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Global dynamics of a differential equation with piecewise constant argument

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ABSTRACT

Several aspects of global dynamics are studied for the scalar differential-difference equation $\varepsilon\dot{x}(t) + x(t) = f(x([t]))$, $0 < \varepsilon \ll 1$, where $[\cdot]$ is the integer part function. The equation is a particular case of the special discretization (discrete version) of the singularly perturbed differential delay equation $\varepsilon\dot{x}(t) + x(t) = f(x(t-1))$. Sufficient conditions for the invariance, global stability of equilibria, existence, stability/instability, and shape of periodic solutions, and the chaotic behavior are derived. The principal analysis is based on the reduction of its dynamics to the one-dimensional map $F : x \to f(x) + [x - f(x)]e^{-1/\varepsilon}$, many relevant properties of which follow from those of the interval map f.

1. Introduction

This paper deals with several aspects of global dynamics of the scalar differential equation with piecewise constant argument

$$\varepsilon\dot{x}(t) + x(t) = f(x([t])), \quad t \in \mathbb{R}^+ \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, $[\cdot]$ is the integer part function, and $\varepsilon > 0$ is a parameter. Eq. (1) is a particular case of the more general equation

$$\varepsilon\dot{x}(t) + x(t) = f(x([t-K])), \quad K \in \mathbb{N} \cup \{0\} \tag{2}$$

which is considered in a companion paper [1]. Eq. (2) falls within the class of the so-called differential equations with piecewise constant argument (EPCA) [2,3]. It is also a special discretization (discrete version) of the singularly perturbed differential delay equation

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)). \tag{3}$$

The latter has been a subject of intensive study in numerous publications for the past 20 years (see review paper [4] and additional references in monographs [5,6]). When $\varepsilon = 0$ Eq. (3) becomes the continuous time difference equation

$$x(t) = f(x(t-1)), \quad t \in \mathbb{R}^+. \tag{4}$$

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The dynamics of Eq. (4) are well understood and studied: they are completely determined by the dynamics of the interval map f; their comprehensive theory and applications are described in the monograph [7].

Both Eqs. (3) and (4) have a great variety of diverse applications; see for example relevant references in [5,6]. One of the applications is that they appear as exact reductions of nonlinear boundary value problems for hyperbolic partial differential equations modeling wave phenomena [7]. When the viscosity/capacity in the oscillating media is disregarded the reduction is to pure difference equations of the form (4) (in simplest cases). However, when the small viscosity/capacity is taken into account one obtains singularly perturbed differential delay equations of the form (3). The latter is a more accurate and realistic differential delay model of the oscillating phenomena than the former. A natural expectation is that many dynamical properties of solutions of Eq. (3) follow from those of Eq. (4) (or interval map f). However, the relationships in dynamics between the two have appeared to be much more involved, even in the case of simple dynamical behavior of the interval map f [4,8,9]. There are two phenomena observed here: (i) the complification, when the dynamics of map f is simple while the dynamics of differential delay equation (3) is complex; and (ii) the simplification, when the dynamics of map f is complicated but the asymptotic dynamics as $t \to \infty$ of solutions of Eq. (3) is simple; see [4] and the introductory part of [1] for more details and examples. There are some further related non-trivial properties of solutions of Eq. (3), such as stable rapidly oscillating solutions and associated structure of the Morse decomposition; see References [4,10,8,11] for additional information.

We show in this paper that, for small $\varepsilon > 0$, many dynamical properties of solutions of equation (1) follow those of interval map f (difference equation (4)). They include the invariance and global stability properties, the existence, uniqueness and stability/instability of periodic solutions corresponding to hyperbolic cycles of map f, and the chaotic behavior derived from the chaotic dynamics present in f. Thus, unlike Eqs. (3) and (4), there exists much more close and natural relationship between the dynamics of the interval map f (Eq. (4)) and singularly perturbed differential-difference equation (1) (for small $\varepsilon > 0$).

This paper can also be viewed as an extension of recently submitted paper [1], as applied to the partial case of Eq. (2). Here we derive some additional and complimentary results to those obtained there. They show, in particular, that the special case $K = 0$ of Eq. (2) is, in a sense, the best and closest reflection of the dynamics of interval map f (Eq. (4)) among both Eq. (3) and the general case of Eq. (2).

