ON THE EXPLOSION PROBLEM IN A BALL

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Abstract. In $\Omega \subset \mathbb{R}^n$ we consider the explosion problem in an incompressible flow introduced in [15]. If Ω is a ball, we show that the explosion threshold can only be increased by addition of an incompressible flow. Further, for any Ω we give a new proof of the $L^p - L^\infty$ estimate for elliptic advection-diffusion problems obtained in [1]. Our proof provides an optimal estimate when Ω is a ball.

Dedicated to George Papanicolaou on the occasion of his 70th birthday.

1. Introduction

The explosion problem concerns existence and regularity of positive solutions of semilinear elliptic equations of the form

$$\begin{cases} -\triangle\phi = \lambda g(\phi) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^n$. Here ϕ is the temperature, and reaction is modeled by the nonlinearity $g(s)$ which is convex, increasing, $g(0) > 0$, and

$$\int_0^\infty \frac{ds}{g(s)} < +\infty. \tag{1.2}$$

The Frank-Kameneckiĭ parameter $\lambda > 0$ measures the relative strength of reaction, compared to diffusion. Physically, absence of positive solutions of (1.1) corresponds to explosion [2]. The model (1.1) was proposed by Zeldovich and Frank-Kameneckiĭ and popularized by Gelfand [11]. When $g(s) = e^s$ and the domain is a ball $\Omega = B_R$, the explosion problem was solved in dimensions $N = 2, 3$ by Frank-Kameneckiĭ and Barenblatt, respectively [10, 11, 21]. A study of (1.1) for general domains and nonlinearities started by Keener and H. Keller [16], Joseph and Lundgren [14], and Crandall and Rabinowitz [7]. They showed there exists a critical Frank-Kameneckiĭ parameter $\lambda_* > 0$ so that (1.1) admits minimal classical positive solutions for $0 < \lambda < \lambda_*$, while no positive solutions exist for $\lambda > \lambda_*$. The regularity of solutions at the critical Frank-Kameneckiĭ parameter λ_* depends on the domain and the type of nonlinearity. Brezis and Vazquez [3] studied the case when the domain is a ball, and $g(s)$ is the exponential e^s or the power $(1+s)^m$ nonlinearity. They showed the solutions at the critical λ_* are uniformly bounded in dimensions less or equal to $N = 9$ and $N = 10$, respectively, while in higher dimensions they are unbounded. For more general nonlinearities $g(s)$ and domains Ω, the first regularity results at

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critical λ_* were established by Nedev [18] in dimensions $N = 2, 3$. and by Cabré [4] in dimension $N = 4$. For further references see [8].

In [1] we began to investigate how the presence of an underlying flow and its properties affect explosion. It is well-known that mixing by an incompressible flow typically enhances diffusion. For example, the effective diffusivity of a periodic incompressible flow is always larger than diffusion in the absence of a flow [9, 19], or that the principal eigenvalue μ_v of the problem

$$\left\{ \begin{array}{ll} -\triangle\phi + v\cdot\nabla\phi = \mu_v\phi & \phi > 0 \text{ in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.3}$$

cannot be smaller than the corresponding eigenvalue μ_0 of (1.13) with $v \equiv 0$. In the explosion context it is natural to conjecture that mixing by an incompressible flow will increase the value of the critical Frank-Kameneckiĭ parameter. More specifically, consider the minimal positive solution of the non-selfadjoint semilinear elliptic problem

$$\left\{ \begin{array}{ll} -\triangle\phi + v\cdot\nabla\phi = \lambda g(\phi) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.4}$$

with a prescribed incompressible flow $v(x)$. Let $\lambda_*(v)$ be the corresponding critical Frank-Kameneckiĭ parameter, and $\lambda_*(0)$ is the same parameter when $v \equiv 0$. Is it true that $\lambda_*(v) \geqslant \lambda_*(0)$?

Berestycki, Kogan, Joulin and Sivashinsky initiated the study of (1.13) in [15] and, surprisingly, showed numerically there were incompressible flows with $\lambda_*(v) < \lambda_*(0)$ if Ω was a two-dimensional long rectangle, and $g(s) = e^s$. This means that addition of a flow (which typically increases $\lambda_*(v)$ due to mixing) can sometimes do the opposite, that is promote the creation of hotspots and inhibit their interaction with the cold boundary $\partial\Omega$. We analyzed in [13] this creation of hotspots on the following linear analog of (1.4). Let B_t be a standard Brownian motion. The expected exit time from Ω of the Itô diffusion

$$dX_t^x = -v(X_t^x)\,dt + \sqrt{2}\,dB_t, \qquad X_0^x = x,$$

is the solution $\tau^v(x)$ of

$$\left\{ \begin{array}{ll} -\triangle\tau^v + v\cdot\nabla\tau^v = 1 & \text{in } \Omega, \\ \tau^v = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.5}$$

We showed that for any domain Ω different from the ball, there was an incompressible flow v such that $\|\tau^v\|_{L^\infty} > \|\tau^0\|_{L^\infty}$. It means that addition of this flow creates a hotter spot in Ω. In a ball however, no (incompressible) stirring will increase this expected exit time beyond the one for $v \equiv 0$ [13].

