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Waiting to settle: Multilateral bargaining with subjective biases

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Abstract

We study multilateral bargaining games where agents disagree over their bargaining power. We show that if agents are extremely optimistic, there may be costly delays in an arbitrarily long finite game but if optimism is moderate, all sufficiently long games end in immediate agreement. We show that the game with extreme optimism is highly unstable in the finite-horizon, and we examine the ramifications of this instability on the infinite-horizon problem. Finally, we consider other voting rules, and show that the majority-rule may be more efficient than the unanimity rule when agents are optimistic.

JEL classification: C72; C78

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"They were so strong in their beliefs that there came a time when it hardly mattered what exactly those beliefs were; they all fused into a single stubbornness."— Louise Erdrich

1. Introduction

Endemic to much of real-world bargaining is subjective uncertainty over the process that determines bargaining power. Typically, that process is amorphous and bargaining power is contingent upon uncertain events. In labor settlements, the value of uncertain outside

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options impacts views on bargaining power. In the negotiation of mergers and acquisitions, the bargaining position of different agents are influenced by each party's expectations of future market values. In oligopolistic collusion, the distribution of profits in the event of a price war induces uncertainty over which firm is in the strongest position to dictate its terms. In a repeated partnership, uncertainty over the abilities and future contribution of each agent will affect the negotiation of wages and shares. At a larger scale, the bargaining power in war and peace negotiations are influenced by expectations of international support. Such subjective uncertainty over the bargaining process—in the absence of clear information signals—may naturally foster conflicting expectations and disagreements over the distribution of bargaining power. Moreover, experimental and empirical evidence lead us to expect that people may cope with subjective uncertainty by forming *optimistic* beliefs whereby each individual expects future uncertainty to be resolved in her favor. [1] If each negotiator believes that future uncertainty will be resolved to her advantage, delayed agreement seems plausible as an outcome. Thus, an informal literature has developed that connects optimistic expectations to delays in bargaining. [2] However, the connection between optimism and delay has not been theoretically established.

It is our aim to provide a sharp characterization of the conditions that lead to delay and examine the bargaining dynamics that emerge when agents are optimistic in a multilateral bargaining game. In particular, we show that extreme optimism can cause delay in arbitrarily long finite horizon games but moderate optimism leads to immediate agreement in every sufficiently long game. Moreover, the bargaining dynamics that emerge in the event of delay bear a crucial resemblance to notions of *stalemates* and *ripe moments for agreement*. Thus, our model formalizes the causal connection between optimistic expectations of bargaining power and dynamics often exhibited by real-world negotiations.

We follow the approach in the literature—particularly, [8,9,16]—in connecting bargaining power to recognition in a random-proposer game. In each round, an agent is randomly recognized to make an offer, and all other agents vote to accept or reject the offer. Akin to the power of an agenda-setter, [3] recognition confers the right upon an agent to ask her opponents to accept a sure payoff or burn money through delay; as such, when an offer is accepted, the proposer captures rents from agreement. In the absence of outside options, agents' continuation values emerge from their ability to capture these agreement rents in future rounds, and thus, recognition directly translates into bargaining power.

Yildiz [16] examines a bilateral bargaining game where agents have optimistic beliefs over the recognition process. The examples below demonstrate that such beliefs can cause delays in short finite-horizon games, but yield immediate agreement in longer games.

Example 1 (*Yildiz [16], Example 0*). Consider a bilateral bargaining game where two risk-neutral players are dividing a dollar, share the common discount factor $\delta \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$, offers can be made at dates $t = 0$, and 1, and it is common knowledge that each player believes with probability 1 that she will be recognized at $t = 1$ independently of recognition at $t = 0$. Since the proposer at $t = 1$ captures the entire surplus, each player anticipates

[1] See [2,13].

[2] See [2,7].

[3] Baron and Ferejohn [3], and Merlo and Wilson [9] make this particular connection.

receiving δ in the event of disagreement at $t = 0$. Since $2\delta > 1$, whoever is recognized at $t = 0$ is unwilling to make an offer the other would be willing to accept. Hence, agreement is delayed.

Surprisingly, Yildiz [16] shows that if agents are persistently optimistic in this fashion, rational, and have common knowledge of optimism, then all sufficiently long bilateral bargaining games end with no delay. This is the Immediate Agreement Theorem and is demonstrated in the example below.

Example 2 (*Yildiz [16], Example 0 continued*). Consider an extension of the above bargaining game where offers can be made at dates $t = 0, 1, 2, 3$, $\delta \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$, and it is common knowledge that each player believes with probability 1 that she will be recognized at each date, independently of the recognition history. By the reasoning in Example 0, each agent knows that if agreement is not possible before $t = 2$, it will not be possible at $t = 2$. As such, at $t = 1$, each agent's discounted continuation value is δ^2. Since $1 - \delta^2 > \delta^2$, and $1 - \delta(1 - \delta^2) > \delta(1 - \delta^2)$, agreement is therefore possible at $t \in \{0, 1\}$, and is thus immediate.

Yildiz [16] establishes that backward induction under persistent optimism yields a stable process whereby if agents know that they will agree before the final round, they will agree immediately. The Agreement Theorem then follows: in sufficiently long games, if bargainers anticipate long periods of fruitless disagreement near the end of the game, they will be willing to accept smaller shares of the pie earlier in the game. Hence, optimism does not create delay in bilateral bargaining if the game is sufficiently long.

We demonstrate that optimism may yield delay in arbitrarily long games if there are more than two players. We consider a canonical multilateral extension to Yildiz's game where agents must unanimously agree to an offer for it to be accepted. We show that while the Immediate Agreement Theorem extends to multilateral bargaining games with moderate optimism, it fails if agents are extremely optimistic. In particular, backward induction is an unstable process where agents disagree because they each optimistically expect to capture future rents from agreement. While future agreement is common knowledge, each agent perceives the future distribution of the surplus to be in her favor, and has an option-value to waiting, despite the shrinking surplus.

Example 3. Consider a bargaining game where three risk-neutral players are dividing a dollar, offers can be made at dates $t = 0, 1, 2, 3, 4$, players share the common discount factor $\delta = 0.57$, and it is common knowledge that each player believes with probability 1 that she will be recognized at each date, independently of the recognition history. By the reasoning in Example 0, agents know that if agreement is not attained before $t = 3$, it shall not be attained at $t = 3$. Hence, at $t = 2$, each agent will agree if offered δ^2. Since $1 - 2\delta^2 > \delta^2$, agreement is attained at $t = 2$. Now let us move back to $t = 1$. Each agent will settle for $\delta(1 - 2\delta^2) \backsimeq 0.2$ since it is common knowledge that agreement is possible in the next period if it is not attained in the current period. As $1 - 2\delta(1 - 2\delta^2) > \delta(1 - 2\delta^2)$, agreement is possible at $t = 1$. Note that at $t = 1$, the proposer captures 60% of the entire surplus. Therefore, at $t = 0$, each agent rejects any offer that provides her with less than

$\delta(1-2\delta(1-2\delta^2)) \simeq 0.34$. Yet, no proposer would be willing to make such an offer since this is greater than what remains if she provided the others with their continuation values: $(1-2\delta(1-2\delta(1-2\delta^2))) \simeq 0.32$. Therefore, any agent recognized would prefer delay to providing the others with their continuation values. This leads to a delayed agreement at $t = 1$.

This example illustrates the dynamics exhibited generally by multilateral bargaining games with extreme optimism. Within the game, certain periods emerge endogenously as *ripe moments for agreement*, where the proposer expects to capture a large share of the surplus. These *ripe moments* lead to *stalemates* in immediately preceding periods. For example, at $t = 1$ in Example 3, the proposer captures 60% of the pie (in contrast to the agreement at $t = 2$ where the proposer captures a mere 35% of the pie) and therefore being the proposer at $t = 1$ is valuable. Since at $t = 0$, each agent expects to be recognized at $t = 1$, each agent perceives a high option-value from settling tomorrow instead of today. There is thus, delayed agreement on the equilibrium path.

Moreover, equilibrium strategies display "cyclical" properties with multiple transitions between agreement and disagreement regimes. In a bargaining game with optimism and many rounds, consider a time period τ where the proposer captures a large share of the surplus (the last date an offer can be made is one such time period). At dates $s < \tau$, each optimistic agent believes that in the event of disagreement until τ, she will be recognized at τ and capture this large surplus. If s is sufficiently close to τ, waiting is not particularly costly, and hence, each optimistic agent will accept only high offers at s. If there are many such agents, the proposer at date s may prefer waiting to settle at τ to acquiescing to each of the other optimistic agents high demands at date s. This precludes agreement in the periods immediately preceding τ. However, in periods in the distant past—i.e., when $\tau - s$ is large—there is common knowledge that agreement shall not be possible in the periods immediately preceding τ. Therefore, the option-value to waiting decreases as the high expectations from recognition at t are moderated by the increasing cost of delay. Agreement must then be possible at least at one date $t < \tau$. The instability of the backward induction process then comes into play, leading to the existence of another period near the date t where the proposer captures a large share of the surplus. This causes delays in even earlier periods.

The bargaining inefficiency represents the impact of irreconcilable beliefs. Agents unanimously agree that the delayed agreement does not lie on the Pareto frontier, but disagree over the set of Pareto-superior allocations. We are able to find a relatively tight bound for the cost of delay and show that it crucially depends upon the time interval between offers. In games where these intervals are long, delay can be very costly destroying over half and even 90% of the pie. On the other hand, frequent offers mitigate the costs of delay, and in the continuous-time limit of the game, agreement is virtually immediate. Therefore, a major implication of the Immediate Agreement Theorem is general: persistent optimism does not create inefficient delay if offers are frequent.

We also study the infinite-horizon game with subjective biases, and find that the cycles exhibited in the finite-horizon game have ramifications on the infinite-horizon equilibria. In the standard infinite-horizon multilateral bargaining games, multiplicity is an issue since every division of the surplus can be supported as a perfect equilibrium outcome. It is thus standard to select the stationary equilibria. The motivation for this refinement is that strategies in the stationary equilibria are usually the only subgame perfect strategy profile that is independent of past history. If agents are moderately optimistic, this equivalence holds making the stationary solution a natural candidate. However, if agents are extremely optimistic, the game possesses a continuum of non-stationary history-independent equilibria, a subset of which Pareto-dominate the stationary equilibrium. This set of equilibria also includes some that are only *slightly* non-stationary, one of which involves delayed agreement. This establishes that delay in the infinite-horizon game requires neither history-dependence nor complexity. Moreover, our finite-horizon results establish that with extreme optimism, the perfect equilibria of finite truncations do not converge to a stationary limit generically. By rendering backward induction unstable, extreme optimism introduces further challenges into the infinite-horizon equilibrium selection problem.

The paper is structured as follows. In Section 2, we characterize the multilateral bargaining environment and the recursive structure of equilibrium continuation values. In Section 3, we derive our main results. In Section 4, we examine the infinite-horizon bargaining game. Section 5 considers an application of our framework to symmetric legislative bargaining games with majority and super-majority voting rules and Section 6 concludes the paper. All proofs, unless otherwise noted, are in the appendix.

2. Model

The model is a random-proposer bargaining game where agents divide a dollar. In every time period, an agent is randomly recognized to make an offer. Based on unanimous consent, this offer is either accepted or rejected by other players. If the offer is rejected, a player is randomly selected in the following round to make an offer, unless the game is terminated.

Let $N = \{1, \ldots, n\}$ be the set of players, $\delta \in (0, 1)$ the common discount factor, and let $T = \{t \in \mathbf{N} : t < \tau\}$ for some positive $\tau \leqslant \infty$, where $\mathbf{N}$ is the set of non-negative integers. At each $t \in T$, a player $i \in N$ is randomly recognized to make an offer. Offers, $x = (x_1, \ldots, x_n)$, denote each player's share, and satisfy the constraint $\sum_{i \in N} x_i \leqslant 1$. We assume that agents vote publicly in a fixed sequential order. If the offer is unanimously accepted, the game ends yielding the payoff vector $(\delta^t x_1, \ldots, \delta^t x_n)$; otherwise, the game proceeds to date $t + 1$, except for $t = \tau - 1$ where the game ends yielding $(0, \ldots, 0)$. If $\tau = \infty$, and there is perpetual disagreement, each player has a payoff of 0. All offers are publicly observable.

