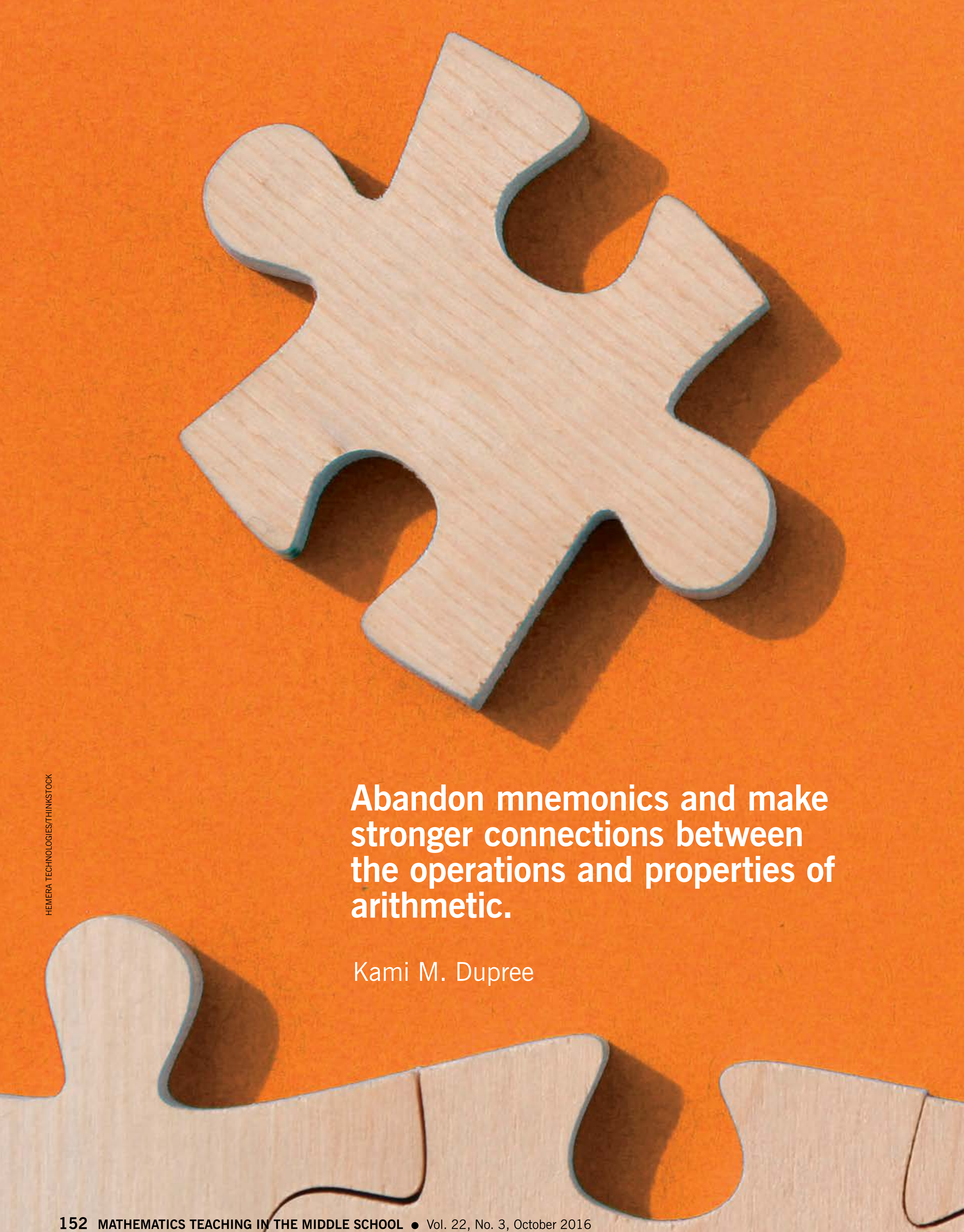

HEMERA TECHNOLOGIES/THINKSTOCK

Abandon mnemonics and make stronger connections between the operations and properties of arithmetic.

Kami M. Dupree

Questioning the Order of Operations

For decades, students have been encouraged to Please Excuse My Dear Aunt Sally as a means of learning the order of operations. Teachers unfamiliar with the Aunt Sally mnemonic are perhaps more familiar with a mnemonic such as PEMDAS. Each mnemonic is intended to convey "parentheses, exponents, multiplication, division, addition, and subtraction." Although many students accurately recall these intended references, some authors have suggested that mnemonics fail to guard against incorrect calculations (e.g., Ameis 2011; Jeon 2012; Wu 2004). Indeed, even some prospective teachers fail to correctly apply the order of operations (Glidden 2008; Jeon 2012) when relying on mnemonics.

