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On the Equality of Modal Damping Power and the Average Rate of Transient Energy Dissipation in a Multimachine Power System

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***Abstract*— This letter establishes a relationship between the concepts of damping torque and the dissipation of transient energy in a multimachine power system with constant power loads. To that end, we present a mathematical proof showing that, for a poorly-damped mode, the total damping power stemming from the interaction of electromagnetic torques and rotor speeds is approximately equal to the average rate of transient energy dissipation in the system corresponding to the modal oscillation. This is verified with numerical studies on the IEEE 2-area 4-machine and 5-area 16-machine test systems.**

***Index Terms*— Lyapunov methods, power systems, stability of linear systems, smart grid**

I. Introduction

DAMPING torque analysis, as introduced by Park in his 1933 paper [1] and furthered by Concordia [2], Shepherd [3], and notable others [4]–[6], help develop insightful understanding of the stabilizing contributions coming from a synchronous machine and its governor and excitation systems. Complementary to this, the Lie derivative of a Lyapunov-like transient energy function, derived in [7], is another control-theoretic measure of system's damping. The concepts of passivity and positive realness have also been used to interpret this complementary notion [8] and there exists a host of publications that have focused on passivity-based damping control [9], [10] of power systems.

Although some of the initial works listed above highlighted an intuitive link between damping torque and dissipation of transient energy (and indirectly with the passivity theory), it is only in the recent works [11] and [12] that a rigorous mathematical connection between the two has been established for a single-machine-infinite-bus system. However, for multi-machine systems such a connection is yet to be confirmed – in this letter, we make a maiden attempt to fill this gap. This is important given the critical emphasis attributed to the theory of damping torque in power system stability analysis, as is evident from classical textbooks like [13] and [14]. To that end, we use a simplified mathematical model for multimachine systems with constant power loads to establish an equivalence between the average power dissipation due to the damping torques on the rotors and the average rate of transient energy dissipation in the system. We emphasize that our focus is on poorly-damped conditions posing stability challenges that form the basic premise for both the theories.

The letter is structured as follows. In Section II we derive a linearized representation of a multimachine system with third-order synchronous generator model, lossless transmission network, and constant power loads. Building on this model, in Sections III and IV, we present a rigorous analytical proof of the approximate equality of total damping power and average rate of transient energy dissipation in the system, for poorly-damped modes, under assumptions of constant mechanical power input and constant field excitation. Case studies extending the proof to higher-order models are presented in Section V, followed by conclusions in Section VI.

Symbols and notations: Quantities V_i, θ_i, P_{L_i}, and Q_{L_i} are respectively, the voltage magnitude, angle, and real and reactive power loads at bus $i \in \{1, \cdots, n\}$. Quantities δ_i, ω_i, E'_{d_i}, E'_{q_i}, T_{m_i}, E_{fd_i}, H_i, x'_{d_i}, x'_{q_i}, x_{d_i}, x_{q_i}, T'_{do_i}, and T'_{qo_i} are respectively, the rotor angle, rotor speed, d- and q-axes induced emfs, mechanical torque input, field circuit excitation voltage, inertia constant, d- and q-axes transient reactances, d- and q-axes synchronous reactances, and d- and q-axes open-circuit transient time-constants of a generator connected to bus $i \in \{1, \cdots, n_g\}$, $n_g < n$. ω_s is the synchronous speed and $Y_{ik}\, e^{j\alpha_{ik}}$ is the $(i,k)^{\text{th}}$-element of the network admittance matrix. Symbol $\Delta\vec{x}_{i,r}$ is the small-signal phasor representation of the r^{th} modal component in x_i. Superscripts T, $*$, and H are respectively the transpose, conjugate, and Hermitian operators. $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real and imaginary parts of a complex entity.

II. System Model: Linearized Representation

Consider a $n-$bus transmission system of which, without loss of generality, first n_g are generator buses. The network is lossless and each synchronous generator is described by a one-axis flux-decay model [7] with manual excitation. In addition, we neglect the stator resistances and assume constant power loads leading to the worst-case damping.

The differential and algebraic equations describing the system, as functions of state variables δ_i, ω_i, and E'_{q_i}, and algebraic variables θ_i and V_i are described below.

(a) Generator buses $(i = 1,\ 2, \dots\ n_g)$:

$$\dot{\delta}_i = \omega_i - \omega_s \tag{1}$$

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$$\frac{\dot{\omega}_i}{\omega_s} = \frac{T_{m_i}}{2H_i} - \frac{E'_{q_i} V_i \sin(\delta_i - \theta_i)}{2H_i\ x'_{d_i}} + \frac{V_i^2 \sin 2(\delta_i - \theta_i)}{4H_i}\Big(\frac{x_{q_i} - x'_{d_i}}{x_{q_i} x'_{d_i}}\Big) \tag{2}$$

$$\dot{E}'_{q_i} = \frac{E_{fd_i}}{T'_{do_i}} - \frac{E'_{q_i}}{T'_{do_i}} - \frac{E'_{q_i} - V_i \cos(\delta_i - \theta_i)}{x'_{d_i}}\Big(\frac{x_{d_i} - x'_{d_i}}{T'_{do_i}}\Big) \tag{3}$$

$$0 = f_i = \frac{E'_{q_i} V_i \sin(\delta_i - \theta_i)}{x'_{d_i}} - \frac{V_i^2 \sin 2(\delta_i - \theta_i)}{2}\Big(\frac{x_{q_i} - x'_{d_i}}{x_{q_i} x'_{d_i}}\Big) + P_{L_i} - \sum_{k=1, k\neq i}^{n} V_i V_k Y_{ik} \sin(\theta_i - \theta_k) \tag{4}$$

$$0 = g_i = \frac{E'_{q_i} V_i \cos(\delta_i - \theta_i)}{x'_{d_i}} - \frac{V_i^2 \cos^2(\delta_i - \theta_i)}{x'_{d_i}} + Q_{L_i} - V_i^2 Y_{ii} - \frac{V_i^2 \sin^2(\delta_i - \theta_i)}{x_{q_i}} + \sum_{k=1, k\neq i}^{n} V_i V_k Y_{ik} \cos(\theta_i - \theta_k) \tag{5}$$

(b) Load buses $(i = n_g + 1,\ n_g + 2, \dots\ n)$:

$$0 = f_i = P_{L_i} - \sum_{k=1, k\neq i}^{n} V_i V_k Y_{ik} \sin(\theta_i - \theta_k) \tag{6}$$

$$0 = g_i = Q_{L_i} + \sum_{k=1, k\neq i}^{n} V_i V_k Y_{ik} \cos(\theta_i - \theta_k) - V_i^2 Y_{ii} \tag{7}$$