At each $t \in T$ and $s \leqslant t$, agent i believes at s that she will be recognized at t with probability p_i. We denote by $p = (p_1, \ldots, p_n)$ the vector of these beliefs across agents. If agents derived their beliefs from a common prior, $\sum_{i \in N} p_i$ would equal 1.

Everything described thus far is common knowledge amongst the agents. The above conditions define the game, and we denote a game as $G(\delta, p, \tau)$.[4]

From each agent's *subjective* point of view, her belief is *true* and the others' beliefs are *optimistic*. If p_i^j denotes the probability agent i ascribes to the event {Agent j is recognized

[4] The reader may be curious as to why the description of the game does not include one agent's beliefs about another's recognition. The subsequent analysis makes clear that for a fixed p_i, the equilibrium is invariant to changes in i's beliefs about j's recognition.

at $t\}$ at all periods $s < t$, then $p_j - p_i^j$ measures Agent j's optimism from Agent i's subjective viewpoint. Then agent i believes the collective optimism to be $\sum_{j\in N\setminus\{i\}}(p_j - p_i^j)$ which equals $\sum_{j\in N} p_j - 1$. Since this is independent of i, all agents agree that this is the collective optimism.

Definition 1 (*Measure of optimism*). $y(p) \equiv \sum_{i\in N} p_i - 1$.

As defined, $y(p)$ lies in the interval $[-1, n-1]$. We assume throughout that the collective is always weakly optimistic and therefore $y \geqslant 0$. [5]

Agents' beliefs are assumed to be stationary. Yildiz [16,17] have shown that if optimism is followed by eventual pessimism, agreement will be delayed in bilateral bargaining; it is straightforward to extend this result to the multilateral case. [6] We impose stationarity in our model to differentiate the delay that we identify in this paper from those that emerge from changing beliefs: our purpose here is to show that in multilateral settings, even persistent optimism can create delay.

Our assumption that beliefs are common knowledge is not innocuous. Agents may be uncertain about both the bargaining process and the beliefs other agents have over that process. Within this context, one seeks to examine the impact of second-order uncertainty (and possibly even higher-orders); this inquiry is left to future research. While it seems likely that private information will accentuate delays, our analysis here shows that extreme optimism in the absence of private information can create delays.

We denote player i's perceived continuation value at the beginning of time t in the game $G(\delta, p, \tau)$ as $V_i^t(\delta, p, \tau)$ and omit the arguments where obvious.

Definition 2 (*Perceived Pie*). $S^t(\delta, p, \tau) \equiv \sum_{i\in N} V_i^t(\delta, p, \tau)$.

S^t provides a natural representation for the total perceived size of the pie at time t, and conveniently maps the n-tuple of continuation values into a scalar. We define V and S in equilibrium recursively.

If $\delta S^{t+1} > 1$, then for an offer x to be accepted, it must be that for each agent j, x_j is greater than δV_j^{t+1}. All such offers are infeasible as $\sum_{j\in N} x_j \geqslant \sum_{j\in N} \delta V_j^{t+1} = \delta S^{t+1} > 1$. Therefore, there must be delay at time t, and $V_i^t = \delta V_i^{t+1}$, and $S^t = \delta S^{t+1}$.

Now take the case where $\delta S^{t+1} \leqslant 1$. Here, if recognized, agent i makes an offer to the group that provides each agent j with her continuation value, δV_j^{t+1} and appropriates $1 - \delta S^{t+1} + \delta V_i^{t+1}$. Therefore we derive $V_i^t = p_i(1 - \delta S^{t+1}) + \delta V_i^{t+1}$. Adding across agents, we generate the agreement-mapping, $S^t = f(S^{t+1}) \equiv 1 + y - \delta y S^{t+1}$. Note that f has a unique fixed point, $\frac{1+y}{1+\delta y}$, which lies in the interval $[1, \frac{1}{\delta}]$ for all (δ, y); we denote this fixed point as S^*.

[5] It can be shown that pessimism leads to immediate agreement in every finite-horizon game.

[6] That fluctuating beliefs can cause delays has a natural interpretation: if individual perceptions change over time, then not only may each agent wait for others to have more "reasonable" expectations, but if each agent is overconfident, she *expects* others to form more reasonable beliefs in the future, and will therefore wait for that event if she is sufficiently patient.

We define the real-valued function Λ on $[0, \infty)$ to summarize backward induction dynamics where

$$S^t = \Lambda(S^{t+1}) = \begin{cases} \delta S^{t+1} & \text{if } S^{t+1} > \frac{1}{\delta}, \\ f(S^{t+1}) & \text{if } S^{t+1} \leqslant \frac{1}{\delta}. \end{cases} \quad (1)$$

We now characterize the subgame perfect equilibria (henceforth SPE). We ignore trivial multiplicities and focus on the SPE payoffs. [7]

Remark 1. In $G(\delta, p, \tau)$, for any $(t, i) \in T \times N$, where $\tau < \infty$, there exists a unique $V_i^t \in [0, 1]$ and $S^t \in [1, 1 + y]$ such that at all SPE strategy profiles, the continuation value of i at the beginning of t is V_i^t and $V_i^t = \max\{\delta V_i^{t+1}, p_i(1 - \delta S^{t+1}) + \delta V_i^{t+1}\}$. Moreover, $S^t = \Lambda(S^{t+1})$.

In the rest of the text, we refer to these unique SPE continuation values and perceived pie size as V_i^t and S^t. Therefore, the criterion to assess whether a game has immediate agreement is the value of S^1: if $S^1 < \frac{1}{\delta}$, the game will end in immediate agreement, and if $S^1 > \frac{1}{\delta}$, there is delay in the first period.

It is straightforward to extend the usual argument (see [11]) to show that subgame perfection need not select a unique outcome in the infinite horizon.

Remark 2. In $G(\delta, p, \infty)$, if $n > 2$, and for all i, $0 < p_i < 1$, then there exists $\delta^* < 1$ such that for all $\delta > \delta^*$, any $x \in X$ can be supported as an SPE of $G(\delta, p, \infty)$.

3. Waiting to settle: Finite horizon delay

Yildiz [16] finds that when agents are optimistic, and have common knowledge of optimism, they will immediately agree in sufficiently long bilateral games. In these bargaining games, one's knowledge of the other's optimism about recognition reduces the perceived gains from trade. This has a moderating effect on one's own optimism; as agents foresee long periods of disagreement near the end of the game, this induces agreement in *some* long game. For us to be assured of agreement in all longer games requires then that the backward induction process be stable: viz., the possibility of agreement in the tomorrow does not induce disagreement today. This property is satisfied by bilateral bargaining games but is violated by multilateral bargaining games, creating the tension between our results and that in [16].

[7] The game allows for multiple SPE, but all SPE are *payoff-equivalent*. Firstly, for any offer that is rejected, there are different voting profiles that reject that offer. Secondly, when $\delta S^{t+1} > 1$, at an SPE, the agent recognized at t can make any offer less than $1 - \delta V_i^{t+1}$. Thirdly, when $\delta S^{t+1} = 1$, equilibrium behavior is consistent with both agreement at t or delayed agreement until $t + 1$.

To examine the dynamics of the game, we examine the payoffs induced by the SPE of each game in the sequence, $\{G(\delta, p, \tau)\}_{\tau \in \mathbf{N}^+}$ for each (δ, p). We use the construction of an *agreement-absorption* set: once the perceived pie lies within this set in the future, agreement is assured in each preceding period.

Definition 3 (*Agreement-absorption*). A set X has the agreement-absorption property for (δ, p) if $X \subseteq [1, \frac{1}{\delta}]$, $X \neq \emptyset$, and $x \in X \Rightarrow f(x) \in X$.

If X is an agreement-absorption set, f maps X to a subset of X. By induction, it follows that if x lies in X, then for each integer n, $f^n(x)$ also lies in X. Trivially, existence of an agreement-absorption set is guaranteed: $\{S^*\}$, the fixed-point of f, satisfies the definition above.

The lemma below shows that the existence of an agreement-absorption set is connected to the possibility of immediate agreement in all sufficiently long games.

Lemma 1. *For every (δ, p), if there exists an agreement-absorption set X and τ such that $S^1(\delta, p, \tau) \in X$, then for every $t \geqslant \tau$ and $t < \infty$, $G(\delta, p, t)$ ends in immediate agreement.*

Therefore, immediate agreement in all sufficiently long games is guaranteed if we can find an agreement-absorption set which contains the perceived surplus for at least one game. We begin with the candidate set, $\left[1, \frac{1}{\delta}\right]$.

Theorem 1. *For every (δ, p), there exists a countably infinite set T^A such that for all $t \in T^A$, $S^1(\delta, p, t) \in [1, \frac{1}{\delta}]$ and the game ends in immediate agreement.*

The above result establishes that for every (δ, p), there exists countably infinitely many games where agreement is possible. As such, by Lemma 1, if $\left[1, \frac{1}{\delta}\right]$ is an agreement-absorption set for (δ, p), immediate agreement is guaranteed in all sufficiently long games. We derive a sufficient condition on (δ, p) for this to be true.

Lemma 2. *For every (δ, p) such that $\delta y \leqslant 1$, the following is true*:

(a) $[1, \frac{1}{\delta}]$ *is an agreement-absorption set.*
(b) *for every $\varepsilon \leqslant \varepsilon^* = \min\{S^* - 1, \frac{1}{\delta} - S^*\}$, the set $\left[S^* - \varepsilon, S^* + \varepsilon\right]$ is an agreement-absorption set.*
(c) *if $\delta y < 1$, then for all $x \in [1, \frac{1}{\delta}]$ and for all $\varepsilon > 0$, there exists $N < \infty$ such that for all $n > N$, $|f^n(x) - S^*| < \varepsilon$.*

Proof. (a) Since $f(x)$ is strictly decreasing in x, and $f(\frac{1}{\delta}) = 1$, then for all $x \in [1, \frac{1}{\delta}]$, $f(x) \in (1, f(1))$. However, $f(1) = 1 + y - \delta y = 1 + y(1 - \delta) \leqslant 1 + \frac{1}{\delta}(1 - \delta) = \frac{1}{\delta}$. Hence, $[1, \frac{1}{\delta}]$ is an agreement-absorption set.

(b) For any $\varepsilon \leqslant \varepsilon^*$, consider the set $X_\varepsilon = \left[S^* - \varepsilon, S^* + \varepsilon\right]$ and $x \in X_\varepsilon$. Then $|f(x) - S^*| = \delta y|x - S^*| \leqslant \delta y \varepsilon \leqslant \varepsilon$. Therefore, $f(x)$ lies in X_ε, establishing that X_ε is an agreement-absorption set.

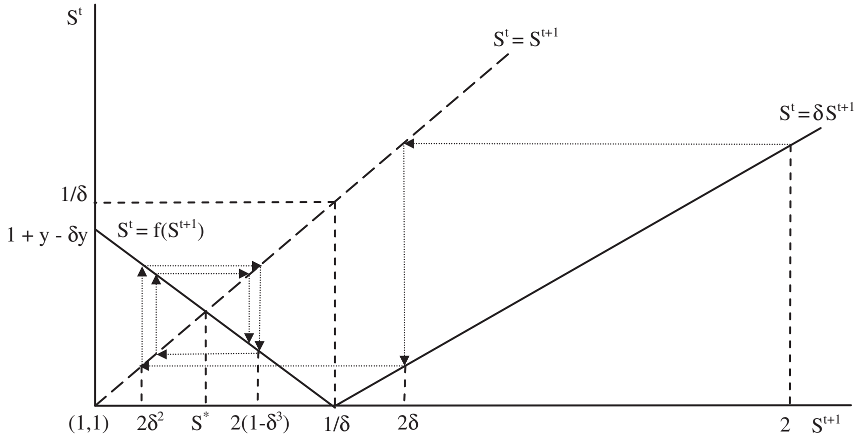

Fig. 1. Moderate optimism ($\delta y < 1$).

(c) If $\delta y < 1$, for any x that lies in $[1, \frac{1}{\delta}]$, $|f(x) - S^*| = \delta y|x - S^*| < |x - S^*|$. By induction then, $|f^n(x) - S^*| = (\delta y)^n(|x - S^*|)$. Since $|x - S^*|$ is fixed and δy is less than 1, for any $\varepsilon > 0$, it suffices to consider $N > \frac{\log \varepsilon - \log(|x - S^*|)}{\log(\delta y)}$. □

Therefore, if $\delta y \leqslant 1$, $\left[1, \frac{1}{\delta}\right]$ is an agreement-absorption set. The Immediate Agreement Theorem follows immediately from Lemma 2 and Theorem 1.