An incompressible flow cannot decrease the critical Frank-Kameneckiĭ parameter $\lambda_*(v)$ too mvch. It was proved in [1], that for any domain Ω and a nonlinearity $g(s)$ there exists $\lambda_0 > 0$ so that $\lambda_*(v) \geq \lambda_0 > 0$ for all incompressible flows $v(x)$ in Ω. The constant λ_0 depends on Ω and the function g, but not on v. The crucial step of the proof in [1] is the following $L^p - L^\infty$ flow-independent estimate: for any $p > n/2$ there exists a constant $C_p(\Omega)$ such that for any incompressible v tangential to $\partial\Omega$ and any $f \in L^p(\Omega)$, the solution of

$$\left\{ \begin{array}{ll} -\triangle\phi + v\cdot\nabla\phi = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.6}$$

satisfies

$$\|\phi\|_{L^\infty} \leqslant C_p(\Omega)\,\|f\|_{L^p}\,. \tag{1.7}$$

Estimate (1.7) turns out to be useful in other contexts [6,12]. For example, in [6] we investigated a steady Stokes-Boussinesq system

$$\begin{cases} -\Delta\phi + v\cdot\nabla\phi = \lambda g(\phi) & \text{in } \Omega, \\ -\Delta v + \nabla p = \rho\,\phi\,\hat{e}_z & \text{in } \Omega, \\ \nabla\cdot v = 0 & \text{in } \Omega, \\ \phi = 0, \text{ and } v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. Here $\mathbf{\hat{e}}_z = (0,1)$, and ρ is the Rayleigh number. The function $g(s)$ is positive, convex and grows polynomially at infinity: $0 < g_0 \leq g(s) \leq C(1+s^m)$ for $s \geq 0$. We used estimate (1.7) to obtain flow-independent estimates for ϕ, that allowed us to prove that for any value of the Frank-Kameneckiĭ parameter λ there exists a critical Rayleigh number ρ_* such that for all $\rho > \rho_*$ there exist positive solutions of (1.8). It means that strong coupling may give rise to flows with very good mixing properties - they prevent explosion for arbitrarily strong reaction.

In this note we give an alternative proof of (1.7), which, in addition, allows to determine an optimal constant $C_p(\Omega)$, when Ω is a ball.

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^1 boundary and $v : \Omega \to \mathbb{R}^n$ a C^1 divergence free vector field tangential to $\partial\Omega$. Then for any $p > n/2$,*

$$\|\phi\|_{L^\infty} \leqslant C_{p,n} R^{2-n/p}\,\|f\|_{L^p}\,, \tag{1.9}$$

where R is the radius of a ball $D \subset \mathbb{R}^n$ with the same Lebesgue measure as Ω. The constants $C_{p,n}$ are given explicitly as follows.

$$C_{p,1} = \frac{1}{2^{1/p}}\left(\frac{p-1}{2p-1}\right)^{1-1/p}, \quad C_{p,2} = \pi^{-1/p}\left(\Gamma\left(\frac{2p-1}{p-1}\right)\right)^{1-1/p}, \tag{1.10}$$

where $\Gamma(x)$ is the usual Γ-function. If $n \geqslant 3$

$$C_{p,n} = \frac{\int_0^1 \rho^{1-n}\int_0^\rho (r^{2-n}-1)^{\frac{1}{p-1}} r^{n-1}\,dr\,d\rho}{\left\|(r^{2-n}-1)^{\frac{1}{p-1}}\right\|_{L^p}}.$$

The estimate (1.9) is optimal - it becomes an equality if Ω is a ball, $v(x)\cdot\vec{x} = 0$, and $f = f(|x|)$ is appropriately chosen, see (2.9), (2.10), (2.11), (2.12) below.

The original proof of Lemma 1.1 in [1] relied on estimates for the corresponding parabolic equation. In the present note we use the rearrangement argument from our proof of Theorem 1.2 in [13]. The same rearrangement argument allows to prove the following Theorem.

