Problems for Mexico Winter School

Curto and Itskov lectures · Jan 25-27, 2017

1. Exercises 1

1. Prove that for any code $\mathcal{C}$ on n neurons, there exists an open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ in $X = \mathbb{R}^1$ such that $\mathcal{C} = \mathcal{C}(\mathcal{U})$.
2. See Figure on the board (Figure 3 from [NR]). Identify $d(\mathcal{C})$ for each of the four codes. (*) Prove that the $d = 2$ codes cannot be realized in $d = 1$.
3. For arrangement of $U_1, \ldots, U_4$ in Example 1 (Figure 2 of [MRC-1]), compute the nerve $\Delta = \mathcal{N}(\mathcal{U})$. Next, compute $\mathcal{N}(\{U_i \cap U_2\}_{i=1,3,4})$ for the restricted cover onto U_2. Verify that the restricted nerve is the link $\mathrm{Lk}_2(\Delta)$.
4. Find $\mathcal{C}_{\min}(\Delta)$ for L6, L7, L18, L22, L25, L26. (Simplicial complex labels from [MRC-1].)

2. Examples of codes

- $\mathcal{C}_0 = \{110, 101, 001, 010, 000\} = \{12, 13, 3, 2, \emptyset\}$.
- $\mathcal{C}_1 = \{110, 101, 011, 000\} = \{12, 13, 23, \emptyset\}$.
- $\mathcal{C}_2 = \{123, 134, 142, 25, 35, \emptyset\}$.
- $\mathcal{C}_3 = \{123, 234, 12, 13, 23, 34, 1, 3, 4, \emptyset\}$.
- $\mathcal{C}_4 = \{2345, 123, 134, 145, 13, 14, 23, 34, 45, 3, 4, \emptyset\}$.
- $\mathcal{C}_5 = \{123, 134, 145, 12, 13, 14, 15, 2, 3, 4, 5, \emptyset\}$.
- $\mathcal{C}_6 = \{123, 124, 12, 13, 14, 23, 24, 2, 3, 4, \emptyset\}$.
- $\mathcal{C}_7 = \{1111, 1011, 1101, 1100, 0011, 0010, 0001, 0000\} = \{1234, 134, 124, 12, 34, 3, 4, \emptyset\}$.

3. Exercises 2

1. Show that $\mathcal{C}_0$ and $\mathcal{C}_1$ are not convex by finding a local obstruction. (We did $\mathcal{C}_0$ in the lecture.)
2. (*) For $n \leq 3$, a code is convex if and only if it has no local obstructions. Find all non-convex codes for $n \leq 3$.
3. Show that $\mathcal{C}_2$ is not convex. What codewords must be added to turn it into a convex code, without altering the simplicial complex? (Hint: compute $\mathcal{C}_{\min}(\Delta(\mathcal{C}))$.)
4. Show that $\mathcal{C}_3$ is a convex code. (Hint: show that it's max $\cap$-complete.)
5. Show that $\mathcal{C}_4$ has no local obstructions. (*) Can you prove it's a closed convex code?

6. Show that $\mathcal{C}_5$ is not max $\cap$-complete. Show that it's a convex code by drawing an explicit realization.

7. Show that $\mathcal{C}_6$ is a convex code, but is not a hyperplane code. (Hint: use the bit flip strategy.)

8. Show that $\mathcal{C}_7$ is convex, with $d(\mathcal{C}) = 2$, by finding an explicit convex realization.

9. (*) Show that any code on n neurons with the all-ones word, $11 \cdots 1$, has a convex realization in dimension 2.

4. Exercises 3: Order Complex

1. Show the third order complex in Figure 2A of the Clique Topology paper (Giusti et. al., PNAS 2015) cannot be realized as the ordering of distances for a configuration of points $\{p_1, \ldots, p_5\}$ in $\mathbb{R}^2$.

2. Construct another, combinatorially distinct, example of a 5×5 order complex that can not be geometrically realized in $\mathbb{R}^2$.

3. Find a general algorithm for determining whether or not an order complex can be geometrically realized in $\mathbb{R}^1$.

4. Prove that any $n \times n$ order complex with distinct entries can be geometrically realized in $\mathbb{R}^{n-1}$.

5. Design an order complex that has a persistent 1-cycle. Design one with a persistent 2-cycle.