Theorem 2 (*Multilateral Immediate Agreement Theorem*). *For every (δ, p) such that $\delta y \leqslant 1$, there exists $\tilde{t} < \infty$ such that for all $t \geqslant \tilde{t}$ and $t < \infty$, $G(\delta, p, t)$ ends with immediate agreement.*

The *Multilateral Immediate Agreement Theorem* shows that if δy is at most 1, then any sufficiently long bargaining game must end in immediate agreement. In particular, our *Multilateral Immediate Agreement Theorem* implies Yildiz's [16] Immediate Agreement Theorem since with two agents, y is at most 1.

We illustrate Theorem 2 using Fig. 1, which shows the dynamics when $y = 1$, and $\delta \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt[3]{2}})$. At the final round, τ, each agent expects to capture the entire dollar if recognized. Therefore, S^τ, the perceived pie-size prior to recognition at τ is $1 + y = 2$. Since 2 is greater than $\frac{1}{\delta}$, expectations of capturing the rents at the end of the game lead to disagreement in the preceding round; $S^{\tau-1}$ is therefore 2δ. Since 2δ is greater than $\frac{1}{\delta}$ as well, there is no agreement at $\tau - 2$, and hence, $S^{\tau-2}$ is $2\delta^2$. However, since $2\delta^2$ is less than $\frac{1}{\delta}$, $S^{\tau-3} = f(2\delta^2) = 2(1 - \delta^3)$. The phase diagram then illustrates the dynamics in the preceding rounds where agreement is possible. Moreover, as seen in the figure, the iterates of *f* gradually converge to S^*. The agreement-absorption property of $\left[1, \frac{1}{\delta}\right]$ follows from $1 + y - \delta y = 2 - \delta \leqslant \frac{1}{\delta}$. The latter two parts of Lemma 2 follow from the slope of

f being $\delta y = \delta < 1$. S^* is thus an attractor in the set $\left[1, \frac{1}{\delta}\right]$, leading to the convergence of perceived pie sizes as we make the game longer.

The bargaining game changes remarkably when δy is greater than 1. It is straightforward to verify that $[1, \frac{1}{\delta}]$ is no longer an agreement-absorption set,[8] but this does not establish delay as illustrated by the following example.

Example 4. Consider $G(\delta, p, \tau)$ where $\tau < \infty$, $N = \{1, 2, 3, 4, 5\}$, $p_i = \frac{19}{20}$ for each agent i, and $\delta = \frac{2}{5}$. Then, at $\tau - 1$, agreement requires that each agent be offered at least $\frac{2}{5}(\frac{19}{20})$. Since this is infeasible, there is no agreement. Hence, at $\tilde{t} - 2$, an agent will agree to an offer of at least $(\frac{2}{5})^2(\frac{19}{20})$. As $1 - 4(\frac{19}{125}) > \frac{19}{125}$, agreement is possible. The reader can verify then that for each agent i, $V_i^{\tilde{t}-2} = \frac{19}{50}$, and $S^{\tilde{t}-2} = \frac{19}{10}$. As $\frac{19}{10} < \frac{1}{\delta}$ and $f(\frac{19}{10}) = \frac{19}{10}$, agreement is possible at all $t \in \{0, 1, 2, \ldots, \tau - 3\}$. Hence, the game ends with immediate agreement. However, $\left[1, \frac{1}{\delta}\right]$ is not an agreement-absorption set as $f(1) = \frac{13}{4} > \frac{5}{2} = \frac{1}{\delta}$.

We prove below that such examples are non-generic: if $\delta y > 1$, then $\{S^*\}$ is the unique agreement-absorption set, and generically, this implies the existence of delay in some arbitrarily long game.

Lemma 3. *For every (δ, p) such that $\delta y > 1$, $\{S^*\}$ is the unique agreement-absorption set.*

Proof. Consider any set $X \neq \{S^*\}$ such that $X \subseteq [1, \frac{1}{\delta}]$ and select $x \in X \backslash \{S^*\}$. Note that $|f(x) - S^*| = \delta y(|x - s|)$. By induction, $|f^k(x) - S^*| = (\delta y)^k |x - S^*|$. It can then be verified that $|f^k(x) - S^*| > |\frac{1}{\delta} - S^*|$ for all $k \geqslant \frac{\log(|\frac{1}{\delta} - S^*|) - \log(|x - S^*|)}{\log(\delta y)}$. Choose a particular such k. If $f^k(x) > S^*$, then $|f^k(x) - S^*| > |\frac{1}{\delta} - S^*|$ implies that $f^k(x) > \frac{1}{\delta}$, and hence, $f^k(x) \notin X$. If $f^k(x) < S^*$, then since f is a decreasing function, it must be that $f^{k+1}(x) > f(S^*) = S^*$. Since $|f^{k+1}(x) - S^*| > |\frac{1}{\delta} - S^*|$, this establishes that $f^{k+1}(x) > \frac{1}{\delta}$, and hence, $f^{k+1}(x) \notin X$. Therefore, X is not an agreement-absorption set. □

When δy is greater than 1, we have shown that the unique agreement-absorption set is $\{S^*\}$. We use this result to construct arbitrarily long games with delay.

Theorem 3. *For almost every (δ, p) such that $\delta y > 1$, there are countably infinite sets T^A and T^D with $T^A \cup T^D = \mathbf{N}^+$, such that for all $\tau \in T^A$, $G(\delta, p, \tau)$ has immediate agreement, and for all $\gamma \in T^D$, $G(\delta, p, \gamma)$ has disagreement in the first round.*

Proof. Consider a particular (δ, p) where $\delta y > 1$. The existence of the set T^A follows from Theorem 1. To establish the existence of T^D, it suffices to show for any $G(\delta, p, t)$ that ends in immediate agreement, there exists $t' > t$ such that the SPE of $G\left(\delta, p, t'\right)$ involves disagreement in the first round. In the appendix, we prove that the set $\{\delta : \exists t$ such

[8] If $\delta y > 1$, then $f(1) = 1 + y(1 - \delta) > 1 + \frac{1}{\delta}(1 - \delta) = \frac{1}{\delta}$.

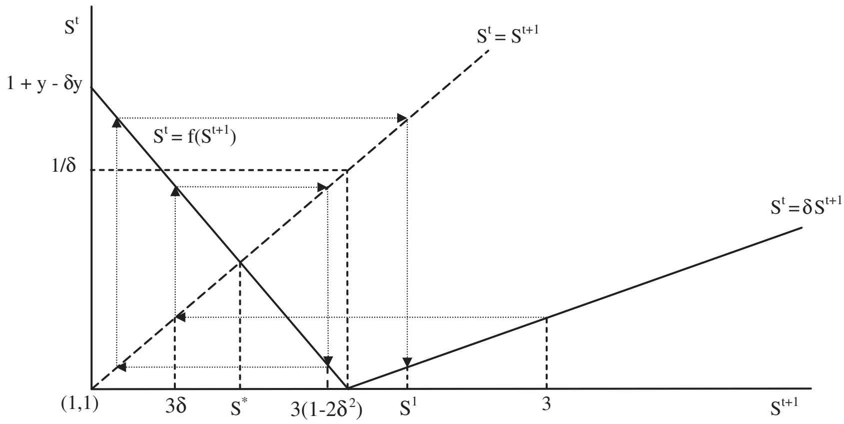

Fig. 2. Extreme optimism ($\delta y > 1$). In this model, $y = 2$ and δ lies between $\frac{1}{2}$ and $(1/3)^{1/2}$ as in Example 3. If t is the date of the penultimate round, $S^{t+1} = 1 + y = 3$ and hence, $S^t = 3\delta$. We use the phase map to iterate the dynamic process and display the subgame perfect continuation values. Since S^1 is greater than $1/\delta$, there is disagreement in the first round.

that $S^1(\delta, p, t) = S^*\}$ is measure zero for each p and can therefore be excluded. Take t such that $S^1(\delta, p, t) \in [1, \frac{1}{\delta}]\backslash\{S^*\}$. Now, note that $G(\delta, p, t)$ is equivalent to the subgame after the rejection of the first offer in $G(\delta, p, t + 1)$. By backward induction, $S^1(\delta, p, t + 1) = 1 + y - \delta y S^1(\delta, p, t)$. If $S^1(\delta, p, t + 1) > \frac{1}{\delta}$, the proof is complete. Otherwise, iteratively repeat the inductive process (considering $G(\delta, p, t + k)$ as a subgame in a larger game, $G(\delta, p, t + k + 1)$) until a game $G(\delta, p, t')$ is found such that $S^1 > \frac{1}{\delta}$. The existence of such a game is established by Lemma 3: if there were no such game, then the set $\{s : \exists t' \geqslant t$ such that $S^1(\delta, p, t') = s\}$ would constitute an agreement-absorption set thereby leading to a contradiction. □

Theorem 3 proves that we can construct arbitrarily long finite-horizon games that will necessarily have delayed agreement on the equilibrium path if agents have extreme persistent optimism. The intuition relies on Lemma 3: the unique agreement-absorption set is $\{S^*\}$, which is too "small" to guarantee immediate agreement for almost every (δ, p).

We illustrate the above result with Fig. 2, where we examine the dynamics of the game in Example 3. Here, we see that agreement is possible at $t \in \{1, 2, 4\}$ but is impossible initially at $t = 0$, as S^1 is larger than $\frac{1}{\delta}$. Unlike Fig. 1, the perceived surplus does not remain within $[1, \frac{1}{\delta}] \times [1, \frac{1}{\delta}]$ leading to delay.

We have shown that it is necessary and sufficient that δy exceed 1 for there to be delay. The intuition for this simple cut-off should be provided. If agreement is possible at period t, the value of δy captures the impact of the perceived surplus in $t + 1$ on the perceived surplus in t. In the case of common priors, for example, δy is 0 and there is no impact whatsoever. In contrast, for almost every positive value of δy, the perceived surplus naturally fluctuates between successive periods: if S^{t+1} is low, the rents from agreement at t are high thereby leading to a high S^t. The magnitude of these fluctuations depend on δy. For δy in (0, 1], the impact of tomorrow's surplus on that today is sufficiently small for the dynamic process to stabilize to a point in the agreement regime, rendering immediate agreement in sufficiently long games. However, when δy exceeds 1, variations in the perceived surplus tomorrow have an even greater impact today: this instability creates the possibility for delay in *some* preceding periods since an inherently unstable process cannot remain bounded by $\left[1, \frac{1}{\delta}\right]$. Note that for δy to be greater than 1, $\sum_{i \in N} p_i$ must be greater than $\frac{\delta+1}{\delta}$ (which is greater than 2); on *average*, agents must be at least twice as optimistic as they would be had they shared a common-prior, and thus are *extremely optimistic*.

As the following example shows, for delay to emerge, it is not necessary that optimism be persistent. In particular, even if agents disagree over the process for only a few initial periods, delay is guaranteed. As such, this illustrates that these bargaining delays do not necessarily emerge from huge rents from agreement at the final round, but that each period within an agreement-regime may induce delay in some preceding period.

As the following examples involve non-stationary beliefs, let p_i^t represent the belief agent i has at all time periods $s \leqslant t$ that she will be recognized at t. Let y^t denote the associated optimism.

Example 5. Let $T = \{0, \ldots, \bar{t}\}$. For $t \in \{0, 1\}$, let agents be extremely optimistic where $\delta y^t > 1$. For all subsequent periods, $t \in \{2, \ldots, \bar{t}\}$, let agents have common priors where $p_i^t = p_i$, and $\sum_{i \in N} p_i = 1$. Then, for all $t > 1$, the reader can verify that $V_i^t = p_i^t$, and so $S^t = 1$. It then follows that $V_i^1 = p_i^1(1-\delta) + \delta p_i$, and $S^1 = (1+y)(1-\delta) + \delta = 1 + y^1(1-\delta)$ which is greater than $\frac{1}{\delta}$. This precludes agreement in the first round.

The reader can verify that our example can be generalized: delayed agreement is possible so long as there is excessive optimism over the bargaining process in the first few rounds. Hence, even if agents come to agree over the bargaining process once they have seen sufficiently many rounds of bargaining, the initial disagreement over recognition suffices to induce delay.