2. Preliminaries

We shall use some basic and well-known definitions, facts, and results from the theory of one-dimensional dynamical systems (interval maps) necessary for the exposition in the present paper. For those and other related additional details and facts we refer the reader to monographs [12,13].

We shall make a standing assumption, to hold for the remainder of this paper, that the map f possesses a closed bounded invariant interval I: $f(x) \in I$ for every $x \in I$.

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ be a cycle of (minimal) period m of the map f: $b_{i+1} = f(b_i)$, $i = 1, 2, \ldots, m-1$, and $b_1 = f(b_m)$. We shall assume that the nonlinearity f is a continuously differentiable function along the cycle $\mathcal{B}$ with the multiplier $\mu(\mathcal{B}) = f'(b_1) \cdot \ldots \cdot f'(b_m)$ satisfying $|\mu(\mathcal{B})| \neq 1$. That is, $\mathcal{B}$ is a hyperbolic cycle.

There are several definitions of chaos (or chaotic dynamics) in one-dimensional maps, such as Li–York chaos, strongly or weakly chaotic maps, etc. We do not discuss specific definitions and relationships between them but rather refer the reader to monographs [12,13] for all those details. Note, however, that the chaotic behavior in one-dimensional maps is closely associated with the famous cycle coexistence ordering of Sharkovsky [13,12]

$$\begin{aligned}
&3 \succ 5 \succ 7 \succ 9 \succ \cdots \succ (2k-1) \succ (2k+1) \succ \cdots \\
&\succ 3\cdot 2 \succ 5\cdot 2 \succ 7\cdot 2 \succ 9\cdot 2 \succ \cdots \succ (2k-1)\cdot 2 \succ (2k+1)\cdot 2 \succ \cdots \\
&\succ 3\cdot 4 \succ 5\cdot 4 \succ 7\cdot 4 \succ 9\cdot 4 \succ \cdots \succ (2k-1)\cdot 4 \succ (2k+1)\cdot 4 \succ \cdots \\
&\succ 3\cdot 2^k \succ 5\cdot 2^9\cdot 2^k \succ 9\cdot 2^k \succ \cdots \succ (2k-1)\cdot 2^k \succ (2k+1)\cdot 2^k \succ \cdots \\
&\cdots \quad \cdots \\
&\cdots \succ 2^{k+1} \succ 2^k \succ \cdots 8 \succ 4 \succ 2 \succ 1.
\end{aligned} \tag{5}$$

For the purpose of this paper we shall call a continuous interval map chaotic if it has a cycle of period not a power of two. Clearly, the chaotic behavior is limited to a subset of the phase space, as a map can have other non-chaotic dynamics at the same time, such as e.g. attracting cycles.

Associated with the cycle $\mathcal{B}$, the continuous time difference equation (4) admits a set of m piecewise constant periodic solutions $P_k^0(t)$:

$$\begin{aligned}
&P_k^0(t) = b_{k+i-1(\bmod m)} \quad \text{for } t \in [i-1, i),\ i, k = 1, 2, \ldots, m; \\
&P_k^0(t) \equiv P_k^0(t+m) \quad \forall t \in \mathbb{R}.
\end{aligned} \tag{6}$$

Note that all $P_k^0(t)$ are shifts of each other. For example, $P_2^0(t) \equiv P_1^0(t+1)$, etc.

The solution $x = x^{\varepsilon}(t, x_0)$ of the initial value problem

$$\varepsilon\dot{x}(t) + x(t) = f(x([t])), \quad x(0) = x_0 \in I \tag{7}$$

is easily found for $t \in [0, 1)$ by direct integration

$$x(t) = f(x_0) + [x_0 - f(x_0)]\,\mathrm{e}^{-t/\varepsilon}, \quad t \in [0, 1). \tag{8}$$

By the continuity,

$$x(1) = x^{\varepsilon}(1, x_0) := \lim_{t\to 1^-} x(t) = f(x_0) + [x_0 - f(x_0)]\,\mathrm{e}^{-1/\varepsilon} := x_1.$$