The importance of correctly performing arithmetic operations cannot be overlooked. The Common Core State Standards for Mathematics (CCSSM) recommends introducing the commutative, associative, and distributive properties as early as third grade (CCSSI 2010, 3.OA.5), advocates the introduction of parentheses and brackets in grade 5 (5.OA.1), and designates direct emphasis on exponents and teaching the order of operations as a key part of the sixth-grade standards (6.EE.2c).

The primary responsibility for instructing students on the order of operations falls in the middle school grades. Teachers can support students in learning the order of operations by building on their early exposure to and understanding of number properties and relationships. However, many teachers learned the order of operations from mnemonics themselves and thus fail to see how something as simple as the commutative property can replace the "left to right" requirement present in most mnemonics. Teachers who understand the fundamental properties and relationships among numbers can abandon mnemonics and truly teach the order of operations.

Several authors have proposed alternatives to mnemonics as a way to teach the order of operations (Ameis 2011; Rambhia 2002; Schrock and Morrow 1993; Zorin and Carver 2015). Even teachers lament the inaccuracy with which students follow provided mnemonics. This article advocates abandoning familiar mnemonics and explains why. It also shares experiences from the preparation and delivery of a one-hour workshop, wherein mathematics teachers, coaches, and curriculum specialists explored the myths perpetuated by traditional mnemonics and discussed possible changes to their instructional approaches regarding the order of operations.

WORKSHOP PARTICIPANTS, DESIGN, AND OUTCOMES

To explore an alternative approach to teaching the order of operations, I conducted a small one-hour workshop aimed at revealing several myths perpetuated by traditional mnemonics. The workshop was conducted with two different teacher groups at two separate times. Elementary school personnel included two curriculum supervisors, five math coaches, two curriculum writers, and two teachers. The group also included four preservice secondary school math teachers as well as three middle school and two high school math teachers—all from various regions of Utah. The first workshop consisted primarily of elementary school professionals; the second consisted of secondary school teachers.

Each workshop was conducted in three phases. In phase 1, teachers were asked to evaluate five expressions without using a calculator. Each of the five expressions was designed to reveal common misconceptions perpetuated by the order of operations mnemonics. Phase 2 of the workshop consisted of a group discussion regarding the misconceptions that each expression revealed and culminated in teachers' realization that their own understanding of basic properties of arithmetic often leads them to violate the word order that well-intentioned mnemonics train students to follow. Phase 3's discussions involved teacher takeaways and implications for more effective teaching of the order of operations.

MISCONCEPTIONS PERPETUATED BY MNEMONICS

The expressions presented in phase 1 of the workshop were designed to reveal the following:

- Misconception 1: Multiplication comes before division.
- Misconception 2: Addition comes before subtraction.
- Misconception 3: Operations must be performed from left to right.
- Misconception 4: Parentheses come first.

Figure 1 shows the expressions used and the misconceptions that each was designed to reveal. The first four expressions were adapted from Jeon's (2012) work with preservice teachers; the final expression was adapted from Ameis's (2011) discussion proposing a hierarchical view of the order of operations.

After teachers had independently evaluated all five expressions, they discussed each expression while working within a group. As needed, clarifications of incorrect procedures that teachers had used took place, and agreement was reached on the correct result for each expression. To

Fig. 1 The intent of presenting these expressions to workshop participants was to reveal misconceptions.

Expression	Misconception
$3 - 6 \bullet 4 + 5$	Misconception 2
$3 - 2 \bullet 4 + 5 \bullet 2 - 3 + 8$	Misconception 2
$7 + 20 \div 2 \bullet 5 + 4$	Misconception 1
$5 - 14 \div 2 \bullet 7 + 3$	Misconceptions 1 and 2
$3 + 2 \bullet 7 + 5 + 3 \bullet (4 + 5 \bullet 2 + 1) + 2 + 10 \bullet 3 - 2$	Misconceptions 2, 3, and 4

initiate the discussion about the final expression, I performed calculations (see **fig. 2**) that purposely violated the widely held beliefs that parentheses come first and that operations must occur from left to right. Because many teachers had also performed some of the operations "out of order," they realized that as content experts, they often unconsciously violated the order they supplied to students when using a mnemonic.