The following result illustrates that the converse is also true: if agents have initial agreement over the bargaining process but conflicting expectations in the future, the game ends in immediate agreement.

Example 6. Let $T = \{0, \ldots, \bar{t}\}$, and let $t' \in [1, \bar{t})$. Let the agents begin the game with common priors: $y^0 = 0$. For $t \leqslant t'$, let $y^t < \frac{1}{\delta}$, and for $t > t'$, let $y^t > \frac{1}{\delta}$. Moreover, let there be $\hat{t} \leqslant t'$ such that $S^{\hat{t}+1} \leqslant \frac{1}{\delta}$. Then the reader can verify that $S^1 < \frac{1}{\delta}$ and there is immediate agreement.

The preceding two examples illustrate the importance of initial optimism. Initial optimism over recognition can create delay even in the presence of future agreement. Similarly, if agents have agreement over the bargaining process for sufficiently many initial rounds, the game is guaranteed to end in immediate agreement.

We have shown that multilateral bargaining can generate delays. We proceed to examine bargaining dynamics and the costs of delay.

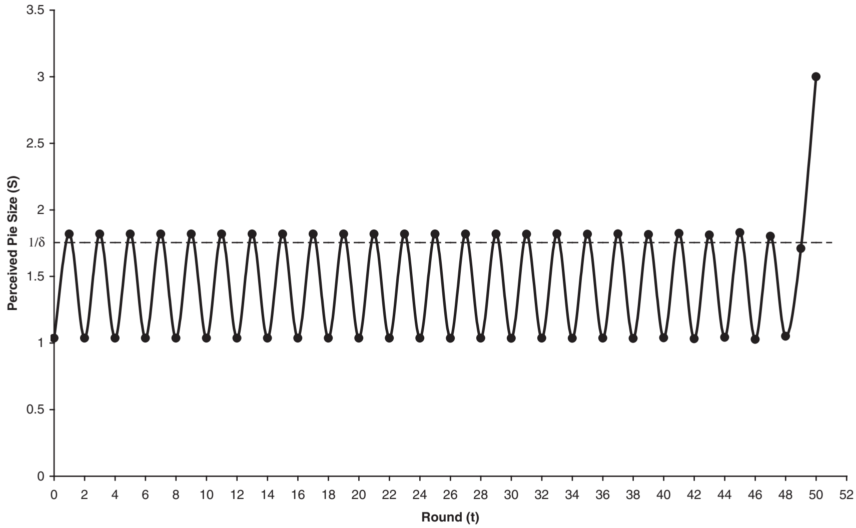

Fig. 3. Bargaining dynamics.

3.1. Bargaining dynamics: Cyclic agreement

Since each subgame after an offer is rejected is isomorphic to a shorter game, Theorem 3 establishes that there are multiple cycles between agreement and disagreement regimes in a long game. This "cycling behavior" may be illustrated using parameters from Example 3.

Example 7. Consider $G(\delta, p.51)$ such that (δ, p) are as in Example 3. Then agreement is delayed until $t = 1$; moreover, agreement is attained at every $t \in \{1, 3, 5, \ldots, 47, 48\}$, and impossible at $t = \{0, 2, 4, \ldots, 46, 49\}$. We illustrate these dynamics in Fig. 3 where we plot S^t and t.

The cycling behavior emerges, as we discussed in the Introduction, from there being multiple moments that are ripe for agreement. In this example, each agreement regime offers the proposer such high rents from agreement that excessive optimism prevents agents from reaching agreement in the preceding round. Since delay in the preceding period is then common knowledge, and delay is very costly, agents are able to agree to a bargaining division two periods before. As such, there is agreement in every alternate period.

This sort of cycling phenomenon is general and is established by Theorem 1. However, it is difficult to calculate the periodicity of the cycles (though the length of delay can be bounded). Here, we show that for a certain class of parameters, in every sufficiently long game, the outcome will alternate between agreement and disagreement regimes.

Let $P = \frac{1+y}{1+\delta^2 y}$, and let $Q = \frac{\delta(1+y)}{1+\delta^2 y}$, and note that $P > \frac{1}{\delta}$, $Q < \frac{1}{\delta}$, $\Lambda(P) = Q$ and $\Lambda(Q) = P$. Thus, $\{P, Q\}$ is an orbit with periodicity 2.

Proposition 1. *For almost every* (δ, p) *such that* $y \in (\frac{1}{\delta}, \frac{1}{\delta^2})$, *and for every* $\varepsilon > 0$, *there exist* $T < \infty$ *such that for all* $t > T$, $|S^0(\delta, p, t) - P| + |S^1(\delta, p, t) - Q| \leqslant \varepsilon$ *or* $|S^0(\delta, p, t) - Q| + |S^1(\delta, p, t) - P| \leqslant \varepsilon$.

The intuition for this is straightforward. When y lies in $\left(\frac{1}{\delta}, \frac{1}{\delta^2}\right)$, the set of periodic points is $\{S^*, P, Q\}$ where S^* is a fixed point and P and Q are points with periodicity 2. However, by Lemma 3, $\{S^*\}$ is unstable and therefore cannot be the limit of continuation values. However, as we show in the appendix, the orbit $\{P, Q\}$ is stable and globally attracting. Thus, with repeated iteration of Λ^2 in sufficiently long games, $\{S^0, S^1\}$ converges to this orbit. We illustrate these dynamics in Fig. 4.

For higher values of y, the mapping Λ seems to exhibit the *period doubling route to chaos*.[9] If y is greater than $\frac{1}{\delta^2}$, P and Q are no longer stable periodic points. Though we do not have a precise characterization of the dynamics in this case, numerical simulations point us towards a general conjecture. Define $y_n = \inf\{y \geqslant 0 : \Lambda$ has a point with periodicity n where n is some integer$\}$.[10] To illustrate $y_1 = 0$ and $y_2 = \frac{1}{\delta}$. It appears that $y_n \leqslant y_{n+1}$ and that as $y \in (y_n, y_{n+1})$, a stable orbit with periodicity n appears, but all orbits with lower periodicity become unstable. Therefore, if the equilibrium sequence converges, it converges to this stable orbit with periodicity n. We conjecture that this phenomenon is general: the orbit n is stable if and only if y lies between (y_n, y_{n+1}) for a fixed δ. Hence, for high values of y, the cycles may converge to orbits of particularly high periodicity, and as y is arbitrarily large, the game appears chaotic.

The unstable dynamics may seem to be an indictment of our equilibrium: backward induction in these games, seemingly, relies on implausibly high orders of sequential rationality. While it is certainly implausible that human beings play these subgame perfect equilibria in a literal fashion (this criticism is equally valid in other long dynamic games), we believe that the game and the cycling of agreement and disagreement regimes provide a useful *metaphor* for *stalemates* and *ripe moments for agreement*. Moreover, we find the multiple transitions between agreement and delay to be characteristic of real-world bargaining where delay and disagreement rear their ugly heads ever so often.

3.2. Costs of delay

The potential for delay is greatest in the rounds near the end of the game, if agreement has not already been reached. Immediately preceding the final round, each optimistic agent expects to capture the entire surplus at the final round while bearing little in the way of time costs. These huge rents at the last round causally lead to disagreement in preceding rounds: if δ is greater than $\frac{1}{1+y}$, and the end of the game is sufficiently close, agents may choose to wait to capture the entire pie in the last round. This delay in the disagreement regime is large: in particular, the delay is as long as $\left[\frac{\log(1+y)}{\log(\frac{1}{\delta})}\right] - 1$ (which we denote as

[9] See Robinson [14, Chapter 3].

[10] It is not clear that y_n is even defined for all integers n.

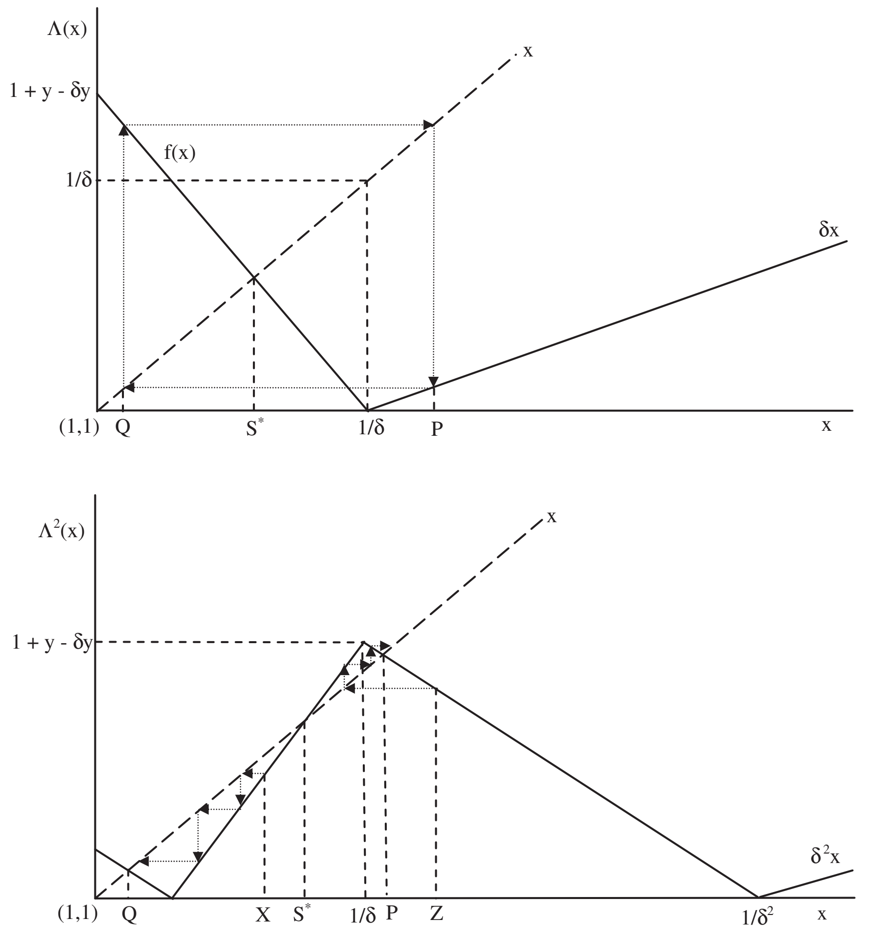

Fig. 4. 2-period cycles. The above two figures illustrate the dynamics when y lies between $1/\delta$ and $1/\delta^2$. The graph of $\Lambda(x)$ displays the periodic points of orbit 2, $\{P, Q\}$. Here $\Lambda(P) = Q$ and $\Lambda(Q) = P$. The graph of $\Lambda^2(x)$ illustrates Proposition 2. As we iterate Λ^2 on the points X and Z, the process converges to Q and P. This shows that P and Q are attracting points, and hence, in longer games, the perceived surpluses generated on the equilibrium path converge to these two points.

$L(\delta, p))$ and entails an efficiency loss, $1 - \delta^{L(\delta,p)} \in [\frac{\delta(1+y)-1}{\delta(1+y)}, \frac{y}{1+y})$.[11] Thus, in the continuous-time limit, upto $\frac{n-1}{n}$ shares of the pie may be lost if agents had to bargain in this final disagreement regime.

In *long games*—$G(\delta, p, t)$ where t is greater than $L(\delta, p) + 2$— the length of the game exceeds that of the disagreement regime near the end of the game. Faced with the prospect of a long disagreement regime, in a sufficiently long game, agents will be able to agree in

[11] The $\lceil\cdot\rceil$ operator finds the smallest integer that is greater than or equal to its argument.

some period before being locked into disagreement until the end. If δy is less than 1, then by Theorem 2, this implies immediate agreement yielding full efficiency. However, if δy is greater than 1, agreement may be delayed. In this section, we examine the efficiency properties of long games with extreme optimism.

While it is impossible to calculate an exact bound for the efficiency loss, we characterize a fairly tight bound, and examine its implications.

Proposition 2. *Consider $G(\delta, p, t)$ such that $t \geqslant L(\delta, p) + 2$. Then, the settlement-time, $\gamma(\delta, p, t)$ is bounded above by $E(\delta, p) = \max\{[\frac{\log(1+y-y\delta)}{\log(\frac{1}{\delta})}]-1, 0\}$. The maximal efficiency loss is bounded above by $1 - \delta^{E(\delta,p)}$ and this bound lies in the set, $[\frac{\delta(1+y-y\delta)-1}{\delta(1+y-y\delta)}, \frac{y(1-\delta)}{1+y-y\delta}]$.*

Using $E(\delta, p)$, we are able to examine the continuous-time limit of $G(\delta, p, t)$ where offers are made frequently.