On the next interval $[1, 2)$ the solution $x(t)$ of Eq. (1) is found by the same formula (8) with x_1 as the initial value:

$$x(t) = x^{\varepsilon}(t) = f(x_1) + [x_1 - f(x_1)]\,\mathrm{e}^{-(t-1)/\varepsilon}, \quad t \in [1, 2),$$
$$x(2) := \lim_{t\to 2^-} x(t) = f(x_1) + [x_1 - f(x_1)]\,\mathrm{e}^{-1/\varepsilon}.$$

The above process of successive step-by-step integration and matching can be continued by induction to any interval $[i-1, i]$, $i \in \mathbb{N}$. Therefore, given arbitrary $x_0 \in I$, the corresponding solution $x = x^{\varepsilon}(t, x_0)$ of Eq. (1) exists for all $t \geq 0$ and is a continuous piecewise exponential function.

Note that in order to solve a more general initial value problem for Eq. (1)

$$\varepsilon\dot{x}(t) + x(t) = f(x([t])), \quad x(t_0) = x_0, \quad t \geq t_0 \geq 0$$

one formally needs an initial function $\phi(s)$, $s \in E_{t_0}$, where E_{t_0} is the initial set associated with the point t_0. In the case when $[t_0] < t_0$, for every integrable function $\phi(s)$, defined on E_{t_0}, the corresponding solution of Eq. (1) is still found on the interval $[t_0, [t_0]+1]$ by direct integration, $x^{\varepsilon}_{\phi}(t) = f(\phi([t_0])) + [x(t_0) - f(\phi([t_0]))]\,\mathrm{e}^{-(t-t_0)/\varepsilon}$. Therefore, $x^{\varepsilon}_{\phi}([t_0]+1)$ is well defined, and the solution is then constructed for $t \geq [t_0]+1$ exactly the same way as it is done above for the case $t_0 = 0$. In view of this, we will only be dealing with the initial value problem (7).

One of our results (stated and proved in next section) deals with the convergence of periodic solutions of differential equation (1) as $\varepsilon \to 0+$ to the above introduced square-wave periodic solutions $P^0_k(t)$ of difference equation (4), defined by formula (6). Since solutions of Eq. (1) are continuous and the functions $P_{0k}(t)$ are discontinuous at the integer values $t_i = i$, $i \in \mathbb{N} \cup \{0\}$, a special definition of the convergence is required.

Definition. The solution $x(t) = x^{\varepsilon}(t, x_0)$ of the initial value problem (7) is said to converge on the interval (t_1, t_2) to a constant A as $\varepsilon \to 0+$ if for arbitrary $\sigma > 0$ and $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that

$$|x(t, x_0) - A| < \sigma \quad \text{for all } t \in [t_1 + \delta, t_2 - \delta], \text{ and all } 0 < \varepsilon < \varepsilon_0.$$

Remark 1. Note that in the case when $x_0 \in I$ is fixed and independent of ε, the solution $x(t)$ converges to the constant $A := f(x_0)$ on the interval $(0, 1)$ as $\varepsilon \to 0+$. Indeed, it follows from the fact that for arbitrary (and fixed) $\sigma > 0$ and $\delta > 0$ the $\varepsilon_0 > 0$ can be chosen to satisfy $|x_0 - f(x_0)|\,\mathrm{e}^{-\delta/\varepsilon_0} < \sigma$. Since the solution $x(t)$ given by (8), $x(t, x_0) = f(x_0) + [x_0 - f(x_0)]\,\mathrm{e}^{-t/\varepsilon}$, is monotone in $[0, 1]$ the required inequality follows

$$|x(t, x_0) - f(x_0)| < \sigma \quad \text{for all } t \in [\delta, 1] \text{ and all } 0 < \varepsilon < \varepsilon_0. \tag{9}$$

Note also that in view of the existence of the invariant interval I for the map f, $f(x) \in I$ for all $x \in I$, the last inequality can be made uniform for all $x_0 \in I$ since $|x_0 - f(x_0)|$ is uniformly bounded.