Linearizing (1) $-$ (5) around an operating point, with V_{i_0} as the voltage magnitude of bus i at that point and defining a new variable $\nu_i = V_i / V_{i_0}$, we obtain

$$\begin{aligned}\big[\,\Delta\dot{\boldsymbol{\delta}}\ \Delta\dot{\boldsymbol{\omega}}\ \Delta\dot{\boldsymbol{E}}'_{\boldsymbol{q}}\,\big]^T &= \mathbf{M}\,[\,\Delta\boldsymbol{\delta}\ \Delta\boldsymbol{\omega}\ \Delta\boldsymbol{E}'_{\boldsymbol{q}}\,]^T + \mathbf{N}\,[\,\Delta\boldsymbol{\theta}\ \Delta\boldsymbol{\nu}\,]^T \\ &\quad + \mathbf{B}\,[\ \Delta\boldsymbol{T_m}\quad \Delta\boldsymbol{E_{fd}}\ ]^T \\ [\ \mathbf{0}\quad \mathbf{0}\ ]^T &= \mathbf{C}\,[\ \Delta\boldsymbol{\delta}\quad \Delta\boldsymbol{\omega}\quad \Delta\boldsymbol{E}'_{\boldsymbol{q}}\ ]^T + \mathbf{D}\,[\ \Delta\boldsymbol{\theta}\quad \Delta\boldsymbol{\nu}\ ]^T\end{aligned} \tag{8}$$

where, $\boldsymbol{\delta}$, $\boldsymbol{\omega}$, $\boldsymbol{E}'_{\boldsymbol{q}}$, $\boldsymbol{\theta}$, and $\boldsymbol{\nu}$ are the vectorized state and algebraic variables of respective type, for instance, $\boldsymbol{\delta} = [\ \delta_i\ \ \dots\ \ \delta_{n_g}\ ]^T$ and $\boldsymbol{\nu} = [\ \nu_i\ \ \dots\ \ \nu_n\ ]^T$, and $\mathbf{M}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are the Jacobian matrices obtained from linearization (see, Appendix II). Finally, eliminating the algebraic variables, we get

$$\big[\,\Delta\dot{\boldsymbol{\delta}}\ \Delta\dot{\boldsymbol{\omega}}\ \Delta\dot{\boldsymbol{E}}'_{\boldsymbol{q}}\,\big]^T = \mathbf{A}\,[\,\Delta\boldsymbol{\delta}\ \Delta\boldsymbol{\omega}\ \Delta\boldsymbol{E}'_{\boldsymbol{q}}\,]^T + \mathbf{B}\,[\,\Delta\boldsymbol{T_m}\ \Delta\boldsymbol{E_{fd}}\,]^T \tag{9}$$

where, $\mathbf{A} = \mathbf{M} - \mathbf{N}\mathbf{D}^{-1}\mathbf{C}$. It follows from the equations above that $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{A_{21}} & \mathbf{0} & \mathbf{A_{23}} \\ \mathbf{A_{31}} & \mathbf{0} & \mathbf{A_{33}} \end{bmatrix}. \tag{10}$$

Apart from the state variables $\Delta\delta_i$, $\Delta\omega_i$, and $\Delta E'_{q_i}$, the output variable ΔT_{e_i}, which is the electromagnetic torque of generator i, is of specific interest to us. From the swing equation this is expressed as $\Delta T_{e_i} = -\frac{2H_i}{\omega_s}\ \Delta\dot{\omega}_i + \Delta T_{m_i}$.

The notions of damping torque and damping power for a given mode originate from the phasor representation (see, Appendix I) of $\Delta\vec{T}_{e_{i,r}}$ in the $\Delta\vec{\delta}_{i,r} - \Delta\vec{\omega}_{i,r}$ plane and the power resulting from the interaction of the torque and the speed. This is explained next.

III. Damping Power in a Multimachine System

Definition 1. *For a mode r, the average damping power of any i^{th} machine due to $\Delta T_{e_{i,r}}(t)$ over a cycle starting from $t = t_0$, denoted by $P_{d_{i,r}}(t_0)$, is defined as*

$$P_{d_{i,r}}(t_0) := \frac{\omega_{d_r}}{2\pi}\int_{t_0}^{t_0 + \frac{2\pi}{\omega_{d_r}}} \Delta T_{e_{i,r}}(t)\ \Delta\omega_{i,r}(t)\ dt. \tag{11}$$

Using the phasor notation described in Appendix I, let $\Delta\vec{T}_{e_{i,r}}(t) = \beta_1\ e^{\sigma_r t}\ \angle\gamma_1$ and $\Delta\vec{\omega}_{i,r}(t) = \beta_2\ e^{\sigma_r t}\ \angle\gamma_2$. Therefore, $P_{d_{i,r}}(t_0) =$

$$\begin{aligned} &= \frac{\omega_{d_r}}{2\pi}\int_{t_0}^{t_0+\frac{2\pi}{\omega_{d_r}}} e^{2\sigma_r t}\ \beta_1 \cos(\omega_{d_r} t + \gamma_1)\ \beta_2 \cos(\omega_{d_r} t + \gamma_2)\ dt \\ &= \frac{\beta_1\ \beta_2\ \omega_{d_r}}{4\pi}\ \cos(\gamma_1 - \gamma_2)\ \frac{e^{2\sigma_r t_0}}{2\ \sigma_r}\ \Big\{e^{\frac{4\pi\sigma_r}{\omega_{d_r}}} - 1\Big\} \\ &\quad + \frac{\beta_1\ \beta_2\ \omega_{d_r}}{4\pi}\int_{t_0}^{t_0+\frac{2\pi}{\omega_{d_r}}} e^{2\sigma_r t}\ \cos(2\omega_{d_r} t + \gamma_1 + \gamma_2)\ dt \end{aligned} \tag{12}$$

Now, considering that our mode of interest is poorly-damped[1], we may expand the exponential $e^{\frac{4\pi\sigma_r}{\omega_{d_r}}}$ and neglect the second and higher order terms. On doing so, the first term in (12) reduces to $\frac{1}{2}\ \beta_1\ \beta_2\ e^{2\sigma_r t_0}\ \cos(\gamma_1 - \gamma_2)$. With the same assumption, the second term in (12) becomes negligible, and can be ignored for mathematical tractability. This is because, with $|\sigma_r| << \omega_{d_r}$, for a complete cycle of $\cos(2\omega_{d_r} t)$, the $e^{2\sigma_r t}$ term remains almost constant, and therefore, the positive and negative half cycles approximately add to zero. Therefore, $P_{d_{i,r}}(t_0) \approx \frac{1}{2}\ \beta_1\ \beta_2\ e^{2\sigma_r t_0}\ \cos(\gamma_1 - \gamma_2) = \frac{1}{2}\ \Re\big\{\Delta\vec{T}_{e_{i,r}}(t_0)\ \Delta\vec{\omega}^*_{i,r}(t_0)\big\}$. Following which, it can be interpreted that $P_{d_{i,r}}(t_0)$ is the average power due to the component of $\Delta\vec{T}_{e_{i,r}}(t_0)$ in the direction of $\Delta\vec{\omega}_{i,r}(t_0)$.

Hereafter, in the letter, assuming all phasors are computed at $t = t_0$ and powers are averaged over a cycle starting at t_0, we shall drop the argument t_0 from our expressions.