Theorem 4. *For every p and for every $\varepsilon > 0$, there exist $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, and $t \geqslant L(\delta, p) + 2$, the loss of efficiency in $G(\delta, p, t)$, $1 - \delta^{\gamma(\delta,p,t)} < \varepsilon$.*

This result shows that an important implication of the Immediate Agreement Theorem generalizes to all multilateral environments: if offers are frequent, persistent optimism cannot explain inefficient delay. One would need to appeal to changing beliefs (as in [17]) to create substantial delays that remain in the continuous-time limit, regardless of the number of agents.

At high values of δ, bargaining power is changing very rapidly; as such, recognition offers little in way of a "temporal monopoly." Recognition offers tremendous bargaining power when offers are less frequent; thus for low values of δ, optimism can result in extremely costly delays, even in long games. The actual cost of delay is very sensitive to the parameters of the game. We present results from numerical simulations to illustrate this dependence on δ. For expository reasons, we consider games where $p_i = p_j$ for all agents $i, j \in N$, and abusing notation we denote the game as $G(\delta, p_i, |N|, t)$.

Game	Delay	Cost of delay
$G(0.9, 0.9, 3, 52)$	1	0.1
$G(0.9, 0.9, 6, 51)$	3	0.27
$G(0.8, 0.9, 7, 52)$	3	0.49
$G(0.7, 1, 10, 51)$	2	0.51

Game	Delay	Cost of delay
$G(0.7, 0.9, 3, 52)$	1	0.3
$G(0.25, 0.9, 6, 52)$	1	0.75
$G(0.2, 0.9, 7, 52)$	1	0.8
$G(0.12, 1, 10, 52)$	1	0.88

As such, optimism can create costly delays in bargaining. Much of the cost of delay emerges from the time between offers: in many of the games above, there is only one period of delay, but that delay is sufficiently costly.

4. Optimism in the infinite-horizon

In this section, we examine plausible equilibria in the infinite-horizon. As mentioned in Remark 5, any division of the dollar in the infinite horizon can be supported as an

SPE. This indeterminacy compels us to select a particular equilibrium. It has been standard within the bargaining literature to focus on the class of stationary equilibria citing history-independence as a motivation. In the standard multilateral bargaining game, the stationary equilibrium is the unique history-independent SPE outcome, and that equivalence holds in the game with moderately optimism. However, extreme optimism breaks the equivalence: the multilateral bargaining game with extreme optimism has a continuum of history-independent equilibria, a subset of which involves delay and Pareto-dominates the stationary equilibrium.

Before analyzing the game, we should briefly discuss the motivation for each solution concept. History-independent equilibria (henceforth HIE) are SPE profiles where actions are not conditioned on past play, but may be conditioned on the calendar date. History-independence captures the desideratum of *anonymity* and *simplicity* where agents are unable to reward or punish any actions in the past. This removes the multiplicity of SPE where *every* division of the surplus can be enforced. Stationary equilibria (henceforth SSPE) are HIE where actions are identical in every equivalent subgame and are usually motivated as being the class of *simplest strategies*.[12] The motivation for stationarity depends crucially on payoffs: if *all* agents benefit from using non-stationary strategies, a modeler has few grounds to preclude such strategies. However, such a discussion is moot if the two solution-concepts are payoff-equivalent.

To proceed, we define our solution concepts informally. The *past history* at each stage consists of the identity of previous proposers, their proposals, and the votes of each voter at each previous stage. We define the *current history* as the standing offer (if there is any) and the sequence of votes on the current offer (if there are any). A strategy profile is a HIE if the action prescribed at any history depends exclusively on the *current history* and a time-specific state variable, and the strategy profile is subgame perfect. A strategy-profile is a SSPE if it is an HIE and the action prescribed at any history depends only on the current history.

We begin by establishing the existence of the SSPE and its uniqueness. The SSPE involves immediate agreement in every game.

Theorem 5. *For every (δ, p), $G(\delta, p, \infty)$ has a unique SSPE outcome. In this equilibrium, all agents agree immediately, and for each t, $S^t = S^* = \frac{1+y}{1+\delta y}$, and $V_i^t = \frac{p_i}{1+\delta y}$.*

We now characterize the set of HIE.

Theorem 6. *For every (δ, p) such that $\delta y < 1$, $G(\delta, p, \infty)$ has a unique HIE outcome which coincides with the SSPE outcome. For every (δ, p) such that $\delta y \geqslant 1$, and for every $t \in T$, $G(\delta, p, \infty)$ has an HIE where $S^t = S$ if and only if $S \in [1, 1 + y - \delta y]$.*

The above theorem establishes that in almost every game of moderate optimism—and including when agents share common priors—the unique history-independent equilibrium is stationary, but in all games of extreme optimism, there is a continuum of non-stationary history-independent equilibria, some of which have delayed agreement. By emphasizing

[12] See, for example, [4].

this divergence, we wish to make clear that the full efficiency and stability properties of the SSPE do not emerge simply from history-independence, but require stationarity if agents are extremely optimistic. In the finite-horizon game in Section 3, the distinction between moderate optimism and extreme optimism was crucial in determining if arbitrarily long games ended in immediate agreement or had delay. It is not coincidental that the equivalence between HIE and SSPE rests on virtually the same knife edge and we should clarify this relationship.

Examining the game with moderate optimism, Lemma 2 establishes that $\left[1, \frac{1}{\delta}\right]$ is an agreement-absorption set. Hence, agreement at any period implies agreement at every preceding period. Therefore, disagreement at any round of an HIE necessitates perpetual disagreement thereafter, which contradicts subgame perfection. This insight establishes that an HIE involves agreement at every round, and since the stationary solution is the only strategy-profile that involves agreement at infinitely many periods when $\delta y < 1$, the equivalence follows. When $\delta y = 1$, there is a continuum of HIE but each HIE ends in immediate agreement.

In the game where $\delta y > 1$, on the other hand, $\left[1, \frac{1}{\delta}\right]$ is *not* an agreement-absorption set, and disagreement at any round of an HIE can be supported by future agreement. This allows us to generate a large set of HIE using each perceived surplus that lies in the set $\left[1, 1 + y - \delta y\right]$, some of which involve delayed agreement. However, the set of HIE is still small relative to that of SPE: as Lemma 13 in the appendix establishes, the space of HIE perceived surpluses at any particular date is a 1-dimensional manifold. Nevertheless, the SSPE is special relative to the set of HIE when it comes to its efficiency properties; to wit, it is the only HIE where agreement is guaranteed in every round, on and off the path of play.

Theorem 7. *For every (δ, p) such that $\delta y > 1$, the unique HIE of $G(\delta, p, \infty)$ where there is agreement at every $t \in T$ is the SSPE.*

While many of the HIE are highly non-stationary and complex, neither multiplicity nor delay is rooted in extreme complexity. In particular, if actions can be conditioned upon simply whether the period is odd or even, there exist multiple equilibria as the following example illustrates. The construction of this equilibrium relies on P and Q, the points of periodicity 2 identified in Section 3.1.

Example 8. Consider $G(\delta, p, \infty)$ such that $\delta y > 1$ and let agents condition their strategies at time t on whether t is odd. We verify that the following is an SPE outcome: at every odd date, agent i is recognized, she offers each agent j an amount $\frac{\delta^2 p_j}{1+\delta^2 y}$ and this is accepted by each agent; at every other date, she offers any offer that leaves her with at least $\frac{\delta p_i}{1+\delta^2 y}$, and this offer is rejected. By backward induction, at odd dates t, $V_i^t = p_i(1 - \frac{\delta^2(1+y)}{1+\delta^2 y}) + \frac{\delta^2 p_i}{1+\delta^2 y}$ $= \frac{p_i}{1+\delta^2 y}$ and $S^t = \frac{1+y}{1+\delta^2 y}$ which is greater than $\frac{1}{\delta}$. Hence, at every even date, there must be delay. Therefore, at any even date τ, $V_i^\tau = \frac{\delta p_i}{1+\delta^2 y}$, which is consistent with equilibrium strategies. Since $t = 0$ is "even," the SPE involves disagreement in the first round.

Though the stationary equilibrium is undeniably the simplest HIE, this example shows that there are other equilibria that are *almost* as simple and yet involve delay.

As we argued earlier, the motivation for stationarity is connected to its payoffs: complex non-stationary strategies should not be precluded if they guarantee higher payoffs. Such an argument can indeed be made for non-stationary HIE: we show that the set of HIE outcomes can be ex ante Pareto-ranked and the stationary equilibrium is Pareto-inferior to a continuum of HIE. Within a game, $G(\delta, p, \infty)$, we define $S^t(\sigma)$ as the perceived surplus at date t if the history-independent strategy profile is σ, and $V_i^t(\sigma)$ as the corresponding perceived continuation value.

Proposition 3. *Consider $G(\delta, p, \infty)$ such that $\delta y > 1$. Then the set of HIE outcomes can be ranked by the ex ante Pareto criterion: for any history-independent equilibria, σ and σ', $S^0(\sigma) > S^0(\sigma') \Leftrightarrow V_i^0(\sigma) > V_i^0(\sigma')$ for each $i \in N$.*

Since individual payoffs are monotonic in S^t, it follows that the SSPE is Pareto-dominated by any history-independent σ where $S^0(\sigma) > \frac{1+y}{1+\delta y}$. The Pareto-dominant solution in particular is any HIE where $S^0 = 1 + y - \delta y$. Moreover, it can be shown that if y exceeds $\frac{1+\sqrt{1+4\delta^2}}{2\delta^2}$, then a history-independent equilibrium with delay can also Pareto-dominate the SSPE. This casts doubt upon the stationary solution: each agent would find it beneficial to play a more complex strategy. Even amongst the history-independent equilibria where agents condition on odd or even dates, there is a non-stationary equilibrium that is Pareto-superior to the stationary solution; hence, even slight non-stationarity can create ex ante Pareto gains.[13]

This leaves the question of predicting infinite-horizon behavior unsolved: while history-independence is too weak to serve as a refinement, stationarity may be too strong. We do not intend to suggest that this is the only setting where the SSPE and HIE diverge, but simply that such a divergence makes selecting a unique equilibrium controversial.

5. Alternative voting rules

So far, we have looked at the unanimity voting rule. In a variety of political institutions, the dominant form of multilateral bargaining involves a majority or super-majority voting rule. Though political institutions do have features that resemble proposal-voting structures, they do not possess stochastic recognition mechanisms with objectively known distributions. Moreover, as an environment where *posture* is critical and excessive optimism is conceivable, legislative bargaining is a natural application of bargaining with heterogeneous priors. Thus, we briefly examine bargaining outcomes with general voting rules and

[13] It also follows from Theorem 3 that if agents are extremely optimistic, the SSPE is not the limit of SPE of finite truncations. Moreover, if $y \in (\frac{1}{\delta}, \frac{1}{\delta^2})$, by Proposition 1, the payoffs of the "alternating" equilibrium of Example 8 provide a limit for those of the SPE of finite truncations (with the appropriate metric). However, it is not clear that continuity at infinity is a desideratum. See [15].

show that in the presence of optimism, the majority-rule is the least likely to cause delay in the class of threshold voting rules.

In the spirit of Baron and Ferejohn [3], we consider symmetric equilibria of symmetric games. [14] Each agent believes that she shall be recognized with probability $p = \frac{1+y}{n}$ where y lies in $[0, n-1]$. When choosing coalitions, each agent randomizes uniformly between all agents of the same continuation value. We consider all quota voting rules where the acceptance of an offer requires the agreement of m agents (including the proposer) where $\frac{n}{2} \leqslant m \leqslant n$. We define the rest of the extensive form as before. We define such a game as $G(\delta, n, m, p, \tau)$, and the reader should note that these games have unique symmetric SPE in the finite-horizon.

The following result is an extension of our approach in the unanimity game.

