AN EQUILIBRIUM EXISTENCE THEOREM WITHOUT COMPLETE OR TRANSITIVE PREFERENCES*

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The Walrasian equilibrium existence theorem is reproved without the assumptions of complete or transitive preferences.

1. Introduction

The aim of this paper is described in its title – to demonstrate that for the general equilibrium Walrasian model to be well defined and consistent (i.e., for it to have a solution), the hypotheses of completeness and transitivity of consumers’ preferences are not needed.

Schmeidler (1969) proved the existence of equilibria in a model with a continuum of traders and incomplete preferences; he asked if completeness could be similarly dropped in the case of a finite number of traders; the result here answers this question in the affirmative. We refer to Aumann (1964, also 1962), for forceful arguments in favor of relaxing completeness assumptions on decision-makers’ preferences.

For the case where preferences are complete, Sonnenschein (1971) showed how it was possible to obtain continuous demand functions without making any use of transitivity.

Let $\Omega$ be the consumption set. Our hypotheses on preferences are: $>$ is an open subset of $\Omega \times \Omega$ (continuity), which is irreflexive (i.e., $x > x$ never holds) and such that, for every $x \in \Omega$, $\{y \in \Omega : y > x\}$ is non-empty and convex. We do not assume that $>$ is asymmetric or transitive and the stringent convexity hypothesis "$\{y \in \Omega : (x, y) \notin >\}$ is convex’ is not made; this last condition lacks

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intuitive appeal in a context where preferences may not be complete. In a few words, the only things of substance we are postulating are non-saturation and the convexity of ‘preferred than’ sets.

The problem at hand seems to require an existence proof of a novel type. Even assuming transitivity and monotonicity of preferences, their incompleteness may severely destroy the convex-valuedness of the demand correspondence [which is an irrelevant consideration in the continuum of traders context; this is the fact exploited by Schmeidler (1969)]; actually, we shall argue in the appendix, by an example, that an attempt to a proof through demand correspondence is completely barren. Of course, the demonstration we give is a fixed-point one, but the mapping constructed does not appear to have been used before. Perhaps the closest relative to the approach taken here, is Smale’s (1974, appendix) existence proof; the specifics are very different, but there is some analogy in the nature of the problems being solved. This will become clear in the text.

For the sake of clarity and conciseness the analysis is limited to pure exchange economies. There is no difficulty in extending the results to, for example, the private ownership economies of Debreu’s Theory of value (1959).

2. The model and statement of theorems

There are $\ell$ commodities, indexed by $h$, and $N$ consumers, indexed by $i$; $\Omega = R_+^\ell$.\(^1\)

In section 2.1, a model where consumers are described by preference relations is given and a theorem is stated. In section 2.2, an alternative model, differing only in the specification of consumers, is described and another theorem stated; it is shown then that the last implies the former.

2.1

Every consumer $i$ is specified by $(X_i, \succ_i, \omega_i)$, where $X_i \subseteq R^1$, $\succ_i \subseteq X_i \times X_i$, and $\omega_i \in R^1$. We denote $(z, v) \in \succ_i$ by $z \succ_i v$.

(C.1) For every $i$, $X_i$ (the consumption set) is a non-empty, bounded below, closed, convex set.

(C.2) For every $i$, $\succ_i$ (the preference relation) is a relatively open set such that,