Definition 2. *For a system with n_g machines, the total damping power of the system for a mode r, denoted by P_{d_r}, is the sum of the average damping powers of all machines.*

$$P_{d_r} = \sum_{i=1}^{n_g} P_{d_{i,r}} \overset{poor\ damp.}{\approx} \frac{1}{2}\sum_{i=1}^{n_g} \Re\Big\{\Delta\vec{T}_{e_{i,r}}\ \Delta\vec{\omega}^*_{i,r}\Big\}. \tag{13}$$

Next, we present the following lemma expressing P_{d_r} in terms of the block matrices in $\mathbf{A}$.

Lemma 1. *Under assumptions of constant power loads, constant mechanical power input, and constant field excitation, for a poorly-damped mode r,*

$$P_{d_r} \approx \frac{1}{2}\ \Delta\vec{\boldsymbol{\omega}}_r^H\ \mathbf{A_{31}}^T\ \mathbf{P}\ (\omega_{d_r}^2 \mathbf{I} + \mathbf{A_{33}}^2)^{-1} \mathbf{A_{31}}\ \Delta\vec{\boldsymbol{\omega}}_r \tag{14}$$

where, $\mathbf{P}$ *is a diagonal matrix with* $\mathbf{P}(i,i) = \frac{T'_{do_i}}{x_{d_i} - x'_{d_i}}$.

Proof. First, we express $\Delta\vec{T}_{e_{i,r}}$ in (13) in terms of the state variables using the swing equation (see, Section II) and the linearized system description obtained in (9) and (10). To

[1] see, [15] pg.4, a mode r is poorly-damped, if $\sigma_r < 0$, with $|\sigma_r| << \omega_{d_r}$ and damping ratio $\zeta_r = -\sigma_r/(\sigma_r^2 + \omega_{d_r}^2)^{0.5} \in (0, 0.03)$.

that end, we take the Laplace transform of ΔT_{e_i}, with the assumption that the inputs $\Delta T_{m_i} = 0$ and $\Delta E_{fd_i} = 0$.

$$\begin{aligned}\Delta \boldsymbol{T_e}(s) &= -\frac{2\,\mathbf{H}}{\omega_s}\Delta\dot{\boldsymbol{\omega}}(s) = -\frac{2\,\mathbf{H}}{\omega_s}\Big\{\mathbf{A_{21}}\ \Delta\boldsymbol{\delta}(s) + \mathbf{A_{23}}\ \Delta\boldsymbol{E'_q}(s)\Big\} \\ &= -\frac{2\,\mathbf{H}}{\omega_s}\ \left\{\mathbf{A_{21}} + \mathbf{A_{23}}\ (s\mathbf{I}-\mathbf{A_{33}})^{-1}\mathbf{A_{31}}\right\}\ \frac{\Delta\boldsymbol{\omega}(s)}{s} \\ &\triangleq\ \mathbf{K}(s)\ \Delta\boldsymbol{\omega}(s).\end{aligned} \tag{15}$$

Next, defining $\mathbf{K}_r = \mathbf{K}(j\omega_{d_r})$, we may re-write (13) as

$$P_{d_r} \approx \frac{1}{2}\Re\Big\{\sum_{i=1}^{n_g}\sum_{j=1}^{n_g}\mathbf{K}_{ij,r}\Delta\vec{\omega}_{j,r}\Delta\vec{\omega}^*_{i,r}\Big\} = \frac{1}{2}\Re\Big\{\Delta\vec{\boldsymbol{\omega}}_r^{H}\mathbf{K}_r\Delta\vec{\boldsymbol{\omega}}_r\Big\} \tag{16}$$

where, $\mathbf{K}_{ij,r}$ is the $(i,j)^{\text{th}}$ element of $\mathbf{K}_r$. Finally, we simplify the expression in (16) using the propositions $(i)-(iv)$ below.

Propositions: $(i)\ \mathbf{P}^{-1}\mathbf{A_{33}}^{T}\mathbf{P} = \mathbf{A_{33}},\ (ii)\ \mathbf{A_{31}}^{T}\mathbf{P} = \frac{2\,\mathbf{H}}{\omega_s}\ \mathbf{A_{23}},$
$(iii)\ 2\ \mathbf{A_{21}}^{T}\mathbf{H} = 2\ \mathbf{H}\ \mathbf{A_{21}},$
$(iv)\ \forall\ \mathbf{x}\in\mathbb{C}^{n_g},\ \Re\{\mathbf{x}^H\mathbf{K}_r\mathbf{x}\} = \mathbf{x}^H\Re\{\mathbf{K}_r\}\ \mathbf{x}.$

Propositions $(i)-(iii)$ are derived using the differential and algebraic equations of the system modeled in Section II. These propositions are then used to establish the symmetry of $\mathbf{K}_r$, which is in-turn used in proving proposition (iv). Proof of these propositions are outlined in Appendix II.

Using (iv) we reduce (16) as follows

$$P_{d_r} \approx \frac{1}{2}\Delta\vec{\boldsymbol{\omega}}_r^{H}\ \Re\{\mathbf{K}_r\}\Delta\vec{\boldsymbol{\omega}}_r. \tag{17}$$

$\Re\{\mathbf{K}_r\}$ is the matrix of multimachine damping torque coefficients for mode r. It is expressed as follows,

$$\begin{aligned}\Re\{\mathbf{K}_r\} &= -\Re\Big\{\frac{2\,\mathbf{H}}{j\omega_{d_r}\omega_s}\Big(\mathbf{A_{21}} + \mathbf{A_{23}}\ (j\omega_{d_r}\mathbf{I}-\mathbf{A_{33}})^{-1}\mathbf{A_{31}}\Big)\Big\} \\ &= \frac{2\,\mathbf{H}}{\omega_s}\ \mathbf{A_{23}}\ (\omega_{d_r}^2\mathbf{I}+\mathbf{A_{33}}^2)^{-1}\mathbf{A_{31}}.\end{aligned}$$

This, along with (ii) when substituted in (17) gives

$$P_{d_r} \approx \frac{1}{2}\ \Delta\vec{\boldsymbol{\omega}}_r^{H}\ \mathbf{A_{31}}^{T}\ \mathbf{P}\ (\omega_{d_r}^2\mathbf{I}+\mathbf{A_{33}}^2)^{-1}\mathbf{A_{31}}\ \Delta\vec{\boldsymbol{\omega}}_r\ .$$

This concludes the proof. □

IV. Consistency of Damping Power with Transient Power Dissipation in the System

For the system model in Section II, the Lyapunov energy function W has been derived in [7] for assessing transient stability[2]. In this section, we deduce an analytical relationship between the Lie derivative of W along the system trajectories and the total damping power of the system.

