Specifically we restate the problems for the familiy $f_\lambda(x) = \lambda x(1-x)$, $0 \le \lambda \le 4$, which map the interval $[0,1]$ into itself.

It is well-known [54] that there exists a sequence of parameter values $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \lambda_* \approx 3,569$ such that for every particular $\lambda \in (\lambda_n, \lambda_{n+1}]$ the map f_λ has cycles with periods $1, 2, 4, \cdots, 2^n$ only and the cycle of period 2^n is a global attractor (it does not attract only repelling cycles of periods $1, 2, 4, \cdots, 2^{n-1}$ and their preimages). For every $\lambda > \lambda_*$ the map f_λ has a cycle of period $(2k+1)2^{i-1}$ for some k, $i \in N$. For an open set of parameter values λ, these cycles are global attractors. There exists a set A in the parameter space [0,4] of positive Lebesgue measure, such that for every particular $\lambda \in A$ the map f_λ has an invariant measure which is absolutely continuous with respect to the Lebesgue measure.

Considering equation (1) with $f(x) = f_\lambda(x) = \lambda x(1-x)$, $0 \le \lambda \le 4$, the specific questions are:

1. Suppose f_λ has a globally attracting cycle of period n. Does equation (1) have a (asymptotically stable) periodic solution "close" to this cycle? If it does, what is the domain of attraction of the periodic solutions?

2. Suppose λ increases within the interval $1 < \lambda < \lambda_* \approx 3,569$ with f_λ going through period doubling bifurcations. Are these bifurcations followed by corresponding changes in the dynamics of equation (1)? That is: when $\lambda \in (\lambda_n, \lambda_{n+1}]$ does equation (1) have asymptotically stable periodic solution close to the particular cycle of period 2^n?

Some computer simulations for a different family [6] suggest this.

3. Do there exist values of the parameter λ for which the semiflow $\mathcal{F}^t$ defined by equation (1) admits the existence of an ergodic invariant measure in the phase space (or on a subset of the phase space)? If so, what is the measure of such λ's.

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