\(^1\)Commodities will be denoted by superscripts while subscripts will be reserved for (consumption, production) vectors; $x \succ y$ means $x^h > y^h$ for all $h$, $x > y$ means $x^h > y^h$ for all $h$ and $x \neq y$, $x \succeq y$ means $x > y$ or $x = y$; $\text{co } D$, $\text{int } D$, $\partial D$ stand for the convex hull, the interior, and the boundary of $D \subseteq R^n$, respectively. The Euclidean norm is $\| \cdot \|$; for $x, y \in R^n$, $xy$ denotes the inner product. If $B, D \subseteq R^n$, $B + D = \{z_1 + z_2 : z_1 \in B, z_2 \in D\}$, $BD = \{zy : z \in B, y \in D\}$. When there is no ambiguity, we write $b$ instead of $\{b\}$. If $B \subseteq R^n$ and $s \in R^n$, $B \succ s$ means $b \succ s$ for every $b \in B$; analogously for $B \geq s$; $[\cdot ]$, $[\cdot ]$, ... denote segments in the usual way.
for every \( x_i \in X_i \), \( \{ z \in X_i : z \succ x_i \} \) is non-empty, convex (non-saturation and convexity) and does not contain \( x_i \) (irreflexivity).

(C.3) For every \( i, \omega_i \succ x_i \) for some \( x_i \in X_i \).

An economy \( \mathcal{E} \) is identified with \( \{(X_i, \succ_i, \omega_i)\}_{i=1}^N \). Let \( \Delta = \{ p \in \Omega : \sum_{k=1}^I p^k = 1 \} \).

Definition. \( (x, p) \in \prod_{i=1}^N X_i \times \Delta \) is an equilibrium for \( \mathcal{E} = \{(x_i, \succ_i, \omega_i)\}_{i=1}^N \) if:

(E.1) \( \sum_{i=1}^N x_i \leq \sum_{i=1}^N \omega_i \);

(E.2) for every \( i, px_i = p\omega_i \);

(E.3) for every \( i, \text{if} z \succ_i x_i, \text{then} pz \succ px_i \).

Theorem 1. If \( \mathcal{E} = \{(x_i, \succ_i, \omega_i)\}_{i=1}^N \) satisfies (C.1), (C.2), (C.3), (C.4), then there is an equilibrium for \( \mathcal{E} \).

Let \( S = \{ x \in R^1 : \|x\| = 1 \} \) and \( \mathcal{E} \) be an economy satisfying the conditions of the theorem. For every \( i \) define \( g_i : X_i \to S \) by \( g_i(x_i) = \{ p \in S : \text{if} z \succ_i x_i, \text{then} pz \succ px_i \} \); then equilibrium condition (E.3), in the presence of (C.3), amounts to requiring \( (1/\|p\|)p \in g_i(x_i) \). This heuristic comment motivates the more general model of the next section.

2.2

A set \( H \subset R^n \) is contractible if the identity map on \( H \) is homotopic to a constant map. For the present purposes it will suffice to know that convex sets and intersections of \( S \) with convex cones which are not linear subspaces, are contractible. A correspondence is said contractible-valued if its values are contractible sets. The product of two contractible sets is contractible.

For every \( i \) let \( g_i : X_i \to S \) be a correspondence and, in the definitions of section 2.1, substitute \( \succ_i \) throughout by \( g_i \).

Replace (C.2) by

(C.2') for every \( i, g_i : X_i \to S \) is an u.h.c., contractible-valued correspondence such that, for every \( x_i \in X_i \), the (possibly empty) set \( \{ p \in S : pz \succ px_i \text{ for all} z \in X_i \} \) is a subset of \( g_i(x_i) \);

and (E.3) by

(E.3') for every \( i, (1/\|p\|)p \in g_i(x_i) \).

We can state:

Theorem 2. If \( \mathcal{E} = \{(X_i, g_i, \omega_i)\}_{i=1}^N \) satisfies (C.1), (C.2'), (C.3), (C.4), then there is an equilibrium for \( \mathcal{E} \).
Theorem 2 implies Theorem 1. Let an economy $\mathcal{E} = \{(X_i, \succ_i, \omega_i)\}_{i=1}^{N}$ satisfying (C.1), (C.2), (C.3), (C.4) be given. For every $i$ and $x_1 \in X_i$ define $g_i(x_1) = \{p \in S : z \succ_i x_1, \text{ then } pz \succeq px_1\}$; since $\succ_i$ is irreflexive and $\{z \in X_i : z \succ_i x_1\}$ is a non-empty, open, convex subset of the convex set $X_i$, the set $\{p \in R^l : z \succ_i x_1, \text{ then } pz \succeq px_1\}$ is a non-empty, convex cone which cannot be a linear subspace; therefore, $g_i(x_1)$ is non-empty and contractible. Obviously, $\{p \in S : pz \succeq px_1 \text{ for all } z \in X_i\} \subseteq g_i(x_1)$.