Theorem 1.2. *Let $\Omega = B_R \subset \mathbb{R}^n$ be a ball of radius R. Given λ, suppose there is a classical positive solution ξ of*

$$-\triangle\xi = \lambda g(\xi),\ \ \xi|_{\partial B_R} = 0. \tag{1.11}$$

Then for any $v \in C^1$ there is a classical solution of

$$\begin{cases} -\triangle\phi + v\cdot\nabla\phi = \lambda g(\phi) & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R. \end{cases} \tag{1.12}$$

Theorem 1.2 implies that adding an incompressible flow could only prevent explosion in a ball: for any v the critical Frank-Kameneckiĭ parameter $\lambda_*(v) \geqslant \lambda_*(0)$, where $\lambda_*(0)$ is the critical Frank-Kameneckiĭ parameter for the explosion problem without a flow. Thus incompressible flows in a ball only improve mixing. This Theorem is essentially a nonlinear analog of our exit-time result [13], that solutions of (1.5) on a ball $\Omega = B_R$ always satisfy $\|\tau^v\|_{L^\infty} \leqslant \|\tau^0\|_{L^\infty}$. This is not surprising; the exit time problem (1.5) and the explosion problem (1.4) are related mathematically in several ways. For example, small solutions of (1.4) are approximately proportional to solutions of (1.5). Rescaled solutions of (1.5) may be used as sub- and super-solutions of (1.4). By further analogy our results in [13] suggest that Theorem 1.2 should not hold for any other domain. Further, we anticipate one can show the contrary: if the domain is different from the ball there is a flow v such that $\lambda_*(v) < \lambda_*(0)$. Similarity of the exit time and the explosion problems also suggests that the respective most poorly mixing flows v might resemble each other. Reasons for poor mixing of certain flows in [13] still remain poorly understood.

We finally note that the value of the critical Frank-Kameneckiĭ parameter is not the only way to characterize good or bad mixing. For example, estimates of the principal eigenvalue (1.13) show that incompressible flows always improve mixing in the sense of rate of decay of solutions

$$
\left\{ \begin{array}{ll} \partial_t \phi + \mathrm{Pe}\ v \cdot \nabla \phi = \nu \triangle \phi & \text{in } \Omega, \\ \phi|_{t=0} = \phi_0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.13}
$$

as $t \to \infty$. Yet another method is called relaxation enhancement [5,22]. It concerns measuring mixing properties of v by asymptotics of $\|\phi\|_{L^\infty(\Omega)}$ for solutions of (1.13) at $t = 1$, and as the Pèclet number $\mathrm{Pe} \to \infty$. One can also look at other Sobolev norms of solutions of (1.13) with $\nu = 0$. This approach is investigated in [17,20] and references therein. All these approaches provide a priori different characterizations of good or bad mixing. The relations between them were not studied systematically yet.

2. Proof of the Lemma

Let $\Omega_h = \{x \in \Omega \,|\, \phi(x) > h\}$, be the h-super-level set of ϕ. Density argument implies that we can assume v and f in (1.6) to be smooth. Then $\phi \in C^\infty(\Omega)$, and by Sard's theorem the set $\mathcal{S}$ of regular values of ϕ has full measure. Thus $\partial\Omega_h$ is a finite union of sufficiently smooth compact manifolds without boundary for each $h \in \mathcal{S}$. Maximum principle implies that we can further assume $f \geqslant 0$.

Let us Ω^* and ϕ^* be the Schwarz symmetric rearrangements of Ω and ϕ. That is, Ω^* is the ball with volume $|\Omega|$ centered at the origin and $\phi^* : \Omega^* \to R_+$ is the non-increasing radial function such that the ball $\Omega^*_h = \{x \in \Omega \,|\, \phi^*(x) > h\}$ satisfies $|\Omega^*_h| = |\Omega_h|$ for each $h \in \mathbb{R}$. We assume the ball Ω^* has radius R.

Let us now consider any $h \in \mathcal{S}$. The isoperimetric inequality gives

$$
|\partial\Omega^*_h| \leqslant |\partial\Omega_h|\,, \tag{2.1}
$$

with equality when Ω_h is a ball.