3. Main results

Like in the case of Eq. (3) there is a number of simple properties of solutions of Eq. (1), which can be viewed as induced by (or "inherited" from) the one-dimensional map f. Some of them are stated and proved in [1]. We list them below, for the sake of completeness, and provide a proof or related reference.

Proposition 1 (*Invariance*). *Suppose $x_0 \in I$. The corresponding solution $x(t) = x^{\varepsilon}(t, x_0)$ of Eq. (1) satisfies $x(t) \in I$ for all $t \geq 0$ and every $\varepsilon > 0$.*

Proof. It can be derived in the same way as for Eq. (3) in paper [4]. Another way to see the invariance property holding for Eq. (1) is from the form of its solutions given by (8). Since $x^{\varepsilon}(t, x_0)$ is monotone for $t \in [0, 1]$ with its range contained in the interval $\langle x(0), f(x(-K))\rangle \subset I$, its range remains within I for all $t \in [0, 1]$ (here, $\langle a, b\rangle$ stands for an interval with the endpoints a and b). The validity of the claim for all $t \geq 0$ follows by induction. □

As it is easily seen every fixed point x_* of map f generates a constant solution $x(t) \equiv x_*$ of Eq. (1). When it is an attracting fixed point for the map f so is the corresponding constant solution for the differential equation (1), as the following statement shows. It is completely analogous to the similar one for differential delay equation (3) established in [4]. We refer the reader to the latter for the proof and additional details; see also [1].

Proposition 2 (*Stability of Equilibria*). *Suppose x_* is an attracting fixed point of the map f with the domain $I_0 \subseteq I$ of immediate attraction. Then the constant solution $x(t) \equiv x_*$ of Eq. (1) is locally asymptotically stable. Moreover, for any initial point $x_0 \in I_0$ the corresponding solution $x(t) = x^\varepsilon(t, x_0)$ satisfies $\lim_{t\to\infty} x(t) = x_*$ (for every $\varepsilon > 0$).*

An immediate implication of Proposition 2 is the following.

Corollary 1 (*Global Stability*). *Suppose x_* is a globally attracting fixed point of the map $f: f^n(x) \to x_*$ as $n \to \infty$ for any $x \in I$. Then the constant solution $x(t) = x_*$ of Eq. (1) is also globally asymptotically stable: $\lim_{t\to\infty} x^\varepsilon(t, x_0) = x_*$ for all $x_0 \in I$ and every $\varepsilon > 0$.*

Theorem 1 (*Existence of Periodic Solutions*). *Suppose interval map f has a hyperbolic cycle $\mathcal{B} = \{b_1, \ldots, b_m\}$ of period m. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ Eq. (1) has a unique periodic solution $x = P_\varepsilon(t)$ of period m generated by the cycle $\mathcal{B}$.*

Proof. As noted above, the solution $x(t) = x^\varepsilon(t, x_0)$ of the initial value problem (7)

$$\varepsilon\dot{x}(t) + x(t) = f(x([t])), \quad x(0) = x_0$$

is given by formula (8)

$$x(t) = f(x_0) + [x_0 - f(x_0)]\,\mathrm{e}^{-t/\varepsilon}, \quad t \in [0, 1).$$

For $x \in I$ set

$$F(x) = F(x, \varepsilon) := f(x) + [x - f(x)]\,\mathrm{e}^{-\frac{1}{\varepsilon}}. \tag{10}$$

Note that when $f(x)$ is continuously differentiable (on I) so is $F(x)$. Moreover, $F(x, 0) = f(x)$, $F'(x, 0) = f'(x)$ and

$$\lim_{\varepsilon\to 0+} F(x, \varepsilon) = f(x), \quad \lim_{\varepsilon\to 0+} F'(x, \varepsilon) = f'(x) \quad \text{uniformly for } x \in I. \tag{11}$$

With $x(1) = F(x_0)$ the asymptotic behavior of the solution $x(t) = x^\varepsilon(t, x_0)$ as $t \to \infty$ is determined by successive iterations of the map F, F^n as $n \to \infty$.