This expert blind spot also manifested itself in teachers' work on the second expression: Many teachers confessed to seeing the expression as

$$3 - 8 + 10 - 3 + 8$$

and then combining the opposites of 3 and –3 and 8 and –8 out of order to arrive at 10. Such shortcuts are possible because as teachers we think beyond the mnemonic to what we know about properties of numbers and operations. This realization not only helped teachers see how mnemonics are misleading for inexperienced students but also opened the door for a deeper look into how to teach the order of operations more effectively.

Fig. 2 This final expression was completed during discussion, demonstrating multiple violations of the traditional order of operations.

$$3 + 2 \cdot 7 + 5 + 3 \cdot (4 + 5 \cdot 2 + 1) + 2 + 10 \cdot 3 - 2$$

$$8 + 2 \cdot 7 + 3 \cdot (4 + 5 \cdot 2 + 1) + 10 \cdot 3$$

$$8 + 14 + 3 \cdot (4 + 5 \cdot 2 + 1) + 30$$

$$8 + 14 + 3 \cdot (5 + 5 \cdot 2) + 30$$

$$52 + 3 \cdot (5 + 10)$$

$$52 + 3 \cdot 15$$

$$52 + 45$$

$$97$$

A NEW APPROACH

Once the misconceptions were revealed, I proposed the following considerations to reshape how we typically think about the order of operations:

- Subtraction is equivalent to adding the opposite (additive inverse). For example,

  $$3 - 6 \bullet 4 + 5, \text{ or } 3 - 24 + 5,$$

  is equivalent to $3 + (-24) + 5$. Thus, every subtraction problem can be viewed as an addition problem, dispelling the misconception that addition comes before subtraction. It also provides a case for teaching students to use the additive inverse to rewrite subtraction as addition.
- Division is equivalent to multiplying by the reciprocal (or multiplicative inverse). For example,

  $$7 + 20 \div 2 \bullet 5 + 4$$

  is equivalent to $7 + 20 \bullet 1/2 \bullet 5 + 4$ and dispels the misconception that multiplication comes before division. Teaching students to rewrite division as multiplication supports the use of fraction arithmetic.
- Taken together, the previous two ideas dispel the misconception that operations must be performed from left to right. When we think of subtraction as addition and division as multiplication, all basic computations become combinations of addition and multiplication, leading to a realization that the only real hierarchy is that multiplication must occur before addition (Wu 2004).
- Parentheses are not an operation; they are a conventional representation of groups in which calculations must be performed separately but not necessarily first. The fact that parentheses are sometimes used to imply multiplication became a topic of some consideration for the teacher groups.

IMPLICATIONS FOR INSTRUCTION

Teachers in both workshops widely agreed that traditional mnemonics can lead student to acquire misconceptions 1–4. Teachers also supported the proposals to rewrite subtraction as addition and division as multiplication. Many teachers acknowledged that making such changes in instruction would have a positive impact on students learning the order of operations. Among noted advantages were reinforcing work with integers and fractions and moving beyond reliance on rules to a deeper understanding of operations with numbers. Most teachers were also surprised to see that the left to right requirement could be abandoned with the correct use of the associative and commutative properties.

However, teachers quickly recognized challenges associated with shifting instructional practices in this way. The primary concerns voiced by teachers in these workshops were these:

1. Rewriting subtraction as addition becomes problematic for sixth graders who will not formally learn operations with negative integers until seventh grade.
2. Since parentheses are not