The correspondence $g_1 : X_i \rightarrow S$ so defined is u.h.c.

Proof. The set $\{(z, x_1, p) \in X_i \times X_i \times S : z \succ_i x_1, pz \succeq px_1\}$ is open. The graph of $g_1$ is the complement of the projection of this set on the last two coordinates; hence it is closed. Q.E.D.

Therefore $\mathcal{E}' = \{(X_1, g_1, \omega_1)\}_{i=1}^{N}$ satisfies the hypothesis of Theorem 2, and so there is an equilibrium for $\mathcal{E}'$.

An equilibrium for $\mathcal{E}'$, is an equilibrium for $\mathcal{E}$.

Proof. Let $(x, y, p)$ be an equilibrium for $\mathcal{E}'$. It has to be shown that, for every $i$, if $z \succ_i x_1$, then $pz > px_1$. Let $pz \leq px_1, z \succ_i x_1$, for some $i$. Pick $\tilde{z} \leq \omega_i, \tilde{z} \in X_i$; then (since $p\tilde{z} < p\omega_i \leq px_i$) for every $z' \in [\tilde{z}, z]$ we have “$z' \in X_i$ and $pz' < px_i$” and, for $z' \in [\tilde{z}, z]$ sufficiently close to $z, z' \succ_i x_1$; hence $p \notin g_i(x_i)$. A contradiction. Q.E.D.

It is worth emphasizing that the model with consumers specified by the $g_i$'s correspondences admits of more interpretations than the one of (global) preference maximization. For example, it encompasses Smale's notion of an 'extended equilibria' for non-convex economies [in which case $g_i$ would be a normalized gradient vector field; see Smale (1974)] or the various concepts of 'local preference satisfaction' and 'preference fields' to be found in the non-integrability literature [see Georgescu-Roegen (1936); Katzner (1970, ch. 6)]. See also Debreu (1972) from where the notation $g_i$ is taken.

3. Proof of the theorems

It suffices to prove Theorem 2.

Let $\mathcal{E} = \{(X_i, g_1, \omega_i)\}_{i=1}^{N}$ satisfy (C.1), (C.2'), (C.3), and (C.4). Denote $e = (1, \ldots, 1) \in R^l$. For every $i$, define $\tilde{g}_i : \Omega \rightarrow S$ by

$$
\tilde{g}_i(x_1) = \begin{cases} 
g_i(x_1), & \text{if } x_1 \in X_i; \\
\{p \in S : pz \succeq px_1 \text{ for all } z \in X_i\}, & \text{if } x_1 \notin X_i, x_i \geq se; \\
\{(1/\|p\|)p : p \in \Delta \text{ and } px_1 = \min_{h} x_h\}, & \text{if } \min_{h} x_h < s.
\end{cases}
$$

The correspondence $\tilde{g}_i$ satisfies (C.2') with respect to $\Omega$. Moreover, if $x_1 \in \partial \Omega$, then $\tilde{g}_i(x_1) = \{p \in S : px_1 = 0\}$. 

If \((x, p) \in \Omega^N \times A\) is an equilibrium for \(\mathcal{E}' = \{(\Omega, \tilde{g}_i, \omega_i)\}_{i=1}^N\), then it is an equilibrium for \(\mathcal{E}\).

**Proof.** For every \(i\), \(px_i \geq p\omega_i > \inf pX_i\). Let \(x_i \not\in X_i\); since \(p \in \tilde{g}_i(x_i)\), it follows by the construction of \(\tilde{g}_i\) that if \(z \in X_i\), then \(pz \geq px_i\), i.e., \(px_i \leq \inf pX_i\); a contradiction. Therefore \(x_i \in X_i\), \(p \in g_i(x_i)\), for every \(i\). Q.E.D.