(a) Assume first that the interval map f has an attracting cycle $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ of period m. Let $U := U_1 \cup U_2 \cup \cdots \cup U_m$ be its domain of immediate attraction. Each U_k is the maximal open interval containing b_k and such that $f(U_k) = U_{k+1(mod\ m)}$. Moreover, each b_k is an attracting fixed point of the map f^m with U_k, $k = 1, 2, \ldots, m$ as its domain of immediate attraction (see [12,13]) for details).

Since b_1 is an attracting fixed point of the map f^m there exists a closed interval $J_1 := [\alpha_1, \beta_1] \subset U_1$ such that $f^m(J_1) \subset int\, J_1$, where $int\, J_1 := (\alpha_1, \beta_1)$ is the interior part of the closed interval J_1. Set $f(J_1) = J_2 := [\alpha_2, \beta_2], f(J_2) = J_3 := [\alpha_3, \beta_3] \ldots, f(J_m) = J_* := [\alpha_*, \beta_*]$. Then $[\alpha_*, \beta_*] \subset int\, J_1$. In view of the form of the map F given by (10), for arbitrary $\sigma_k > 0$ there exists ε_k such that $F(J_k) \subset [\alpha_{k+1} - \sigma_k, \beta_{k+1} + \sigma_k]$, $k = 1, 2, \ldots, m-1$ for all $0 < \varepsilon < \varepsilon_k$, and $F(J_m) \subset [\alpha_* - \sigma_k, \beta_* + \sigma_k]$ for all $0 < \varepsilon < \varepsilon_m$. In view of this and the continuity of F, for arbitrary $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that $F^m(J_1) \subseteq J_1$ for all $0 < \varepsilon < \varepsilon_0$. Therefore, F^m has a fixed point $x_* \in J_1$. Since, by the construction, all points $F^k(x_*)$, $k = 0, 1, 2, \ldots, m-1$ are disjoint they correspond to an m-periodic solution $x_*^\varepsilon(t)$ of Eq. (1).

(b) Assume next that the interval map f has a hyperbolic repelling cycle $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ of period m with the multiplier $|\mu(\mathcal{B})| = |f'(b_1) \cdot f'(b_2) \cdot \ldots \cdot f'(b_m)| \neq 1$. Then $f'(b_k) \neq 0$ for all $k = 1, 2, \ldots, m$, – otherwise $\mathcal{B}$ would be attracting. Moreover, since $f'(x)$ is continuous in a neighborhood of the cycle $\mathcal{B}$ there exist intervals $J_k \ni b_k$ such that $f'(x) \neq 0$, $\forall x \in J_k$, $k = 1, \ldots, m$, and $|\mu_*| = |f'(x_1) \cdot f'(x_2) \cdot \ldots \cdot f'(x_m)| > 1$ for arbitrary selection of $x_1 \in J_1, \ldots, x_m \in J_m$. Thus, there exists a closed interval $J_0 \ni b_1$ such that $|\frac{\mathrm{d}}{\mathrm{d}x} f^m(x)| > 1$ for all $x \in J_0$. Due to the continuity of both $F(x; \varepsilon)$ and $F'(x, \varepsilon)$ in $\varepsilon > 0$ at $\varepsilon = 0+$, and since $F(x; 0) = f(x)$, $F'(x; 0) = f'(x)$ (see (11)), there exists an interval $J_* \subseteq J_0$ such that it is mapped by F^m onto itself, $F^m(J_*) \supset J_*$, and $|\frac{\mathrm{d}}{\mathrm{d}x} F^m(x)| > 1$ for all $x \in J_*$. Therefore, F^m has a (unique) fixed point $x_* \in J_*$. The interval J_* can be chosen small enough, so that the points $x_*, F(x_*), \ldots, F^{m-1}(x_*)$ are disjoint. They form a cycle of period m of the map F, which corresponds to a m-periodic solution of Eq. (1). As in part (a), the initial value problem $x(0) = x_*$ for Eq. (1) gives rise to a periodic solution $P^\varepsilon(t) = x_*^\varepsilon(t)$ of period m. This completes the proof of existence of periodic solutions.