As derived[3] in [7], the derivative of W along the trajectories of the system is expressed as $\dot{W} = -\sum_{i=1}^{n_g}\frac{T'_{do_i}}{x_{d_i}-x'_{d_i}}(\Delta\dot{E}'_{q_i})^2$. Following which, the average value of this derivative over a cycle can be calculated as $\bar{\dot{W}} = -\frac{1}{T}\sum_{i=1}^{n_g}\frac{T'_{do_i}}{x_{d_i}-x'_{d_i}}\int_{t_0}^{t_0+T}(\Delta\dot{E}'_{q_i})^2\ dt$. Assuming, in $\Delta\dot{E}'_{q_i}$ the modal components $\Delta\dot{E}'_{q_{i,r}}$ are poorly-damped sinusoids of different frequencies, the average value can be decoupled as $\bar{\dot{W}} \approx -\frac{\omega_{d_r}}{2\pi}\sum_{r=1}^{m}\sum_{i=1}^{n_g}\frac{T'_{do_i}}{x_{d_i}-x'_{d_i}}\int_{t_0}^{t_0+\frac{2\pi}{\omega_{d_r}}}(\Delta\dot{E}'_{q_{i,r}})^2\ dt$ (see, (23)−(25) in [11]). We use this to define the transient power dissipation of a mode, discussed next.

Definition 3. *For a mode r, its average rate of transient energy dissipation over a cycle (or simply, 'transient power dissipation'), denoted by P_{W_r}, is defined as*

$$P_{W_r} := \frac{\omega_{d_r}}{2\pi}\sum_{i=1}^{n_g}\frac{T'_{do_i}}{x_{d_i}-x'_{d_i}}\int_{t_0}^{t_0+\frac{2\pi}{\omega_{d_r}}}(\Delta\dot{E}'_{q_{i,r}})^2\ dt. \tag{18}$$

Theorem 1. (Main Result) *Under assumptions of constant power loads, constant mechanical power input, and constant field excitation, for a poorly-damped mode r, P_{W_r} is approximately equal to P_{d_r}.*

Proof. Using the phasor notation in Appendix I, and with the assumptions and approximations as before (see, algebraic manipulations in (12) and discussions following it), for a poorly-damped mode, (18) can be expressed as follows,

$$P_{W_r} \approx \frac{1}{2}\sum_{i=1}^{n_g}\frac{T'_{do_i}}{x_{d_i}-x'_{d_i}}\Big|\Delta\vec{\dot{E}}'_{q_{i,r}}\Big|^2. \tag{19}$$

Further, using notations from (29), we may write $\Delta\vec{\dot{E}}'_{q_{i,r}} = (\sigma_r + j\omega_{d_r})\Delta\vec{E}'_{q_{i,r}} = 2(\sigma_r + j\omega_{d_r})c_r e^{\sigma_r t_0}\psi_{E'_{q_{i,r}}}$. Next, substituting this in (19) we get,

$$P_{W_r} \approx 2|\hat{c}_r|^2\ (\sigma_r^2+\omega_{d_r}^2)\ \Psi_{E'_{q_r}}^{H}\ \mathbf{P}\ \Psi_{E'_{q_r}} \tag{20}$$

where, $\Psi_{E'_{q_r}} = \Big[\psi_{E'_{q_{1,r}}}\cdots\ \psi_{E'_{q_{n_g,r}}}\Big]^T$ and $\hat{c}_r = c_r\ e^{\sigma_r t_0}$.

Since λ_r is an eigenvalue of the system, we may write $\mathbf{A}\Psi_r = \lambda_r\Psi_r$, where, the right eigenvector $\Psi_r = \Big[\ \Psi_{\delta_r}\quad \Psi_{\omega_r}\quad \Psi_{E'_{q_r}}\ \Big]^T$. Next, using the structure of $\mathbf{A}$ in (10), we can split this into the following equations

$$\Psi_{E'_{q_r}} = (\lambda_r\mathbf{I}-\mathbf{A_{33}})^{-1}\ \mathbf{A_{31}}\ \Psi_{\delta_r}\quad\text{and}\quad \Psi_{\delta_r} = \frac{1}{\lambda_r}\Psi_{\omega_r}. \tag{21}$$

Using these, along with (28) describing $\Delta\vec{\boldsymbol{\omega}}_r = 2\ \hat{c}_r\ \Psi_{\omega_r}$, we may re-write (20) as follows

$$\begin{aligned}P_{W_r} \approx 2\ |\hat{c}_r|^2\ (\sigma_r^2{+}\omega_{d_r}^2)\ &\frac{\Delta\vec{\boldsymbol{\omega}}_r^{H}}{2\ \hat{c}_r^*\ \lambda_r^*}\ \mathbf{A_{31}}^{T}\ (\lambda_r^*\mathbf{I}-\mathbf{A_{33}}^{T})^{-1} \\ &\mathbf{P}\ (\lambda_r\mathbf{I}-\mathbf{A_{33}})^{-1}\ \mathbf{A_{31}}\ \frac{\Delta\vec{\boldsymbol{\omega}}_r}{2\ \hat{c}_r\ \lambda_r}.\end{aligned} \tag{22}$$

Using proposition (i), this reduces to

$$P_{W_r} \approx \frac{1}{2}\ \Delta\vec{\boldsymbol{\omega}}_r^{H}\ \mathbf{A_{31}}^{T}\ \mathbf{P}\ (|\lambda_r|^2\mathbf{I}+\mathbf{A_{33}}^2)^{-1}\mathbf{A_{31}}\ \Delta\vec{\boldsymbol{\omega}}_r\ . \tag{23}$$

Finally, considering that the mode of interest is poorly-damped, we substitute $\lambda_r = j\omega_{d_r}$ in (23). This leads to $P_{W_r} \approx \frac{1}{2}\ \Delta\vec{\boldsymbol{\omega}}_r^{H}\ \mathbf{A_{31}}^{T}\ \mathbf{P}\ (\omega_{d_r}^2\mathbf{I}+\mathbf{A_{33}}^2)^{-1}\mathbf{A_{31}}\ \Delta\vec{\boldsymbol{\omega}}_r$, which from Lemma 1 is P_{d_r}. This concludes the proof. □

Extensions to higher-order models: Next, we present the generalized expressions of P_{W_r} and P_{d_r} for higher-order machine models relaxing some of the assumptions made in Theorem 1. To that end, we do the following − (1) introduce an exciter and a power system stabilizer (PSS) in the model for modulating ΔE_{fd_i}, (2) include a q-axis damper winding, and (3) introduce a speed governor for modulating[4] ΔT_{m_i}.