Proposition 4. *For (δ, n, p, m) where $\delta y \leqslant \frac{n-1}{m-1}$, then there exists $\tau < \infty$, such that for all $t \geqslant \tau$, $G(\delta, n, m, p, t)$ ends in immediate agreement. Generically, for (δ, n, p, m) where $\delta y > \frac{n-1}{m-1}$, there are countably infinite sets, T and T' such that for all $\tau \in T^A$, $G(\delta, n, m, p, \tau)$ ends in immediate agreement, and for all $\gamma \in T'$, $G(\delta, n, m, p, \gamma)$ has disagreement in the first round.*

Recall that with a unanimity rule, we are ensured immediate agreement in all sufficiently long games so long as δy is less than 1. Therefore disagreement necessitates that agents be at least twice as optimistic as if they shared a common prior. However, if we have a majority-voting rule ($\frac{n-1}{m-1} = 2$), then we have immediate agreement in all sufficiently long games so long as δy is less than 2. Disagreement in any long game would therefore require that $p \geqslant \frac{2}{\delta n} + \frac{1}{n} > 3(\frac{1}{n})$, or agents be more than *three times* as optimistic as if they shared a common prior. Hence, by weakening the voting rule, we provide greater possibility for immediate agreement.

Therefore, in legislative bargaining games with optimism, the likelihood of delay with majority-voting rules may be less than with unanimity voting rules (if the symmetric equilibria are implemented). This draws a contrast to the research on bargaining with a stochastic surplus (with a commonly known distribution). In that environment, the unanimity rule implements the *optimal stopping rule*, but as Eraslan and Merlo [6] establish, the possibility of exclusion from the winning coalition drives agents to agree *too quickly* in a majority-voting rule. Within our framework, the threat of exclusion from future coalitions has an identical effect: it compels agents to mitigate their disagreements over bargaining shares, and accept smaller bargaining shares today, resulting in earlier agreement. Since the bargaining surplus is constant, this undoubtedly enhances efficiency. Coalition-formation, thus, is a double-edged sword in a world of political uncertainty and disagreement.

[14] This restriction simplifies our analysis, but is not without loss of generality: as Norman [10] shows, the symmetric game has many asymmetric equilibria with differing continuation payoffs, and asymmetric games have a unique asymmetric equilibrium.

6. Conclusion

We have examined agreement and disagreement in multilateral bargaining games. We have seen that the immediate agreement theorem does not hold for excessively optimistic players, and that delay in bargaining games may be explained by subjective biases once we have at least 3 agents. If offers are frequent, this generates a small efficiency loss and hence, the efficiency implications of the Immediate Agreement Theorem remain in the continuous-time limit. However, if there are significant time-costs, optimism can create costly bargaining delays in arbitrarily long games.

Our model suggests that when bargaining is over a finite time-horizon and delay is costly, optimism over the bargaining process will foster bargaining delay. Insofar as rational agents may agree to disagree, the bargaining delay that we identify need not be rooted in irrationality. However, as experimental evidence corroborates the prevalence of optimism, self-serving biases, and the use of imprecise heuristics to form subjective beliefs, there are reasons beyond rational disagreement to expect conflicting views over bargaining power. Our model suggests a strong causal connection between this collective optimism and delayed agreement.

The bargaining protocol that we have studied in this paper is one where each agent has veto power over all offers, and can thereby restrict the payoff of any other agent. As such, no partial agreements are possible. This protocol is natural in those contexts where unanimous agreement is necessary before the surplus may be divided and agents have decision-rights over its distribution. In other settings, it may be more plausible to allow for partial offers and agreement. It would be useful to understand the impact of optimism on such bargaining protocols and examine whether there is any detail-free efficient procedure when agents disagree over the distribution of bargaining power; this task is left to future research.

A theoretical implication of this paper is that once we allow for extreme optimism, the stationary equilibrium in the infinite-horizon is truly a *knife-edge* equilibrium. We have shown that allowing for even a small "*amount*" of memory allows for multiple equilibria in the infinite-horizon, all of which need not have immediate-agreement. For behavioral predictions, an equilibrium refinement is necessary, but our analysis of the stationary equilibrium calls into question whether it is the *right* refinement.

Our paper is not the only transferable utility stochastic bargaining model to emerge with delay. Merlo and Wilson [9] analyze a multilateral bargaining game where the surplus and recognition are stochastic but where agents agree over the distribution of the stochastic process. It finds that the unique stationary equilibrium may involve delayed agreement but such delays are necessarily Pareto-efficient as the unanimity rule implements the optimal stopping-time. [15] Cripps [5] analyzes a similar bilateral bargaining game where the surplus is stochastic but the buyer and seller may have different levels of patience. This creates a disagreement between the agents about the optimal stopping time and interestingly, this disagreement leads to inefficient subgame perfect equilibria. Ponsati and Sakovics [12] analyze a stochastic model of bilateral bargaining where the availability of outside options

[15] In Merlo and Wilson [8], it is shown that this result does not extend to NTU games; they present an example where the unique stationary equilibrium outcome is inefficient.

is random, and finds that when the outside-options are moderate, agents can use the *credible* threat of taking an outside-option to generate a continuum of subgame perfect equilibrium outcomes. The possibility for delay arises in their model but only for particular parameters is this delay inefficient. As such, all these papers have convincingly shown that bargaining in a stochastic environment may be inefficient. Our paper provides one particular stochastic model of bargaining where efficiency is guaranteed if agents agree over the distribution of the stochastic process, but is lost when there is excessive disagreement.

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Appendix A. Omitted Proofs

Proof of Lemma 1. Consider X, an agreement-absorption set, and τ such that $S^1(\delta, p, \tau) \in X$, and denote $x = S^1(\delta, p, \tau)$. For all $t > \tau$, the game $G(\delta, p, \tau)$ is equivalent to the subgame in $G(\delta, p, t)$ where the offers in the first $t - \tau$ periods have been rejected. Hence, $S^{1+t-\tau}(\delta, p, t) = x \in [1, \frac{1}{\delta}]$. Therefore, by induction, $S^1(\delta, p, t) = f^{t-\tau}(x) \in X \subseteq [1, \frac{1}{\delta}]$, and $G(\delta, p, t)$ ends in immediate agreement. □

Proof of Theorem 1. This is an immediate extension of Theorem 2 in [16]. It suffices to show that for any τ, we can find $t \geqslant \tau$ such that $S^1(\delta, p, t) \in \left[1, \frac{1}{\delta}\right]$. Take any $\tau \in \mathbf{N}^+$ and consider $G(\delta, p, \tau)$. If $S^1(\delta, p, \tau) \in [1, \frac{1}{\delta}]$, the proof is complete. If $S^1(\delta, p, \tau) > \frac{1}{\delta}$, note that the subgame following the rejection of the first-offer of $G(\delta, p, \tau + 1)$ is isomorphic to $G(\delta, p, \tau)$. Since $S^2(\delta, p, \tau + 1) = S^1(\delta, p, \tau) > \frac{1}{\delta}$, we calculate $S^1(\delta, p, \tau + 1) = \delta S^2(\delta, p, \tau + 1) = \delta S^1(\delta, p, \tau)$. By induction then, $S^1(\delta, p, t) = \delta^{t-\tau} S^1(\delta, p, \tau)$ if $S^1(\delta, p, \hat{t}) > \frac{1}{\delta}$ for each $\hat{t} \in (\tau, t) \cap \mathbf{N}$. Since $S^1(\delta, p, \tau)$ is finite, and $\delta < 1$, we are guaranteed agreement in $G(\delta, p, t)$ for some $t > \tau + \frac{\log S^1(\delta, p, t)}{\log(\delta)} - 1$. □

Proof of Theorem 3. Here we show that the set $\{(\delta, p)$ where $\delta y > 1 : \exists t$ where $S^1(\delta, p, t) = S^*(\delta, p)\}$ has measure 0. Pick a fixed $t < \infty$. Note that $S^{t-1}(\delta, p, t) = 1 + y \neq S^*(\delta, p)$ for all (δ, p) where $\delta y > 1$. $S^1(\delta, p, t) = \Lambda^{t-2}(1 + y(p))$ is a polynomial in δ, y, and t, and the set $\left\{\delta : S^1(\delta, p, t) - S^*(\delta, p, t) = 0\right\}$ is, at most, countable for each t. Therefore, $\cup_{t \in \mathbf{N}^+} \{\delta : S^1(\delta, p, t) - S^*(\delta, p, t) = 0\}$ is countable and has Lebesgue measure 0. □

Proof of Proposition 1. By iterating Λ, we can summarize

$$\Lambda^2(x) = \begin{cases} \delta^2 x & \text{if } x > \frac{1}{\delta^2}, \\ 1 + y - \delta^2 yx & \text{if } \frac{1}{\delta} < x \leqslant \frac{1}{\delta^2}, \\ (1+y)(1-\delta y) + \delta^2 y^2 x & \text{if } \frac{\delta(1+y)-1}{\delta^2 y} < x \leqslant \frac{1}{\delta}, \\ \delta(1+y-\delta yx) & \text{if } 1 \leqslant x \leqslant \frac{\delta(1+y)-1}{\delta^2 y}. \end{cases} \tag{A.1}$$

We restrict attention to $(\delta, p) \in D = \{(\delta, p) : y \in (\frac{1}{\delta}, \frac{1}{\delta^2})\}$. It can be verified that the set of fixed points of Λ^2 is $\{S^*, P, Q\}$ where $P = \frac{1+y}{1+\delta^2 y} \in [\frac{1}{\delta}, \frac{1}{\delta^2}]$, and $Q = \frac{\delta(1+y)}{1+\delta^2 y} \in [1, \frac{\delta(1+y)-1}{\delta^2 y}]$. We prove the proposition through a series of results. We first show that for any point $x \in [1, S^*)$, $\lim_{n\to\infty} \Lambda^{2n}(x) = Q$. Then we proceed to show that for almost every $x \in [1, \infty)$, there exists k such that $\Lambda^k(x)$ lies in $[1, S^*)$, and hence, for some k, $\lim_{n\to\infty} \Lambda^{2n+k}(x) = Q$. This allows us to prove the proposition.

Lemma 4. *For every $x \in [1, S^*)$, and for each $\varepsilon > 0$, there exists $N < \infty$ such that for all $n > N$, $\left|\Lambda^{2n}(x) - Q\right| < \varepsilon$.*

Proof. We first show that $x \in [1, \frac{\delta(1+y)-1}{\delta^2 y}] \Rightarrow \Lambda^2(x) \in [1, \frac{\delta(1+y)-1}{\delta^2 y}]$. Pick an arbitrary x. Since Λ^2 is decreasing on this part of the domain, $\Lambda^2(x) \leqslant \Lambda^2(1) = \delta(1 + y - \delta y)$. It can be directly verified that $\Lambda^2(1) = \frac{\delta(1+y)-1}{\delta^2 y} + (1-\delta)(\delta^2 y - 1)(\delta y - 1) < \frac{\delta(1+y)-1}{\delta^2 y}$. Hence, $\Lambda^2([1, \frac{\delta(1+y)-1}{\delta^2 y}]) \subset [1, \frac{\delta(1+y)-1}{\delta^2 y}]$. Furthermore, for $x \in [1, \frac{\delta(1+y)-1}{\delta^2 y}]$, we can show that $|\Lambda^2(x) - Q| = (\delta^2 y)|x - Q| < |x - Q|$. We can then prove by induction that $|\Lambda^{2n}(x) - Q| = (\delta^2 y)^n |x - Q|$ for every integer n. Therefore, for every $n > [\frac{\log \varepsilon - \log(|x-Q|)}{\log(\delta^2 y)}]$ [16] we obtain $|\Lambda^{2N}(x) - Q| < \varepsilon$.

Now consider $x \in (\frac{\delta(1+y)-1}{\delta^2 y}, S^*)$. Since Λ^2 is increasing on this restricted domain, $\Lambda^2(x) < \Lambda^2(S^*) = S^*$. Moreover, it can be shown that $|\Lambda^2(x) - S^*| = (\delta^2 y^2)|x - S^*|$, or $\Lambda^2(x) = x - (\delta^2 y^2 - 1)(S^* - x)$ and hence $\Lambda^2(x) < x$ for all x. By induction then, for all integers n where $\Lambda^{2n-2}(x) \in [\frac{\delta(1+y)-1}{\delta^2 y}, S^*)$, we have $|\Lambda^{2n}(x) - S^*| = (\delta^2 y^2)^n |x - S^*|$. Therefore, for $N = [\frac{\log(|\frac{\delta(1+y)-1}{\delta^2 y} - S^*|) - \log(|x - S^*|)}{2\log(\delta y)}]$, it is true that $\Lambda^{2N}(x) \leqslant \frac{\delta(1+y)-1}{\delta^2 y}$. By applying the argument above, if we choose any $n > N + [\frac{\log \varepsilon - \log(|\frac{\delta(1+y)-1}{\delta^2 y} - Q|)}{\log(\delta^2 y)}]$, then $|\Lambda^{2n}(x) - Q| < \varepsilon$. □

16 $[\cdot]$ is the operator that finds the smallest integer greater than or equal to its argument.