Note here that by choosing the interval J_1 in part (a) of the proof, or the interval J_0 in part (b), to contain any other point $b_k \neq b_1$ of the cycle $\mathcal{B}$, one obtains a "shifted" cycle $\{b_k^*, b_{k+1}^*, \ldots, b_{k-1}^*\}$ for the map F. This results in a formally different periodic solution of Eq. (1) which is, however, a mere shift of the periodic solution $x = P^\varepsilon(t)$ (obtained via the original choice of b_1).

(c) Uniqueness: In the case of hyperbolic cycle $\mathcal{B}$, $\mu(\mathcal{B}) \neq 1$, the uniqueness of the periodic solution $P^\varepsilon(t) = x_*^\varepsilon(t)$ of period m follows from the uniqueness of the fixed point x_* of map F^m. The latter is deduced from the following observation. There exists a closed interval $J_0 \ni x_*$ about the fixed point $x = x_*$ of the map F^m such that $|\frac{\mathrm{d}}{\mathrm{d}x}(F^m(x))| \neq 1$ for all $x \in J_0$. This follows from (11). Therefore, the interval J_0 can be chosen in such a way that it is mapped into itself (when $|\mu(\mathcal{B})| < 1$) or onto itself (when $|\mu(\mathcal{B})| > 1$), in either case guaranteeing the uniqueness of the fixed point x_*. The unique fixed point x_* generates a unique periodic solution $P^\varepsilon(t) = x_*^\varepsilon(t)$ of period m of differential Eq. (1). This completes the proof of the theorem. □

Remark 2. As it can be easily seen from the above proof, part (a), the existence of a periodic solution of Eq. (1) can also be shown in the case of a non-hyperbolic attracting cycle $\mathcal{B}$ of the map f. Indeed, in this case there still exists a closed interval $J_1 = [\alpha, \beta] \ni b_1$ such that $f^m(J_1) \subset intJ_1$. Therefore, the construction of the fixed points for map F can be done the same way as in the proof of Theorem 1. The uniqueness of the periodic solution cannot be claimed in this case, however.

Remark 3. Note that the existence of an invariant interval I for the map f is not necessary for the existence of periodic solutions established by Theorem 1. In each of the cases (a) or (b) of the proof, this follows from the construction of the fixed point x_* for the map F^m on the interval J_* for all $0 < \varepsilon < \varepsilon_0$.

Remark 4. Fixed point x_* of the map F generates a cycle $\mathcal{B}^* = \{b_1^*, \ldots, b_m^*\}$, where $b_1^* := x_*, b_k^* := F(b_{k-1}^*), k = 1, \ldots, m-1$, and $b_1^* := F(b_m^*)$. We claim that the cycle $\mathcal{B}^*$ converges to the cycle $\mathcal{B}$ of the map f as $\varepsilon \to 0+$. Namely, that for arbitrary $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that $|b_k - b_k^*| < \sigma$ for all $0 < \varepsilon < \varepsilon_0$. This follows from the continuity of $F(x, \varepsilon)$ in ε at $\varepsilon = 0+$ (see (11)), the fact that $F(x, 0) = f(x)$, and that the interval J_1 in the proof of Theorem 1 can be chosen sufficiently small.

Theorem 2 (*Asymptotic Shape as $\varepsilon \to 0+$*). *The periodic solution $P^\varepsilon(t)$ of Theorem 1 converges, as $\varepsilon \to 0+$, to the piecewise constant periodic solution $x = P_1^0(t)$ of difference equation (4) defined by the cycle $\mathcal{B}$.*

Proof. Let x_* be the fixed point of the map F^m, which existence is established by Theorem 1. Consider the initial value problem

$$\varepsilon\dot{x}(t) + x(t) = f(x([t])), \quad x(0) = x_* = b_1^*.$$

Its solution $x(t) = x^\varepsilon(t, x_*)$ is the corresponding periodic solution $P^\varepsilon(t)$ of differential Eq. (1). It is given on the interval $[0, 1]$ by formula (8)

$$x(t) = f(x_*) + [x_* - f(x_*)]\,\mathrm{e}^{-t/\varepsilon}, \quad t \in [0, 1).$$