[2] for the expression of W, see, eqns (3.14) and (3.15) in [7]

[3] in eqn (3.17) of [7], for our model $D_i = 0$

[4] assumed, $\Delta T_{m_i} = D_{gov_i}\Delta\omega_i$

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Accounting for the damping contribution from these, the expression of $\dot{W}$ and by extension P_{W_r} in (19) will be modified. The modified expression for P_{W_r} is shown in (24). In (24), the first, second, and the last terms are respectively, the transient power dissipations in the field winding $P_{W_r}^{field}$, damper winding $P_{W_r}^{damp}$, and governor $P_{W_r}^{gov}$.

$$P_{W_r} \approx \sum_{i=1}^{n_g} \frac{T'_{do_i}}{2\left(x_{d_i}-x'_{d_i}\right)}\left\{\left|\Delta\vec{E}'_{q_{i,r}}\right|^2 - \frac{1}{T'_{do_i}}\Re\left(\Delta\vec{E}'_{q_{i,r}}\Delta\vec{E}^*_{fd_{i,r}}\right)\right\} + \sum_{i=1}^{n_g} \frac{T'_{qo_i}}{2\left(x_{q_i}-x'_{q_i}\right)}\left|\Delta\vec{E}'_{d_{i,r}}\right|^2 + \sum_{i=1}^{n_g} D_{\mathrm{gov}_i}\left|\Delta\vec{\omega}_{i,r}\right|^2. \tag{24}$$

For the damping power, its expression P_{d_r} in (17) would remain the same with $\Re\{\mathbf{K}_r\}$ modified as shown below.

$$\Re\{\mathbf{K}_r\} = \frac{2\mathbf{H}}{\omega_s}\left\{\mathbf{A_{23}}\left(\omega_{d_r}^2\mathbf{I}+\mathbf{A_{33}}^2\right)^{-1}\left(\mathbf{A_{31}}+\mathbf{A_{33}}\mathbf{A_{32}}\right)-\mathbf{A_{22}}\right\}. \tag{25}$$

Note that, in (25), the block matrices $\mathbf{A_{23}}$, $\mathbf{A_{31}}$, and $\mathbf{A_{33}}$ are larger in dimensions compared to that in (17) to account for the additional state variables like $\Delta \boldsymbol{E'_d}$, $\Delta \boldsymbol{E_{fd}}$, etc. which are now concatenated to the vector $\Delta \boldsymbol{E'_q}$ as the third entry. Further, due to the speed feedback to governor and the washout block in PSS, the terms $\mathbf{A_{22}}$ and $\mathbf{A_{32}}$ are non-zero.

In Section V, we will numerically verify the equality between P_{d_r} and P_{W_r}, for higher-order models, with case studies on two different IEEE test systems with damper windings, different types of excitation systems, and PSSs.

V. Case Studies

A. IEEE 2-area 4-machine Kundur Test System

Consider the 4-machine system[5] in [13] with two-axis machine model (includes field and q-axis damper winding) and DC1A excitation system [13] for each synchronous generator. The network is lossless and the total load of the system is $2,734$ MW. Under nominal conditions, the system has a poorly-damped mode at 0.67 Hz with a damping of 2.9%. For this poorly-damped mode, in Fig. 1, we show that for small perturbation[6] in the system, across operating points, the total damping power is numerically equal to the sum of the average rates of transient energy dissipations in the field and damper windings. The operating point is varied by progressively reducing the tie flow between buses 7 and 9 from 433 MW under nominal condition to -400 MW while maintaining the total load of the system constant.

Validation under large disturbances: Next, we consider the same system, but with flux-decay machine model (only field winding) and IEEE ST1A excitation system [13] on all generators. Additionally, generator 1 is equipped with a PSS. We simulate a 5-cycle three-phase self-clearing fault near bus 8. From the detrended time-domain responses, using $\Delta\omega_1$ as the reference, we next estimate [16] the relative modeshapes for all $\Delta\omega_i$, ΔT_{e_i}, $\Delta E'_{q_i}$, $\Delta E'_{d_i}$, and ΔE_{fd_i}-s for the poorly-damped mode. The $P_{d_{i,r}}$ and $P_{W_{i,r}}$ of all 4 generators as computed using these modeshapes are shown in Table I. As can be seen, for the small-signal (linearized) model,

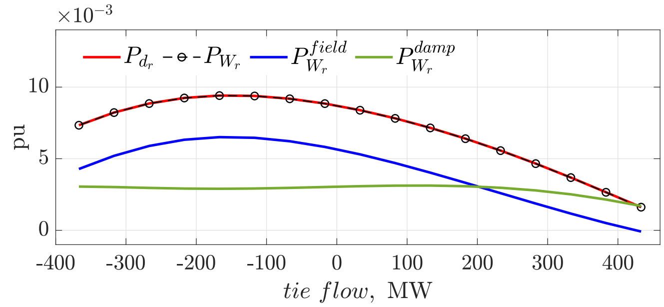

Fig. 1: Equality of P_{d_r} and $P_{W_r} (= P_{W_r}^{field} + P_{W_r}^{damp})$ for the poorly-damped mode, across different operating points in the $4-$machine system.

TABLE I: Damping and Dissipative Powers in $4-$Machine System Model for the Poorly-Damped Mode

	small-signal model		time-domain responses	
	$\frac{P_{d_{i,r}}}{2\,\lvert\hat{c}_r\rvert^2}$	$\frac{P_{W_{i,r}}}{2\,\lvert\hat{c}_r\rvert^2}$	$\frac{P_{d_{i,r}}}{2\,\lvert\hat{c}_r\rvert^2}$	$\frac{P_{W_{i,r}}}{2\,\lvert\hat{c}_r\rvert^2}$
G1	0.0382	0.2294	0.0361	0.2274
G2	0.0321	-0.1830	0.0340	-0.1877
G3	0.0435	0.0794	0.0405	0.0754
G4	0.0521	0.0401	0.0472	0.0390
Sum	0.1659	0.1659	0.1578	0.1541

the equality of P_{d_r} and P_{W_r} is verified. Further, the values estimated from post-fault time-domain responses indicate that the approximate equality of P_{d_r} and P_{W_r} holds even under large disturbances.

B. IEEE 5-area 16-machine NY-NE Test System

Next, consider the $16-$machine New York-New England test system in [17] with two-axis machine model. Generators G1 $-$ G8 have DC1A exciters, G9 is equipped with a ST1A exciter and a power system stabilizer (PSS), and the remaining generators have manual excitation. The machine and the network data can be obtained from [17]. The system has a poorly-damped mode at 0.56 Hz with 2.8% damping. In Fig 2, corresponding to this particular mode, the consistency of total damping power and sum of power dissipation in machine windings is shown for pulse disturbances[6].

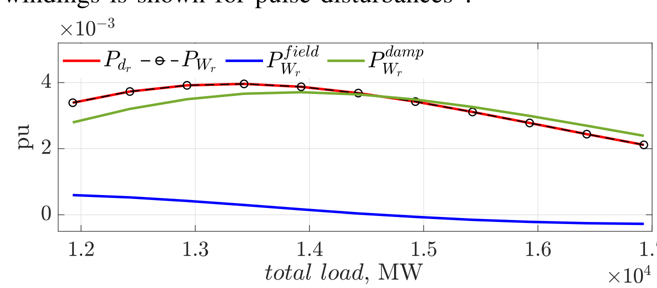

Fig. 2: Equality of P_{d_r} and $P_{W_r} (= P_{W_r}^{field} + P_{W_r}^{damp})$ for the poorly-damped mode, across different operating points in the $16-$machine system.