With an abuse of notation, for any function $g : \Re \to \Re$, we define the set-valued function g on all subsets of $\Re$ where for $A \in 2^{\Re}$, $g[A] = \{x \in \Re : \exists a \in A \text{ such that } g(a) = x\}$.

Lemma 5. *Consider the infinite sequence of sets,* $\{X_i\}_{i=0}^{\infty}$ *where* $X_0 = [1, S^*)$ *and for* $i \geqslant 1$, $X_i = (\frac{S^*}{\delta^{i-1}}, \frac{S^*}{\delta^i})$. *Then for* $i > 1$, $\Lambda[X_i] \subseteq X_{i-1}$.

Proof. For $i \geqslant 2$, x lies in X_i implies that $x > \frac{1}{\delta}$. Hence $\Lambda(x) = \delta x \in X_{i-1}$. For $i = 1$, consider $X_1 = (S^*, \frac{1}{\delta}] \cup [\frac{1}{\delta}, \frac{S^*}{\delta})$. It can be verified that $\Lambda[(S^*, \frac{1}{\delta}]] = \Lambda[[\frac{1}{\delta}, \frac{S^*}{\delta})] = [1, S^*) = X_0$. $\square$

For each $x \in \cup_{i=0}^{\infty} X_i$, let $i(x) = \{i \in \mathbf{N} : x \in X_i\}$. The following result shows that for each point within this infinite sequence of sets, there is a subsequence of iterates that converge to Q.

Lemma 6. *For each* $x \in \cup_{i=0}^{\infty} X_i$, *for each* $\varepsilon > 0$, *there exists* N *such that for all* $n > N$, $|\Lambda^{2n+i(x)}(x) - Q| < \varepsilon$.

Proof. Since $x \in X_{i(x)}$, by induction, it can be shown that $\Lambda^i[X_i] \subset X_0$, and hence $\Lambda^{i(x)}(x) \in X_0$. By Lemma 4, for each $\varepsilon > 0$, there exists $N < \infty$ such that for all $n > N$, $\left|\Lambda^{2n}(\Lambda^{i(x)}(x)) - Q\right| = |\Lambda^{2n+i(x)} - Q| < \varepsilon$. $\square$

Lemma 7. *For each* $x \in \cup_{i=0}^{\infty} X_i$, *for each* $\varepsilon > 0$, *there exists* $N < \infty$ *such that for all* $n > N$, $\left|\Lambda^{2n+i(x)}(x) - Q\right| + \left|\Lambda^{2n+i(x)+1}(x) - P\right| < \varepsilon$, *and* $\left|\Lambda^{2n+i(x)}(x) - Q\right| + \left|\Lambda^{2n+i(x)-1}(x) - P\right| < \varepsilon$.

Proof. By Lemma 6, for each $\varepsilon > 0$, there exists $N < \infty$ such that for all $n > N$, $\left|\Lambda^{2n+i(x)}(x) - Q\right| < \frac{\varepsilon}{2y} < \frac{\varepsilon}{2}$. It can be verified that $\left|\Lambda^{2n+i(x)+1}(x) - P\right| = |\Lambda(\Lambda^{2n+i(x)}(x)) - \Lambda(Q)| = (\delta y)\left|\Lambda^{2n+i(x)}(x) - Q\right| < \frac{\delta\varepsilon}{2} < \frac{\varepsilon}{2}$. Therefore, $\left|\Lambda^{2n+i(x)}(x) - Q\right| + \left|\Lambda^{2n+i(x)+1}(x) - P\right| < \varepsilon$. Moreover, $\left|\Lambda^{2n+i(x)-1}(x) - P\right| \leqslant \frac{1}{\delta}\left|\Lambda^{2n+i(x)}(x) - Q\right| < \frac{\varepsilon}{2\delta y} < \frac{\varepsilon}{2}$, and hence $\left|\Lambda^{2n+i(x)}(x) - Q\right| + \left|\Lambda^{2n+i(x)-1}(x) - P\right| < \varepsilon$. $\square$

We now use Lemma 7 to prove the proposition. We first restrict our analysis to the set $\tilde{D} = \{(\delta, p) \in D : \forall t \in \mathbf{N}, 1 + y \neq \frac{S^*}{\delta^t}\}$.

Lemma 8. *For each* $(\delta, p) \in \tilde{D}$, *for each* $\varepsilon > 0$, *there exists* $T < \infty$ *such that for all* $t > T$, $|S^0(\delta, p, t) - P| + |S^1(\delta, p, t) - Q| < \varepsilon$ *or* $|S^0(\delta, p, t) - P| + |S^1(\delta, p, t) - Q| < \varepsilon$.

Proof. Pick any positive ε. Since $1 + y \neq \frac{S^*}{\delta^t}$ for each $t \in \mathbf{N}$, it is established that $1 + y \in \cup_{i=0}^{\infty} X_i$. Let $i(1 + y) = i$. By Lemma 7, there exists $N < \infty$ such that for all $n > N$, $\left|\Lambda^{2n+i}(1 + y) - Q\right| + \left|\Lambda^{2n+i+1}(1 + y) - P\right| < \varepsilon$. Define $T = 2(N + 1) + i$. Picking any

$t > T$, it is clear that $S^0(\delta, p, t) = \Lambda^{t-1}(1 + y(p))$, and $S^1(\delta, p, t) = \Lambda^{t-2}(1 + y(p))$. If $i \bmod 2 = t \bmod 2$, then $S^0(\delta, p, t) = \Lambda^{t-1}(1 + y(p)) = \Lambda^{2n+i+1}(1 + y(p))$ and $S^1(\delta, p, t) = \Lambda^{t-2}(1 + y(p)) = \Lambda^{2n+i}(1 + y(p))$ where $n = \frac{t-i}{2} - 1 \in \mathbf{N}$, and $n > N$. Therefore $|S^0(\delta, p, t) - P| + |S^1(\delta, p, t) - Q| < \varepsilon$. If $i \bmod 2 = t \bmod 2 + 1$, then $S^0(\delta, p, t) = \Lambda^{t-1}(1 + y(p)) = \Lambda^{2n+i}(1 + y(p))$ and $S^1(\delta, p, t) = \Lambda^{2n+i-1}(1 + y(p))$ where $n = \frac{t-i-1}{2} \in \mathbf{N}$, and $n > N$. Therefore $|S^0(\delta, p, t) - Q| + |S^1(\delta, p, t) - P| < \varepsilon$. □

Lemma 9. *The set* $D \backslash \tilde{D}$ *has Lebesgue measure* 0.

Proof. Note that $(\delta, p) \in D \backslash \tilde{D}$ if and only if there exists $t \in \mathbf{N}$ such that $1 + y = \frac{S^*}{\delta^t}$. Hence, for each p, the set $\{\delta : (\delta, p) \in D \backslash \tilde{D}\} = \cup_{t \in \mathbf{N}^+} \{\delta : \delta^t + \delta^{t+1} y = 1\}$, which is a countable union of countable sets. $D \backslash \tilde{D}$ is thus of Lebesgue measure 0. □

The proposition follows from the preceding lemmas.

Proof of Proposition 2. Consider $T = \{t \in \mathbf{N} : t < \tau\}$ as in the definition of the game and restrict attention to all $G(\delta, p, \tau)$ where $\delta y > 1$ and $\tau \geqslant L(\delta, p) + 2$.

Lemma 10. *There is agreement at period* $\tau - L(\delta, p) - 2$.

Proof. It suffices to show that $S^{\tau - L(\delta, p) - 1} \leqslant \frac{1}{\delta}$. Since $S^{\tau-1} = 1 + y > \frac{1}{\delta}$, we can recursively construct the set $D = \{t \in T : t + 1 \in D \cup \{\tau\}$ and $S^t > \frac{1}{\delta}\}$, which is the set of all dates of disagreement regimes near the end of the game. For each $t \in D$, $S^t = \delta^{\tau-1-t}(1 + y)$. Now $t \in D$ implies that $\delta^{\tau-1-t}(1 + y) > \frac{1}{\delta}$, and therefore $(\tau - t) < \frac{\log(1+y)}{\log(\frac{1}{\delta})} \leqslant L(\delta, p) + 1$. The conclusion then follows since $\tau - L(\delta, p) - 1 \notin D$, and $\tau - L(\delta, p) \in D$. □

Lemma 11. S^t *is bounded above by* $1 + y - \delta y$ *for all* $t \leqslant \tau - L(\delta, p) - 1$.

Proof. We know that $S^{\tau - L(\delta, p) - 1} \leqslant \frac{1}{\delta} < 1 + y - \delta y$ so the bound holds for $t = \tau - L(\delta, p) - 1$. By way of contradiction, consider $t < \tau - L(\delta, p) - 1$ where $S^t > 1 + y - \delta y$. Note that $S^{t+1} \in [1, \frac{1}{\delta}]$ implies that $S^t = f(x) \leqslant f(1) = 1 + y - \delta y$. Hence, $S^t > 1 + y - \delta y$ implies that $S^{t+1} > \frac{1}{\delta}$. Therefore $S^{t+1} = \frac{1}{\delta} S^t > \frac{1}{\delta}(1 + y - \delta y) > 1 + y - \delta y$. Then by induction, for all $t' \in \{t, t + 1, \ldots, \tau - 1\}$, $S^{t'} > 1 + y - \delta y$. Yet this is a contradiction since $\tau - L(\delta, p) - 1 \in \{t, t + 1, \ldots, \tau - 1\}$ and $S^{\tau - L(\delta, p) - 1} < 1 + y - \delta y$. □

It is straightforward to then calculate $E(\delta, p)$. If $S^1 < \frac{1}{\delta}$, the game ends in immediate agreement with full efficiency. If $S^1 > \frac{1}{\delta}$, then there is delay. Consider the set $D_1 = \{t \in T \backslash D : t \geqslant 1$ and $S^t > \frac{1}{\delta}$ and $S^{t+1} \leqslant \frac{1}{\delta}\}$. Since $1 \leqslant \tau - L(\delta, p) - 1$, and $S^{\tau - L(\delta, p) - 1} \leqslant \frac{1}{\delta}$, D_1 is non-empty. Let $\gamma = \min_{t \in D_1} t$. By Lemma 11, $S^\gamma \leqslant 1 + y - \delta y$, and therefore, $\gamma < E(\delta, p)$. Hence, $1 - \delta^\gamma < 1 - \delta^{E(\delta, p)}$. The reader can verify then that $\frac{1}{1+y-\delta y} < \delta^{E(\delta, p)} \leqslant \frac{1}{\delta(1+y-\delta y)}$, and hence, $1 - \delta^{E(\delta, p)} \in [\frac{\delta(1+y-\delta y)-1}{\delta(1+y-\delta y)}, \frac{y(1-\delta)}{1+y-\delta y})$. □

Proof of Theorem 4. Pick $\varepsilon > 0$. For all $G(\delta, p, t)$ where $t \geqslant L(\delta, p) + 2$, the maximal loss of efficiency is $1 - \delta^{E(\delta,p)}$ that lies in the set $[\frac{\delta(1+y-y\delta)-1}{\delta(1+y-y\delta)}, \frac{y(1-\delta)}{1+y-y\delta})$. Then for $\delta > 1 - \frac{\varepsilon}{y(1-\varepsilon)}$, we can establish that $1 - \delta^{\gamma(\delta,p,t)} < 1 - \delta^{E(\delta,p)} < \frac{y(1-\delta)}{1+y-y\delta} < \varepsilon$. □

Proof of Theorem 5. By stationarity, and since all subgames prior to recognition are isomorphic to each other, for all i and t, $V_i^t = V_i^{t+1}$. Thus, $S^t = S^{t+1}$. Since perpetual disagreement is not an SPE outcome, $S^t = \Lambda(S^t)$ implies that $S^t = S^* = \frac{1+y}{1+y\delta}$, and $V_i^t = \frac{p_i}{1+\delta y}$. It is easy to confirm that the following is the essentially unique SSPE: if an agent is recognized, she offers all other agents their discounted continuation values, and other agents accept this offer and refuse anything less. □

Proof of Theorem 6. By definition, an SSPE is an HIE. For each i, let $\{V_i^t\}_{t=0}^{\infty}$ represent i's continuation value at each period in the game when the strategy-profile is σ and for each t, let $S^t = \sum_{i \in N} V_i^t$.