Given arbitrary $\sigma > 0$ and $\delta > 0$, and in view of formula (9) (Remark 1) and Remark 4, one has on the interval $[\delta, 1]$

$$|b_2 - x(t, x_*)| \le |f(x_*) - b_2| + |x_* - f(x_*)|\,\mathrm{e}^{-t/\varepsilon} \le \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma \quad \text{for all } 0 < \varepsilon < \varepsilon_0.$$

This proves the desired convergence on the interval $(0, 1)$. The same reasoning can be repeated for the initial value problem

$$\varepsilon\dot{x}(t) + x(t) = f(x([t])), \quad x(0) = x_* = b_2^*$$

to prove the convergence on the next interval $(1, 2)$, and then for all the remaining intervals $(2, 3), \ldots, (m-1, m)$. This completes the proof of convergence. □

Theorem 3 (*Stability of Periodic Solutions*). *Suppose that the interval map f has a hyperbolic cycle $\mathcal{B}$ of period m. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the periodic solution $x = P^\varepsilon(t)$ of Theorem 1 is* (i) *asymptotically stable with the asymptotic phase if the cycle $\mathcal{B}$ is attracting, and* (ii) *unstable if the cycle $\mathcal{B}$ is repelling.*

Proof. Since the periodic solution $x = P^\varepsilon(t)$ of differential Eq. (1) is generated by the cycle $\mathcal{B}^* = \{b_1^*, \ldots, b_m^*\}$ of the map F its stability is determined by the corresponding multiplier $\mu(\mathcal{B}^*) = F'(b_1^*) \cdot \ldots \cdot F'(b_m^*)$. Since the cycle $\mathcal{B}^*$ of the map F converges to the cycle $\mathcal{B}$ of the map f (see Remark 4), and in view of the continuity of F and F' in ε at $\varepsilon = 0+$ given by (11), for arbitrary $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that

$$|\mu(\mathcal{B}^*) - \mu(\mathcal{B})| = |F'(b_1^*) \cdot \ldots \cdot F'(b_m^*) - f'(b_1) \cdot \ldots \cdot f'(b_m)| < \sigma$$

for all $0 < \varepsilon < \varepsilon_0$. Therefore, an attracting cycle $\mathcal{B}$ of the map f with $|\mu(\mathcal{B})| < 1$ yields an asymptotically stable periodic solution $P^\varepsilon(t)$ of differential Eq. (1). Likewise, a repelling cycle $\mathcal{B}$ with $|\mu(\mathcal{B})| > 1$ gives rise to an unstable periodic solution of Eq. (1). □

Next theorem concerns the chaotic behavior in differential Eq. (1). We say that its solutions exhibit chaotic dynamics on a subset $S_0 \subseteq \mathbb{R}$ of initial values x_0, if there exists a chaotic one-dimensional map whose dynamics are equivalent to those of differential-difference equation (1) on the set S_0.

Theorem 4 (*Chaotic Behavior*). *Suppose interval map f has a cycle $\mathcal{B}$ of period $m \neq 2^i$. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, solutions of Eq. (1) exhibit chaotic dynamics (on a subset of initial values x_0).*

Proof. Since the asymptotic dynamics of solutions of Eq. (1) as $t \to \infty$ is completely determined by those of the iterations of map F, we shall deduce the chaos from the latter.

Suppose the interval map f has a cycle $\mathcal{B}$ of period $m \neq 2^i, i \in \mathbb{N} \cup \{0\}$. According to Sharkovsky's ordering (5) it also has a cycle of any other period n following m, that is when $m \succ n$. Due to a result by L. Block [14], if a continuous map f of an interval I has a cycle of period m then the map $F(x) := f(x) + \epsilon g(x), g \in C(I, I)$ has a cycle of every period n such that $m \succ n$ in the ordering (5) for all $0 \le \epsilon \le \epsilon_0$ and some $\epsilon_0 > 0$. Therefore, we apply Block's result to our map $F(x) = f(x) + [x - f(x)]\,\mathrm{e}^{-1/\varepsilon}$ and use (11) to conclude the existence of chaos in Eq. (1) for all sufficiently small $\varepsilon > 0$. □

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