VI. Conclusions

A mathematical proof was presented for the approximate equality of total damping power and the average rate of transient energy dissipation in a multimachine power system, under assumptions of poor-damping like lossless transmission network, constant power loads, constant mechanical power input, and constant excitation voltage. Numerical studies showed

[5] slightly modified to include a third line between buses $7-8$ and $8-9$
[6] one-time 0.2 s unit pulse disturbance in the excitation system reference

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the equality holds true even when some of these assumptions are relaxed. Future work will focus on establishing this connection for more realistic load models.

Appendix I
Phasor Notation

For an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, assume there are m oscillatory modes in the response, each due to a complex-conjugate eigenvalue pair λ_r $(= \sigma_r + j\omega_{d_r})$ and λ_r^* of $\mathbf{A}$. The time evolution of any i^{th} state variable $x_i(t)$ can then be expressed as the sum of m modal constituents, as in (26)

$$x_i(t) = \sum_{r=1}^{m} x_{i,r}(t) = \sum_{r=1}^{m} \left\{ e^{\lambda_r t} c_r \psi_{i,r} + e^{\lambda_r^* t} c_r^* \psi_{i,r}^* \right\} \tag{26}$$

where, $c_r = \boldsymbol{\phi}_r^T \mathbf{x}(0)$, and $\boldsymbol{\phi}_r^T$ and Ψ_r are respectively the left and right eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue λ_r with $\psi_{i,r}$ as the i^{th} entry of Ψ_r. Denoting $2c_r\psi_{i,r} \triangleq \beta_{i,r} e^{j\gamma_{i,r}}$, $x_{i,r}(t)$ reduces to

$$x_{i,r}(t) = 2\,\Re\{e^{\lambda_r t} c_r \psi_{i,r}\} = \beta_{i,r}\; e^{\sigma_r t} \cos(\omega_{d_r} t + \gamma_{i,r}). \tag{27}$$

This sinusoidal variation is represented in the dynamic phasor (mentioned as 'phasor' going forward) notation using the magnitude and phase of the signal, as shown in eqn (28).

$$\vec{x}_{i,r}(t) \;\triangleq\; \beta_{i,r}\; e^{\sigma_r t} \angle\gamma_{i,r} \;=\; 2\; c_r\; \psi_{i,r}\; e^{\sigma_r t} \tag{28}$$

The phasor $\vec{x}_{i,r}(t)$ is rotating at the modal frequency ω_{d_r} with its amplitude having an exponential decay. Next, from (27), $\dot{x}_{i,r}(t) = \beta_{i,r}\; e^{\sigma_r t}\; \{\sigma_r\; \cos(\omega_{d_r} t + \gamma_{i,r}) - \;\omega_{d_r}\; \sin(\omega_{d_r} t + \gamma_{i,r})\}$. Therefore,

$$\vec{\dot{x}}_{i,r}(t) = (\sigma_r + j\; \omega_{d_r})\; \vec{x}_{i,r}(t). \tag{29}$$

Appendix II
Proof of Propositions

Observe that, from (8), the matrices can be structured as

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{M_{21}} & \mathbf{0} & \mathbf{M_{23}} \\ \mathbf{M_{31}} & \mathbf{0} & \mathbf{M_{33}} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{N_{21}} & \mathbf{N_{22}} \\ \mathbf{N_{31}} & \mathbf{N_{32}} \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} \mathbf{C_{11}} & \mathbf{0} & \mathbf{C_{13}} \\ \mathbf{C_{21}} & \mathbf{0} & \mathbf{C_{23}} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{D_{11}} & \mathbf{D_{12}} \\ \mathbf{D_{21}} & \mathbf{D_{22}} \end{bmatrix}.$$

Following the notation that $\mathbf{D_{kl}}(i,j)$ is the $(i,j)^{\text{th}}$ element of the $(k,l)^{\text{th}}$ submatrix $\mathbf{D_{kl}}$ of $\mathbf{D}$, we may write $\forall\; j \neq i$

$$\mathbf{D_{12}}(i,j) = V_{j0} \frac{\partial f_i}{\partial V_j}\bigg|_0 = -V_{i0}V_{j0}Y_{ij}\sin(\theta_{i0} - \theta_{j0}) \tag{30a}$$
$$\mathbf{D_{21}}(i,j) = \frac{\partial g_i}{\partial \theta_j}\bigg|_0 = V_{i0}V_{j0}Y_{ij}\sin(\theta_{i0} - \theta_{j0}) = \mathbf{D_{12}}(j,i) \tag{30b}$$
$$\mathbf{D_{11}}(i,j) = \frac{\partial f_i}{\partial \theta_j}\bigg|_0 = V_{i0}V_{j0}Y_{ij}\cos(\theta_{i0} - \theta_{j0}) = \mathbf{D_{11}}(j,i) \tag{30c}$$
$$\mathbf{D_{22}}(i,j) = V_{j0} \frac{\partial g_i}{\partial V_j}\bigg|_0 = V_{i0}V_{j0}Y_{ij}\cos(\theta_{i0} - \theta_{j0}) = \mathbf{D_{22}}(j,i). \tag{30d}$$

Similarly, it can be shown that,

$$\mathbf{D_{12}}(i,i) = V_{i0} \frac{\partial f_i}{\partial V_i}\bigg|_0 = \frac{\partial g_i}{\partial \theta_i}\bigg|_0 = \mathbf{D_{21}}(i,i). \tag{31}$$

Therefore, from eqns (30a), (30b), and (31), it can be inferred that $\mathbf{D_{12}}^T = \mathbf{D_{21}}$. Additionally, from eqns (30c) – (30d), $\mathbf{D_{11}}^T = \mathbf{D_{11}}$ and $\mathbf{D_{22}}^T = \mathbf{D_{22}}$. Thus, $\mathbf{D}^T = \mathbf{D}$. Further, $\mathbf{D}$ being real and symmetric implies $\mathbf{D}^{-1}$ is also real and symmetric, $\mathbf{D}^{-T} = \mathbf{D}^{-1}$.

Proof of Prop. (i) : Recall, $\mathbf{A} = \mathbf{M} - \mathbf{N}\mathbf{D}^{-1}\mathbf{C}$

$$\begin{aligned} &\implies \mathbf{A_{33}} = \mathbf{M_{33}} - \begin{bmatrix} \mathbf{N_{31}} & \mathbf{N_{32}} \end{bmatrix} \mathbf{D}^{-1} \begin{bmatrix} \mathbf{C_{13}} \\ \mathbf{C_{23}} \end{bmatrix} \\ &\implies \mathbf{A_{33}}^T = \mathbf{M_{33}}^T - \begin{bmatrix} \mathbf{C_{13}}^T\, \mathbf{C_{23}}^T \end{bmatrix} \mathbf{D}^{-T} \begin{bmatrix} \mathbf{N_{31}}^T \\ \mathbf{N_{32}}^T \end{bmatrix}. \end{aligned} \tag{32}$$