In view of this we assume for the rest of the proof that \(\mathcal{E}\) satisfies:

\[(C.5) \quad \text{For every } i, X_i = \Omega \text{ and, for all } x_i \in \partial \Omega, g_i(x_i)x_i \leq 0.\]

The fixed-point theorem to be used (an immediate corollary of the Eilenberg-Montgomery theorem) is contained in the:

**Lemma.** If \(K \subset \mathbb{R}^n\) is a non-empty, convex, compact set and \(F: K \rightarrow \mathbb{R}^n\) is an u.h.c. contractible-valued correspondence, then there exist \(x \in K\) and \(y \in F(x)\) such that \((y-x)(z-x) \leq 0\) for all \(z \in K\).

**Proof.** Let \(F(K) \subset H\), where \(H\) is taken convex and compact. Since \(K\) is closed and convex, we can define a continuous function \(\sigma_K: H \rightarrow K\) by letting \(\sigma_K(x) \in K\) be the \(\| \cdot \|\)-nearest element to \(x\) in \(K\). For every \(x \in H\), \(z \in K\), one has \((x - \sigma_K(x))(z - \sigma_K(x)) \leq 0\). The correspondence \(F \circ \sigma_K: H \rightarrow H\) is u.h.c. and contractible-valued. By the Eilenberg-Montgomery fixed-point theorem [Eilenberg (1946); see also, Debreu (1952)], there is \(x \in H\) such that \(x \in F(\sigma_K(x))\).

Call \(y = \sigma_K(x)\). Then \(x \in F(y)\), \(y \in K\) and \((x - y)(z - y) \leq 0\) for every \(z \in K\). Q.E.D.\(^2\)

Pick an arbitrary \(\varepsilon > 0\) and let \(r > \sum_{i=1}^N \omega_i + \varepsilon e\), \(k = r\ell(\ell N + 2)\) and \(\varphi: A \rightarrow \mathbb{R}^1\) be a continuous function such that: (i) \(p\varphi(p) \geq 0\) for all \(p \in A\); (ii) \(\varphi(A) \leq \varepsilon e\); (iii) if \(p^k = 0\) then \(\varphi(p) \leq -k\), for all \(p \in A\). It is clear that such a function exists.

By an obvious limiting argument the theorem will be proved if we show the existence of \((x, p) \in \Omega^N \times A\) such that \(\sum_{i=1}^N x_i \leq \sum_{i=1}^N \omega_i + \varepsilon e\) and \(p \in g_i(x_i)\), \(|px_i - p\omega_i| < \varepsilon\) for all \(i\).

Denote \(x_i(p) = p\omega_i + (1/N)p\varphi(p)\) and define the u.h.c., contractible-valued correspondences \(\Phi_i: [0, r]^i \times A \rightarrow \mathbb{R}^1\), \(1 \leq i \leq N\), \(\Phi_A: [0, r]^N \times A \rightarrow \mathbb{R}^1\) by

\[
\Phi_i(x_i, p) = x_i + x_i(p)g_i(x_i) - px_i \frac{p}{\|p\|},
\]

\[
\Phi_A(x, p) = p + \sum_{i=1}^N (x_i - \omega_i) - \varphi(p).
\]

\(^2\)The lemma can be proved by appealing only to the Brouwer fixed-point theorem. Suppose \(F\) is a function. Then there is \(x \in K\) such that \(x = \sigma_K(F(x))\), i.e., \((F(x) - x)(z - x) \leq 0\) for all \(z \in K\); the result follows then by a continuity argument and the fact [proved, for example, in Mas-Colell (1974b)] that, with the hypothesis made, there is for any \(\varepsilon > 0\) a continuous function \(f: K \rightarrow \mathbb{R}^n\) which graph is contained in the \(\varepsilon\)-neighborhood of the graph of \(F\).
Let then the u.h.c., contractible-valued correspondence \( \Phi : [0, r]^n \times \Delta \rightarrow R^{n+1} \) be given by

\[
\Phi(x, p) = \Phi_1(x_1, p) \times \ldots \times \Phi_n(x_n, p) \times \varphi_\Delta(x, p).
\]