Since v is divergence-free and ϕ is constant on $\partial\Omega_h$, we have

$$
\int_{\partial\Omega_h} |\nabla\phi|\, d\sigma = -\int_{\partial\Omega_h} \frac{\partial\phi}{\partial\nu} d\sigma = \int_{\Omega_h} (-\triangle\phi + v\cdot\nabla\phi) dx = \int_{\Omega_h} f(x) dx. \tag{2.2}
$$

The co-area formula gives

$$-\int_{\partial\Omega_h}\frac{1}{|\nabla\phi|}\,d\sigma=\frac{\partial}{\partial h}\,|\Omega_h|=\frac{\partial}{\partial h}\,|\Omega_h^*|=-\int_{\partial\Omega_h^*}\frac{1}{|\nabla\phi^*|}\,d\sigma. \tag{2.3}$$

Since $|\nabla\phi^*|$ is constant on each level-set $\partial\Omega_h^*$ the isoperimetric inequality (2.1) and the Cauchy-Schwarz inequality imply

$$\int_{\partial\Omega_h^*}|\nabla\phi^*|\;d\sigma\int_{\partial\Omega_h^*}\frac{1}{|\nabla\phi^*|}\,d\sigma=|\partial\Omega_h^*|^2\leqslant|\partial\Omega_h|^2\leqslant\int_{\partial\Omega_h}|\nabla\phi|\;d\sigma\int_{\partial\Omega_h}\frac{1}{|\nabla\phi|}\,d\sigma.$$

Using (2.2) and (2.3) in the last inequality we obtain

$$\int_{\partial\Omega_h^*}|\nabla\phi^*|\;d\sigma\leqslant\int_{\partial\Omega_h}|\nabla\phi|\;d\sigma=\int_{\Omega_h}f(x)dx,$$

with equality when Ω_h is a ball and $|\nabla\phi|=\partial\phi/\partial\nu$ is constant on $\partial\Omega_h$.

Suppose ρ is the radius of Ω_h^*, $h=h(\rho)$. Denoting the surface area of the $(n-1)$-dimensional unit sphere by A_n, we have

$$\int_{\partial\Omega_{h(\rho)}^*}|\nabla\phi^*|\;d\sigma=A_n\rho^{n-1}\,|\nabla\phi^*(\rho)|\,.$$

Thus

$$|\nabla\phi^*(\rho)|\leqslant\frac{1}{A_n}\rho^{1-n}\int_{\Omega_{h(\rho)}}f(x)dx, \tag{2.4}$$

for any $h(\rho)\in\mathcal{S}$ with equality when Ω is a ball, and $v(x)\cdot\vec{x}=0$.

Suppose $f^*(|x|)$ is the symmetric rearrangement of f. Then for any ρ

$$\int_{\Omega_{h(\rho)}}f(x)dx\leqslant\int_{B_\rho}f^*(|x|)dx=A_n\int_0^\rho f^*(r)r^{n-1}dr.$$

Denote

$$G_n^R(f^*)=\int_0^R\rho^{1-n}\int_0^\rho f^*(r)r^{n-1}drd\rho. \tag{2.5}$$

Since $\mathcal{S}$ has full measure and ϕ^* is continuous, we have

$$\|\phi\|_{L^\infty}=\|\phi^*\|_{L^\infty}\leqslant\int_0^R\rho^{1-n}\int_{\Omega_{h(\rho)}}f(x)dxd\rho\leqslant G_n^R(f^*), \tag{2.6}$$

with the equality sign when Ω is a unit ball, $v(x)\cdot\vec{x}=0$, and $f=f^*$.

Since $\|f\|_{L^p}=\|f^*\|_{L^p}$, it remains to find an optimal f^* (or a sequence $\{f_k\}_{k=1}^\infty$) that maximizes the right-hand side of (2.6) for each p. If $p>n/2$, the right-hand side is bounded by Hölder's inequality:

$$\begin{aligned}G_n^R(f^*)&\leqslant\frac{1}{A_n}\int_0^R\rho^{1-n}\,\|f^*\|_{L^p(B_\rho)}\,|B_\rho|^{1-1/p}\,d\rho\\&\leqslant\frac{\|f\|_{L^p}}{(A_n)^{1/p}}\int_0^R\rho^{1-n/p}d\rho\leqslant\frac{R^{2-n/p}\,\|f\|_{L^p}}{(2-n/p)(A_n)^{1/p}}.\end{aligned}$$

Therefore

$$\|\phi\|_{L^\infty}\leqslant C_{p,n}R^{2-n/p}\,\|f\|_{L^p}\,,\;C_{p,n}\leqslant\frac{1}{(2-n/p)(A_n)^{1/p}}. \tag{2.7}$$