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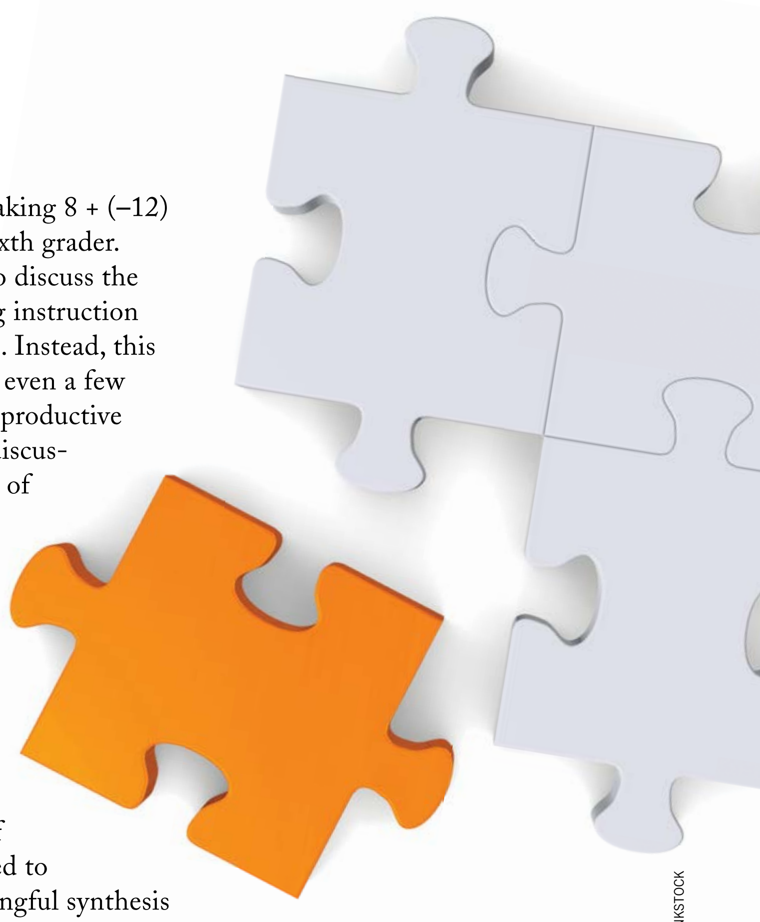
Parentheses are not an operation; they are a conventional representation of groups in which calculations must be performed separately but not necessarily first.

themselves an operation but can be used to imply multiplication, how should the use of parentheses be approached in the early grades? More specifically, should students in the early grades be exposed to parentheses as implied multiplication, or should this instruction be delayed until the later grades?

In addressing the first of these concerns, full responsibility for teaching the order of operations falls on sixth-grade teachers, as outlined in the CCSSM, yet operations with integers is not introduced until seventh grade. Although approaching subtraction as addition of an opposite is a logical and perhaps even better instructional approach to teaching the order of operations, teachers questioned whether students in sixth grade are prepared to work with such calculations as 3 + (–24). Arriving at a firm solution to this challenge was beyond the scope of the workshop, but some valuable ideas surfaced during discussion.

For example, one middle school teacher provided the following problem: 8 – 12. If 8 – 12 is rewritten as 8 + (–12), then students in sixth grade, who are unfamiliar with calculations involving negative integers, may be challenged beyond their current knowledge. However, students are taught very early in their mathematical experience to decompose numbers like 14 to 10 + 4, and students in sixth grade are exposed to the concept of a negative integer in relation to its distance from zero. Thus, teachers wondered if drawing on students' early experience decomposing numbers might offer a solution that would allow a teacher to lead sixth-grade students from 8 + (–12) to 8 + (–8 + –4) to 8 + (–8) + (–4). If so, then it may not be unreasonable to think that sixth graders could understand that 8 + (–8) is equivalent to zero, making 8 + (–12) fully accessible to a sixth grader.

My intent is not to discuss the merits of approaching instruction in this way in grade 6. Instead, this article illustrates that even a few teachers can produce productive conversations. Such discussions led to questions of whether sole responsibility for teaching the order of operations should be placed in the hands of sixth-grade teachers. If so, how might the focus of calculations requiring the order of operations be modified to allow for more meaningful synthesis of the skills that students acquire in the early grades? These are key questions that should be addressed in future work and conversations among K–grade 12 teachers desiring to improve student understanding of the order of operations.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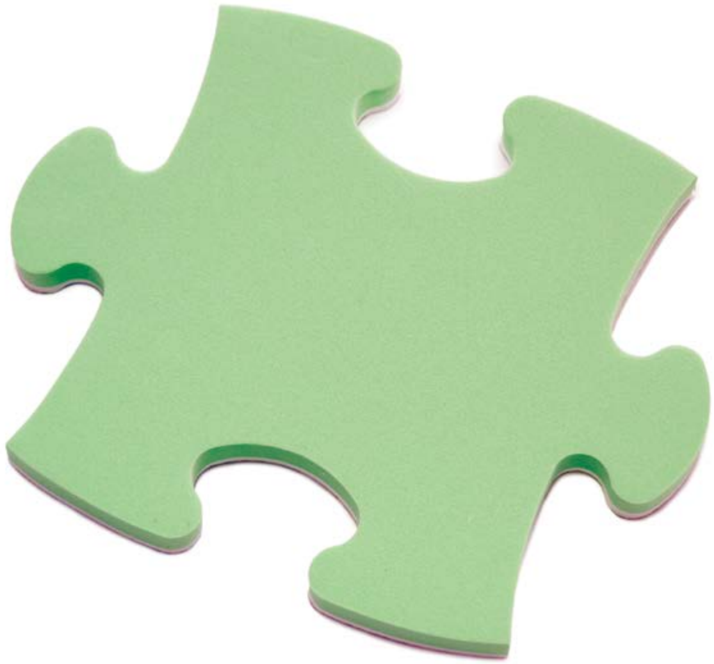
The discussion about parentheses proved to be extremely enlightening for participants. They questioned whether parentheses meant multiplication and whether students generally hold the perception that "parentheses mean multiply." These ideas highlighted problems that algebra students have when differentiating other notation with parentheses, such as $f(x)$ and $\sin(x)$, from multiplication. Many teachers felt that our conventional use of parentheses to indicate multiplication, such as in 3(8), was likely a byproduct of simplifying otherwise complex notational representations. In 3×8, the $\times$ may be mistaken for an *x;* with $3 \bullet 8$, the $\bullet$ may be mistaken for a decimal point. Teachers seemed genuinely puzzled about the merits of using the various representations when introducing students to multiplication in the early grades. Again, the intent of this article is not to resolve this puzzlement. However, conversations initiated by these teachers indicate a need to address further the role that parentheses play in mathematical operations and algebra.