Applying the lemma to \( \Phi \) we obtain \( p \in \Delta, x \in \Omega^n \), and, for every \( i, p_i^* \in g(x_i) \) such that, denoting \( \hat{x} = \sum_{i=1}^{n} x_i, \hat{\omega} = \sum_{i=1}^{n} \omega_i \),

(a) for every \( i, \) if \( z \in [0, r]^i \), then \( (z - x_i) \left( \alpha_i(p)p_i^* - px_i \|p\| \right) \leq 0; \)

(b) if \( q \in \Delta \), then \( (q - p)(\hat{x} - \hat{\omega} - \varphi(p)) \leq 0. \)

We show that \((x, p)\) is as desired; let \( y = \hat{\omega} + \varphi(p). \)

If \( p \in \partial \Delta \), then

\[
\left( \frac{1}{\lambda} e - p \right)(\hat{x} - y) \geq -p\hat{x} - \frac{1}{\lambda} ey \geq -\lambda N_r - r + \frac{r}{\lambda} = \frac{r}{\lambda} > 0,
\]

which contradicts (b). Therefore \( p \geq 0 \) and, by (b), \( \hat{x} - y = \lambda e \) for some \( \lambda \in R. \)

If \( x_i \in \partial \Omega \), then by taking \( z = 0 \) (resp. \( z = re \)) if \( x_i \neq 0 \) (resp. \( x_i = 0 \)), we contradict (a). Therefore, for every \( i, x_i > 0 \) and so, by (a), \( \alpha_i(p)p_i^* \geq (px_i/\|p\|)p. \)

Hence \( \alpha_i(p) \geq px_i \) for every \( i, \) which implies \( \lambda \leq 0. \) From this we get, for every \( i, \) \( x_i \leq \hat{x} \leq y < re, \) and so, again by (a), \( \alpha_i(p)p_i^* \leq (px_i/\|p\|)p. \) Therefore, for every \( i, \) \( \alpha_i(p)p_i^* = (px_i/\|p\|)p \) which yields \( \alpha_i(p) = px_i, \) \( p = p_i^*. \) Since, then, \( px = py, \) we also have \( \lambda = 0. \) Hence \( \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i + \varepsilon e \) and \( p \in g_i(x_i), \)

\( \omega_1 \leq px_1 \leq \omega_1 + \varepsilon \) for all \( i. \) This concludes the proof.

4. Remarks

4.1

Let \( \succ \) be a preference relation on \( \Omega \) (to make things specific) satisfying the hypotheses of Theorem 1, i.e., (C.2). For \( p \in \Delta, p \geq 0, \) \( w \in [0, \infty) \) define \( h^\succ (p, w) = \{ x \in \Omega : y \succ x, \) then \( py > px \}; \) this set is non-empty [this follows from Sonnenschein's proof in (1971); although his result is phrased in terms of a complete preorder \( \succeq, \) the proof of the non-emptiness of \( h^\succ (p, w) \) uses only the convexity of the induced \( \succ \}. \) Hence a demand correspondence \( h^\succ : \text{Int } \Delta \times [0, \infty) \rightarrow \Omega \) is well defined; it is also u.h.c. Given initial endowments \( \omega \in \Omega, \)