Rescaling, we assume $R = 1$, $\|f\|_{L^p} = 1$, suppress the superscript of G_n^1, and concentrate on obtaining the optimal constant

$$C_{p,n} = \sup_{\|g\|_{L^p}=1} G_n(g). \tag{2.8}$$

For $n = p = 1$ we can find a sequence that shows the estimate on $C_{1,1}$ in (2.7) is already optimal. Indeed, on $B_1 = (-1, 1)$ let

$$f_k(x) = \begin{cases} k/2, |x| \leqslant 1/k, \\ 0, \text{ otherwise}. \end{cases} \tag{2.9}$$

If $\phi_k'' = f_k$, $\phi(\pm 1) = 0$, then

$$\|\phi_k\|_{L^\infty} = G_1(f_k) = \int_0^1 \int_0^\rho f_k(r) dr d\rho = \frac{1}{2} - \frac{1}{4k} \to \frac{1}{2}, \quad \text{as } k \to \infty.$$

For other $C_{p,n}$ note that $p > 1$. Therefore, there exists a function $f(|x|)$, such that

$$G_n(f) = C_{p,n}, \ \|f\|_{L^p} = 1.$$

Indeed, suppose

$$\lim_{k\to\infty} G_n(f_k) = C_{p,n}$$

for a sequence $\{f_k\}_{k=1}^\infty$ with $\|f_k\|_{L^p} = 1$. Then there exists a subsequence, still denoted by $\{f_k\}_{k=1}^\infty$ that converges to f in L^p weakly and in L^1 strongly. Thus

$$\lim_{k\to\infty} G_n(f_k) = G_n(f),$$

and $\|f\|_{L^p} \leqslant 1$. If $\|f\|_{L^p} < 1$, then $g = f/\|f\|_{L^p}$ satisfies $G_n(g) > C_{p,n}$, that contradicts (2.8). Therefore $\|f\|_{L^p} = 1$, and f_n that converges to f in L^p strongly.

A necessary condition for f to be an extremal point of the functional G_n is

$$G_n(g) = 0, \text{ for any } g(|x|) \in L^p \text{ such that } \int_0^1 g(r) f^{p-1}(r) r^{n-1} dr = 0.$$

Approximating f by smooth functions the last condition implies that

$$G_n(g) = 0, \text{ for any } g(r) = \delta(r-a)\frac{1}{f^{p-1}(a)a^{n-1}} - \delta(r-b)\frac{1}{f^{p-1}(b)b^{n-1}},$$

where $\delta(r)$ is the Dirac δ-function, and $0 \leqslant a, b \leqslant 1$. Therefore

$$f^{p-1}(r) = C \int_r^1 \rho^{1-n} d\rho.$$

For $n = 1$

$$f = \left(\frac{2(p-1)}{2p-1}\right)^{-1/p} (1-r)^{\frac{1}{p-1}}, \tag{2.10}$$

for $n = 2$

$$f = \left(2^{-\frac{p}{p-1}} \pi \Gamma\left(\frac{2p-1}{p-1}\right)\right)^{-1/p} (-\ln r)^{\frac{1}{p-1}}. \tag{2.11}$$

Finally, for $n \geqslant 3$

$$f = A_{p,n}(r^{2-n} - 1)^{\frac{1}{p-1}}, \ A_{p,n} = \left\|(r^{2-n} - 1)^{\frac{1}{p-1}}\right\|_{L^p}^{-1}. \tag{2.12}$$

□

3. Proof of the Theorem

Let $g_n(s) = \max(g(s), n)$. For each fixed n there is a bounded solution of

$$\begin{cases} -\triangle\phi_n + v\cdot\nabla\phi_n = \lambda g_n(\phi_n) & \text{in } \Omega, \\ \phi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Suppose ϕ_n^* is the symmetric rearrangement of ϕ_n. As in the proof of Lemma 1.1, we obtain

$$|\nabla\phi_n^*(\rho)| \leqslant \frac{\lambda}{A_n}\rho^{1-n}\int_{B_\rho} g_n(\phi_n^*)dx, \tag{3.2}$$

for each $\rho \in [0, R] \cap \mathcal{S}$, where $\mathcal{S}$ is the set of regular values of ϕ_n. Inequality (3.2) implies that ϕ_n^* is a subsolution of (1.11) for any n. Thus

$$\|\phi_n\|_{L^\infty} = \|\phi_n^*\|_{L^\infty} \leqslant \|\xi\|_{L^\infty}.$$

Therefore, for any $n \geqslant \|\xi\|_{L^\infty}$, we have $g_n(\phi_n) \equiv g(\phi_n)$, and $\phi_n \equiv \phi$, where ϕ is a classical solution of (1.12), that satisfies

$$\|\phi\|_{L^\infty} \leqslant \|\xi\|_{L^\infty}.$$

□

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