SUGGESTIONS FOR INSTRUCTION

Many a mathematics teacher laments the inaccuracy with which students follow the order of operations despite teachers' best efforts. These efforts often include an appeal to mnemonics, leading to mistaken ideas that multiplication happens before division, that addition happens before subtraction, or that parentheses must be completed first. Teachers often insist that students somehow miss the instruction that multiplication and division (as well as addition and subtraction) occur together from left to right. However, the real problem is that the left-to-right requirement is not a byproduct of mathematical properties and can be eliminated if students understand subtraction as



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Should students in the early grades be exposed to parentheses as implied multiplication, or should this instruction be delayed until the later grades?

addition of an opposite and division as multiplication by a reciprocal.

Teacher conversations that developed in the course of this workshop indicated at least three advantages in abandoning the use of mnemonics in teaching the order of operations. First, abandoning the use of mnemonics allows, even requires, students to use concepts typically undervalued and underappreciated in mathematics, such as additive and multiplicative inverses, as an aid in rethinking subtraction and division problems and the use of commutative and associative properties of arithmetic. Second, when we help students view subtraction as addition of an opposite, we reinforce work with integers, perhaps reducing the amount of rule acquisition currently used when teaching operations with integers. Third, viewing division as multiplying by the reciprocal reinforces fraction arithmetic and supports students in developing deeper understanding and fluency with multiplication of fractions.

Of course, abandoning such long-held instructional tools as Please Excuse My Dear Aunt Sally requires a more strategic focus on what, when, and to what depth we teach students about operations with numbers. Opening the door to these conversations presents challenges, but the potential benefits outweigh the challenges.

To begin the conversation, see the following suggestions that surfaced from the workshop:

- Help teachers at all grade levels see the benefit of not only teaching the properties of arithmetic but also using them when making computations. For example, many students are currently taught that 4 + 17 + 6 can be easily rethought of as 4 + 6 + 17, aiding a quick combination of 4 and 6 as 10. However, at question should be (a) whether these teachers understand why they are teaching such an approach and (b) why teachers in the later grades are not using and building on those early approaches.
- Consider carefully how the use of parentheses is presented to students. Far too many students are under the impression that parentheses mean multiply, a fallacy perpetuated by mnemonics suggesting that parentheses are part of the order of operations. The means by which students come to recognize when parentheses are signifying implied multiplication and when they are not is a crucial consideration in the early grades, with lasting implications into algebra and beyond.
- More conversations should focus on the vertical alignment of aspects of CCSSM to provide a foundation for understanding the order of operations and to support its continued focus beyond grade 6. Articulating such an alignment should allow for an initial exposure to the order of operations in grade 6, and continued and deepened understanding in later grades.

It is an exciting time to be an educator; much of what CCSSM contains is pressing the boundaries of teacher understandings of mathematics and challenging educators to understand mathematics differently, and more deeply, than the way we learned it. This boundary pressing has the potential to move more students toward a comprehensive understanding of mathematics. One important boundary that we should press is abandoning the well-intentioned mnemonics that are, in truth, not as helpful as we have come to believe.

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