From eqns (3) and (4) observe that, $\forall\; j \neq i$,

$$\mathbf{N_{31}}(i,j) = \frac{\partial \dot{E}'_{q_i}}{\partial \theta_j}\bigg|_0 = 0\;; \qquad \mathbf{C_{13}}(i,j) = \frac{\partial f_i}{\partial E'_{q_j}}\bigg|_0 = 0\;; \tag{33}$$

and for elements on the principal diagonal,

$$\mathbf{N_{31}}(i,i) = \frac{\partial \dot{E}'_{q_i}}{\partial \theta_i}\bigg|_0 = \frac{V_{i0}\sin(\delta_{i0} - \theta_{i0})}{x'_{d_i}} \left(\frac{x_{d_i} - x'_{d_i}}{T'_{do_i}}\right) \tag{34a}$$
$$\mathbf{C_{13}}(i,i) = \frac{\partial f_i}{\partial E_{q_i}}\bigg|_0 = \frac{V_{i0}\sin(\delta_{i0} - \theta_{i0})}{x'_{d_i}} \tag{34b}$$

Further note, $\mathbf{N_{31}}$ and $\mathbf{C_{13}}$ are rectangular matrices of dimensions $\mathbb{R}^{n_g \times n}$ and $\mathbb{R}^{n \times n_g}$ respectively. Therefore, combining eqns (33) and (34) we get

$$\mathbf{P}^{-1}\mathbf{C_{13}}^T = \mathbf{N_{31}} \tag{35}$$

Similarly, from eqns (3) and (5), $\forall\; j \neq i$,

$$\mathbf{N_{32}}(i,j) = V_{j0}\frac{\partial \dot{E}'_{q_i}}{\partial V_j}\bigg|_0 = 0\;; \quad \mathbf{C_{23}}(i,j) = \frac{\partial g_i}{\partial E'_{q_j}}\bigg|_0 = 0\;; \quad \text{and}$$

$$\mathbf{N_{32}}(i,i) = V_{i0}\frac{\partial \dot{E}'_{q_i}}{\partial V_j}\bigg|_0 = \frac{V_{i0}\cos(\delta_{i0} - \theta_{i0})}{x'_{d_i}}\left(\frac{x_{d_i} - x'_{d_i}}{T'_{do_i}}\right)$$
$$\mathbf{C_{23}}(i,i) = \frac{\partial g_i}{\partial E_{q_i}}\bigg|_0 = \frac{V_{i0}\cos(\delta_{i0} - \theta_{i0})}{x'_{d_i}}$$

Therefore, following arguments as before,

$$\mathbf{P}^{-1}\mathbf{C_{23}}^T = \mathbf{N_{32}} \tag{36}$$

Finally, observe that $\mathbf{M_{33}} \in \mathbb{R}^{n_g \times n_g}$ with $\mathbf{M_{33}}(i,j) = \frac{\partial \dot{E}'_{q_i}}{\partial \delta_j}\bigg|_0 = 0 \quad \forall\; j \neq i$. This implies $\mathbf{M_{33}}$ is diagonal.

Using $\mathbf{D}^{-T} = \mathbf{D}^{-1}$, and the results (35) and (36), eqn (32) can be rewritten as

$$\mathbf{P}^{-1}\mathbf{A_{33}}^T\mathbf{P} = \mathbf{M_{33}} - \begin{bmatrix} \mathbf{N_{31}} & \mathbf{N_{32}} \end{bmatrix} \mathbf{D}^{-1} \begin{bmatrix} \mathbf{C_{13}} \\ \mathbf{C_{23}} \end{bmatrix} = \mathbf{A_{33}}.$$

This concludes the proof. □

Proof of Prop. (ii) : It can be seen from eqns (2) and (3) that blocks $\mathbf{M_{31}}$ and $\mathbf{M_{23}}$ are diagonal. Also,

$$\mathbf{M_{31}}(i,i) = \frac{\partial \dot{E}'_{q_i}}{\partial \delta_i}\bigg|_0 = -\frac{V_{i0}\sin(\delta_i - \theta_i)}{x'_{d_i}}\left(\frac{x_{d_i} - x'_{d_i}}{T'_{do_i}}\right)$$
$$\mathbf{M_{23}}(i,i) = \frac{\partial \dot{\omega}_i}{\partial E'_{q_i}}\bigg|_0 = -\frac{V_{i0}\sin(\delta_i - \theta_i)}{2H_i\; x'_{d_i}}\;\omega_s$$

Therefore, we my write

$$\mathbf{M_{31}}^T\,\mathbf{P} = \frac{2\,\mathbf{H}}{\omega_s}\,\mathbf{M_{23}} \tag{37}$$

where $\mathbf{H}$ is diagonal with $\mathbf{H}(i,i) = H_i$.
Now, as before, for $\mathbf{N}$ and $\mathbf{C}$ matrices,

$$\mathbf{N_{21}}(i,j) = \frac{\partial \dot{\omega}_i}{\partial \theta_j}\bigg|_0 = 0\ ; \qquad \mathbf{C_{11}}(i,j) = \frac{\partial f_i}{\partial \delta_j}\bigg|_0 = 0. \quad \text{Also,}$$

$$\begin{aligned}\mathbf{N_{21}}(i,i) = \frac{\partial \dot{\omega}_i}{\partial \theta_i}\bigg|_0 &= \frac{\omega_s\ E'_{q_{i0}} V_{i0}\cos(\delta_{i0}-\theta_{i0})}{2H_i\ x'_{d_i}} \\ &\quad - \frac{\omega_s V_{i0}^2 \sin 2(\delta_{i0}-\theta_{i0})}{2H_i}\Big(\frac{x_{q_i}-x'_{d_i}}{x_{q_i}x'_{d_i}}\Big) \\ \mathbf{C_{11}}(i,i) = \frac{\partial f_i}{\partial \delta_i}\bigg|_0 &= \frac{E'_{q_{i0}} V_{i0}\cos(\delta_{i0}-\theta_{i0})}{x'_{d_i}} \\ &\quad -V_{i0}^2 \sin 2(\delta_{i0}-\theta_{i0})\Big(\frac{x_{q_i}-x'_{d_i}}{x_{q_i}x'_{d_i}}\Big) = \frac{2\ H_i}{\omega_s}\ \mathbf{N_{21}}(i,i)\end{aligned}$$

Combining these with the fact that, $\mathbf{N_{21}} \in \mathbb{R}^{n_g \times n}$ and $\mathbf{C_{11}} \in \mathbb{R}^{n \times n_g}$ we may write,

$$\mathbf{C}_{\mathbf{11}}^T = \frac{2\ \mathbf{H}}{\omega_s}\ \mathbf{N_{21}}. \tag{38}$$

Similarly, it can be shown that

$$\mathbf{C}_{\mathbf{21}}^T = \frac{2\ \mathbf{H}}{\omega_s}\ \mathbf{N_{22}}. \tag{39}$$