Lemma 12. *Consider (δ, p) such that $\delta y \leqslant 1$. Then for any HIE σ, $S^t \in [1, \frac{1}{\delta}]$ for all t.*

Proof. Pick any arbitrary t. By definition of HIE, $S^t = \Lambda(S^{t+1})$ for each t. This implies that S^t is at least 1. Since $\left[1, \frac{1}{\delta}\right]$ is an agreement-absorption set, if $S^t > \frac{1}{\delta}$, then for all $\tau > t$, it must be that $S^\tau > \frac{1}{\delta}$. However, this would entail perpetual disagreement after period t, which is not an SPE outcome. Therefore, $S^t \in [1, \frac{1}{\delta}]$ for all t. □

If $\delta y = 0$, then by the preceding lemma, $S^t = 1 + y - \delta y S^{t+1} = 1$, and therefore, for each integer τ, $S^\tau = 1$. Therefore, $V_i^t = p_i(1-\delta) + \delta V_i^{t+1} = \sum_{\tau=0}^{\infty} \delta^\tau p_i(1-\delta) = p_i$. Hence, the HIE outcome is stationary.

If $\delta y \in (0, 1)$. Consider a HIE and its induced perceived surplus, $\{S^t\}_{t=0}^{\infty}$. If the HIE is non-stationary, there exists date τ where $S^\tau \neq S^*$. Since by the preceding lemma, $S^{\tau+1} \in [1, \frac{1}{\delta}]$, it must be true that $S^{\tau+1} = f^{-1}(S^\tau) = \frac{1+y-S^\tau}{\delta y}$, where ε is some arbitrarily small number. Then it can be shown by induction for all $k > 0$ that $|S^{\tau+k} - S^*| = (\delta y)^{-k}|S^\tau - S^*|$. Since $\delta y < 1$, for all $k > \frac{\log(|\frac{1}{\delta} - S^*|) - \log(|S^\tau - S^*|)}{\log(\frac{1}{\delta y})}$, it must be the case that $|S^{\tau+k} - S^*| > |\frac{1}{\delta} - S^*|$. Pick a particular such k. If $S^{\tau+k} > S^*$, this implies that $S^{\tau+k} > \frac{1}{\delta}$. If $S^{\tau+k} < S^*$, since f^{-1} is a decreasing function, this implies that $S^{\tau+k+1} = f^{-1}(S^{\tau+k}) > f^{-1}(S^*) = S^*$, and since $|S^{\tau+k+1} - S^*| > (\delta y)^{-1}|S^{\tau+k} - S^*| > |\frac{1}{\delta} - S^*|$, it is established that $S^{\tau+k+1} > \frac{1}{\delta}$. In either case, this poses a contradiction to Lemma 12.

Now consider (δ, p) such that $\delta y \geqslant 1$. For $x \in [1, 1 + y - \delta y]$, consider the correspondence $\Lambda^{-1}(x) = \{z : \Lambda(z) = x\} \cap [1, 1 + y - \delta y]$, and its upper-envelope, $g(x) = \max_{z \in \Lambda^{-1}(x)} z$.

We can now construct the equilibrium: let $S^t = S$, for all $\tau \in \{0, \ldots, t-1\}$, let $S^\tau = \Lambda^{t-\tau}(S)$, and for all $\tau > t$, let $S^\tau = g^{\tau-t}(S)$. By construction, for all periods t, $S^t = \Lambda(S^{t+1})$. Associate with each S^τ a vector $\{V_1^\tau, \ldots, V_n^\tau\}$ where $V_i^\tau = \frac{p_i}{1+y} S^\tau$. Note that if $S^{\tau+1} < \frac{1}{\delta}$, then $V_i^\tau = \frac{p_i}{1+y}(1 + y - \delta y S^{\tau+1}) = p_i(1 - \delta S^{\tau+1}) + \delta V_i^{\tau+1}$ and that if

$S^{\tau+1} > \frac{1}{\delta}$, $V_i^\tau = \frac{p_i}{1+y}\delta S^{\tau+1} = \delta V_i^{\tau+1}$. Thus, the following is an HIE: if i is recognized at time τ and $S^{\tau+1} < \frac{1}{\delta}$, she offers each agent j her continuation value, $\delta V_j^{\tau+1}$, and j accepts any proposal that offers her at least this continuation value. If $S^{\tau+1} > \frac{1}{\delta}$, agent i makes any proposal that does not leave her with less than $\delta V_i^{\tau+1}$, and on the equilibrium path, this offer is rejected. This completes the construction.

We now show that for every history-independent equilibrium, for every t, $S^t \in [1, 1 + y - \delta y]$. Since perpetual disagreement can never be an equilibrium outcome following any finite history, for every t, there exists $t' > t$ such that $S^{t'} \in [1, \frac{1}{\delta}]$. Then it can be shown by induction that $S^t \leqslant \Lambda(1) = 1 + y - \delta y$. If $S^{t+1} > \frac{1}{\delta}$, then $S^t > \delta S^{t+1} > 1$, and if $S^{t+1} < \frac{1}{\delta}$, then $S^t > \Lambda(\frac{1}{\delta}) = 1$. □

Proof of Theorem 7. Consider a history-independent equilibrium, σ, and its induced perceived surplus, $\{S^t(\sigma)\}_{t=0}^{\infty}$. If there is agreement at every round t, then the set $\{S : \exists t$ such that $S = S^t\}$ is an agreement-absorption set. By Lemma 3, it must be that $S^t(\sigma) = S^*$. Hence, σ is the SSPE. □

Proof of Proposition 3. We prove the following lemma.

Lemma 13. *Consider any HIE σ of $G(\delta, p, \infty)$. Then for each t, and for each $i \in N$, we have $V_i^t(\sigma) = \frac{p_i}{1+y} S^t(\sigma)$.*

Proof. Define for each t, $g^t = \sum_{i=t+1}^{\infty} \delta^{i-t}(1 - \delta S^i(\sigma))\mathbf{1}_{\{S^i(\sigma) \leqslant \frac{1}{\delta}\}}$. Since $\delta < 1$, and for each i, $(1 - \delta S^i(\sigma))\mathbf{1}_{\{S^i(\sigma) \leqslant \frac{1}{\delta}\}} \leqslant 1$, it is guaranteed that the limit exists. The result then follows from recognizing that $V_i^t = p_i g^t$. □

Proposition 3 follows immediately from Lemma 13: $S^0(\sigma) > S^0(\sigma')$ implies that for each $i \in N$, $V_i^0(\sigma) > V_i^0(\sigma')$. The equivalence then follows since if for each $i \in N$, $V_i^0(\sigma) > V_i^0(\sigma')$, then $S^0(\sigma) = \sum_{i \in N} V_i^0(\sigma) > \sum_{i \in N} V_i^0(\sigma') = S^0(\sigma')$. □

Section 5: We restrict n to be at least 3. We first characterize the difference equations that govern continuation values. Since agents are symmetric, we can write for each agent i, $V_i^t = V^t$. We define $S^t = mV^t$.

As before, if $S^{t+1} > \frac{1}{\delta}$, then for an offer x to be accepted, it must be true for at least m agents that $x_j \geqslant \delta V^{t+1}$, or $x \geqslant \delta S^{t+1} > 1$. Hence, all offers are infeasible and $\{S^t, V^t\} = \{\delta S^{t+1}, \delta V^{t+1}\}$. If $S^{t+1} \leqslant \frac{1}{\delta}$, then $V^t = \frac{1+y}{n}(1-(m-1)\delta V^{t+1}) + (1 - \frac{1+y}{n})(\frac{m-1}{n-1})\delta V^{t+1} = \frac{1+y}{n} - \frac{m-1}{n-1}\delta y V^{t+1}$, and $S^t = g(S^{t+1})$ where $g(x) = \frac{m}{n}(1+y) - \frac{m-1}{n-1}\delta y x$. It may be verified that on the equilibrium path, S^t may be less than 1, but S^t is at least $g(\frac{1}{\delta}) = \frac{m}{n} + \frac{n-m}{n(n-1)} y \leqslant 1$ since $y \leqslant n - 1$.

Proof of Proposition 4. Analogous to Theorem 1, we know that there exists a countably infinite set T^A such that for all $t \in T^A$, $G(\delta, n, m, p, t)$ ends in immediate agreement. Hence, it suffices to show that if $\delta y \leqslant \frac{n-1}{m-1}$, then $x \in A = [g(\frac{1}{\delta}), \frac{1}{\delta}]$ implies $g(x) \in A$

since this proves a version of the Immediate Agreement Theorem. Note that an important difference here is that the lower bound of the set is $g(\frac{1}{\delta}) \leqslant 1$.

Since g is a decreasing function, for all $x \in A$, $g(x) \leqslant g(g(\frac{1}{\delta})) = g^2(\frac{1}{\delta})$. It may be verified by direct calculation that if $y = \frac{n-1}{\delta(m-1)}$ then $g^2(\frac{1}{\delta}) = \frac{1}{\delta}$. For all (δ, y), consider $I(\delta, y) = g^2(\frac{1}{\delta})$, and $h(\delta, y) = \frac{\partial}{\partial y} I(\delta, y) = \frac{m}{n}(1 - \delta\frac{(m-1)}{(n-1)}) - 2\delta y \frac{(n-m)(m-1)}{n(n-1)^2}$. Note that $h(\delta, y)$ is decreasing in y, and therefore for $y \leqslant \frac{n-1}{\delta(m-1)}$, $h(\delta, y) \geqslant h(\delta, \frac{n-1}{\delta(m-1)}) = \frac{m}{n}(1 - \delta\frac{(m-1)}{(n-1)}) - \frac{2(n-m)}{n(n-1)}$. Since $h(\delta, \frac{n-1}{\delta(m-1)})$ is decreasing in δ, for each δ, $h(\delta, \frac{n-1}{\delta(m-1)}) > h(1, \frac{n-1}{\delta(m-1)}) = \frac{(m-2)(n-m)}{n(n-1)} \geqslant 0$. This establishes that $h(\delta, y) > 0$ for each (δ, y) where $\delta y \leqslant \frac{n-1}{m-1}$. Since $I(\delta, y)$ is continuous and differentiable in y, by the First Fundamental Theorem of Calculus, this yields that for all $y < \frac{n-1}{\delta(m-1)}$, we can calculate $I(\delta, y) = I(\delta, \frac{n-1}{\delta(m-1)}) - \int_y^{\frac{n-1}{\delta(m-1)}} h(\delta, t)\, dt < I(\delta, \frac{n-1}{\delta(m-1)}) = \frac{1}{\delta}$. Therefore, if $S^{t+1} \leqslant \frac{1}{\delta}$, then $S^t \leqslant g^2(\frac{1}{\delta}) < \frac{1}{\delta}$. Therefore, if $\delta y \leqslant \frac{n-1}{m-1}$, all sufficiently long games end in immediate agreement.

To prove that if $\delta y > \frac{n-1}{m-1}$, that generically there are arbitrarily long games with delay, by an analogous result to Theorem 3, it suffices to shows that the unique agreement-absorption set is $\{\frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}\}$, where $\frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}$ is the unique fixed point of g. Assume by way of contradiction that there exists another agreement-absorption set A where $x \in A \backslash \{\frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}\}$. Then as $\delta y > \frac{n-1}{m-1}$, we know that $|g(x) - \frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}| > \lambda |x - \frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}|$ where $\lambda \in (1, \frac{m-1}{n-1}\delta y)$, and by induction, $|g^t(x) - \frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}| > \lambda^t |x - \frac{m(n-1)(1+y)}{n(n-1+\delta my-\delta y)}|$. Therefore for sufficiently high t, we are guaranteed to find $g^t(x) > \frac{1}{\delta}$, and hence A is not an agreement-absorption set. □

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