Now, recall $\mathbf{A} = \mathbf{M} - \mathbf{N}\mathbf{D}^{-1}\mathbf{C}$. Therefore,

$$\begin{aligned}\mathbf{A_{31}} &= \mathbf{M_{31}} - \begin{bmatrix}\mathbf{N_{31}} & \mathbf{N_{32}}\end{bmatrix}\mathbf{D}^{-1}\begin{bmatrix}\mathbf{C_{11}} \\ \mathbf{C_{21}}\end{bmatrix} \\ \implies \mathbf{A}_{\mathbf{31}}^T\mathbf{P} &= \mathbf{M}_{\mathbf{33}}^T\mathbf{P} - \begin{bmatrix}\mathbf{C}_{\mathbf{11}}^T & \mathbf{C}_{\mathbf{21}}^T\end{bmatrix}\mathbf{D}^{-T}\begin{bmatrix}\mathbf{N}_{\mathbf{31}}^T \\ \mathbf{N}_{\mathbf{32}}^T\end{bmatrix}\mathbf{P}\end{aligned} \tag{40}$$

Next, using $\mathbf{D}^{-T} = \mathbf{D}^{-1}$ and results $(35) - (39)$ in (40)

$$\begin{aligned}\mathbf{A}_{\mathbf{31}}^T\mathbf{P} &= \frac{2\ \mathbf{H}}{\omega_s}\ \mathbf{M_{23}} - \frac{2\ \mathbf{H}}{\omega_s}\ \begin{bmatrix}\mathbf{N_{21}}\ \mathbf{N_{22}}\end{bmatrix}\mathbf{D}^{-1}\begin{bmatrix}\mathbf{C_{13}} \\ \mathbf{C_{23}}\end{bmatrix} \\ &= \frac{2\ \mathbf{H}}{\omega_s}\ \mathbf{A_{23}}\end{aligned}$$

This concludes the proof. □

Proof of Prop. (iii) : As before, observe from (2) that $\mathbf{M_{21}}$ is diagonal. Therefore, we may write

$$\mathbf{H}\ \mathbf{M_{21}} = \mathbf{M}_{\mathbf{21}}^T\ \mathbf{H}. \tag{41}$$

$$\begin{aligned}\text{Also,} \quad \mathbf{A_{21}} &= \mathbf{M_{21}} - \begin{bmatrix}\mathbf{N_{21}} & \mathbf{N_{22}}\end{bmatrix}\mathbf{D}^{-1}\begin{bmatrix}\mathbf{C_{11}} \\ \mathbf{C_{21}}\end{bmatrix} \\ \implies 2\ \mathbf{A}_{\mathbf{21}}^T\mathbf{H} &= \mathbf{M}_{\mathbf{21}}^T\mathbf{H} - \begin{bmatrix}\mathbf{C}_{\mathbf{11}}^T\ \mathbf{C}_{\mathbf{21}}^T\end{bmatrix}\mathbf{D}^{-T}\begin{bmatrix}2\ \mathbf{N}_{\mathbf{21}}^T\ \mathbf{H} \\ 2\ \mathbf{N}_{\mathbf{22}}^T\ \mathbf{H}\end{bmatrix}\end{aligned} \tag{42}$$

Finally, substituting (38), (39), and (41) in (42)

$$\begin{aligned}2\ \mathbf{A}_{\mathbf{21}}^T\mathbf{H} &= 2\ \mathbf{H}\ \mathbf{M_{21}} - 2\ \mathbf{H}\ \begin{bmatrix}\mathbf{N_{21}}\ \mathbf{N_{22}}\end{bmatrix}\mathbf{D}^{-1}\begin{bmatrix}\mathbf{C_{11}} \\ \mathbf{C_{21}}\end{bmatrix} \\ &= 2\ \mathbf{H}\ \mathbf{A_{21}}.\end{aligned}$$

This concludes the proof. □

Proof of Prop. (iv) : From the definition of $\mathbf{K}_r$ and (15),

$$\begin{aligned}\mathbf{K}_r^T &= -\frac{2}{\omega_s\ j\omega_{d_r}}\Big\{\mathbf{A}_{\mathbf{21}}^T\mathbf{H} + \mathbf{A}_{\mathbf{31}}^T\ (j\omega_{d_r}\mathbf{I} - \mathbf{A}_{\mathbf{33}}^T)^{-1}(\mathbf{A}_{\mathbf{23}}^T\ \mathbf{H})\Big\} \\ &= -\frac{2}{\omega_s\ j\omega_{d_r}}\Big\{\mathbf{A}_{\mathbf{21}}^T\mathbf{H} + \mathbf{A}_{\mathbf{31}}^T\ \mathbf{P}\ (j\omega_{d_r}\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}_{\mathbf{33}}^T\mathbf{P})^{-1} \\ &\qquad \mathbf{P}^{-1}(\mathbf{H}\ \mathbf{A_{23}})^T\Big\}.\end{aligned}$$

Using propositions $(i) - (iii)$ we can re-write $\mathbf{K}_r^T$ as

$$\mathbf{K}_r^T = -\frac{2\ \mathbf{H}}{\omega_s\ j\omega_{d_r}}\ \Big\{\mathbf{A_{21}} + \mathbf{A_{23}}\ (j\omega_{d_r}\mathbf{I} - \mathbf{A_{33}})^{-1}\mathbf{A_{31}}\Big\} = \mathbf{K}_r. \tag{43}$$

Next, for any $\mathbf{x} = \mathbf{x}_1 + j\mathbf{x}_2$, with $\mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^{n_g}$

$$\mathbf{x}^H\mathbf{K}_r\mathbf{x} = (\mathbf{x}_1^T - j\mathbf{x}_2^T)\Big(\Re(\mathbf{K}_r) + j\Im(\mathbf{K}_r)\Big)(\mathbf{x}_1 + j\mathbf{x}_2). \tag{44}$$

Also, since $\mathbf{K}_r = \mathbf{K}_r^T$, we may write

$$\mathbf{x}_1^T\Im(\mathbf{K}_r)\mathbf{x_2} = \mathbf{x}_2^T\Im(\mathbf{K}_r)\mathbf{x}_1 \text{ and } \mathbf{x}_1^T\Re(\mathbf{K}_r)\mathbf{x_2} = \mathbf{x}_2^T\Re(\mathbf{K}_r)\mathbf{x}_1. \tag{45}$$

The identities in (45) reduces the real part of (44) as follows

$$\begin{aligned}&\Re\{\mathbf{x}^H\mathbf{K}_r\mathbf{x}\} = \mathbf{x}_1^T\Re(\mathbf{K}_r)\mathbf{x}_1 + \mathbf{x}_2^T\Re(\mathbf{K}_r)\mathbf{x}_2 \\ &= \mathbf{x}_1^T\Re(\mathbf{K}_r)\mathbf{x}_1 + \mathbf{x}_2^T\Re(\mathbf{K}_r)\mathbf{x}_2 + j\mathbf{x}_1^T\Re(\mathbf{K}_r)\mathbf{x}_2 - j\mathbf{x}_2^T\Re(\mathbf{K}_r)\mathbf{x}_1 \\ &= \mathbf{x}^H\Re\{\mathbf{K}_r\}\ \mathbf{x}.\end{aligned} \tag{46}$$

This concludes the proof. □

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