\( h^\succ \omega : \text{Int } \Delta \rightarrow \Omega \) stands for the excess demand correspondence generated by \( h^\succ, \) i.e., \( h_{\omega}^\succ (p) = h(p, \omega). \)
In the appendix we give an example of a quite simple $\succ$ which is transitive, monotone and satisfies (C.2), but for some $\omega \in \Omega$, no u.h.c. subcorrespondence of $h^{\succ \omega}$ is connected-valued (say, that a correspondence is connected-valued if its values are connected sets; $F : A \rightarrow B$ is a subcorrespondence of $G : A \rightarrow B$ if $F(t) \subseteq G(t)$ for all $t \in A$); also, $\succ$ possesses a continuous utility function [i.e., a function $u : \Omega \rightarrow \mathbb{R}$ such that if $x \succ y$, then $u(x) > u(y)$; this is Aumann's term (1964)], but not a quasi-concave one. Thus, the example shows that, even if transitivity is assumed, Theorem 1 cannot be obtained as a corollary of available existence results and, also, that a proof by the way of demand correspondences is not possible.

4.2

Schmeidler (1969) proved that, in the continuum of agents case, equilibria exist for economies which (in addition to other hypotheses) have consumers with transitive, incomplete, not necessarily convex, preferences. Trivial examples [every consumer has preferences on $\mathbb{R}^2$ given by \{$(x, y) : 'x_1 > y_1 - 1 \text{ and } x_2 > y_2$, or 'x_1 > y_1 \text{ and } x_2 > y_2 - 1$\}] show that, unless convexity of preferences is assumed, this result cannot be improved upon by dropping transitivity. This is, we believe, a very good reason to keep transitivity ($\succ$) among the standard assumptions of equilibrium analysis.

Appendix

We give here an example of a relation $\succ$ on $\Omega$ such that (i) $\succ$ is transitive, monotone, and satisfies (C.2); (ii) there is a continuous utility for $\succ$; (iii) there is no quasi-concave utility for $\succ$; and (iv) for a $\omega \in \Omega$, there is no u.h.c. connected-valued subcorrespondence of $h^{\succ \omega}$.

We define $\succ$ first in $\mathbb{R}^2_+$ and show then that there is no u.h.c., connected-valued subcorrespondence of the demand correspondence $h^{\succ}$. This suffices since defining a $\succ'$ in $\mathbb{R}^2_\ell$ by \{$(x, y) : 'x_1 > y_1 - 1 \text{ and } x_2 > y_2$, or 'x_1 > y_1 \text{ and } x_2 > y_2 - 1$\}' and taking $\omega = (0, 0, 1)$ what was true of the demand correspondence of $\succ$ will be true of the excess demand correspondence of $\succ'$.

Hence, let $\ell = 2$. Define (utility functions) $u', u'' : \Omega \rightarrow \mathbb{R}$ by $u'(x) = \min \{x_1, 2(x_1 + x_2)^2 - 2\}$, $u''(x) = \min \{x_2, 2(x_1 + x_2)^2 - 2\}$; see fig. 1.

Define $\succ \subset \Omega \times \Omega$ by (see fig. 2) $\succ = \{(x, y) \in \Omega \times \Omega : 'x \succ y$ or $u'(x) > u'(y)$ and $y_1 > \frac{3}{2}$, $y_2 < \frac{1}{2}$ or $u''(x) > u''(y)$ and $y_1 < \frac{3}{2}$, $y_1 < \frac{3}{2}$ or $y_2 > \frac{3}{2}$, and $x_1 + x_2 > y_1 + y_2$}. In the definition of $\succ$ only strict inequalities enter, and so the relation is open; clearly, it is also monotone and irreflexive. It is easily checked that:

- (a) if $y_1 + y_2 < \frac{3}{2}$ and $x \succ y$, then $x_1 + x_2 > y_1 + y_2$;
- (b) if $y_1 + y_2 \geq \frac{3}{2}$, $y_1 < \frac{3}{2}$ (resp. $y_2 < \frac{3}{2}$), then $x \succ y$ implies $x_2 > \frac{3}{2}$ (resp. $x_1 > \frac{3}{2}$) and $x_1 + x_2 > \frac{3}{2}$. 
The fact that $\succ$ is convex is a consequence of (a) and the monotonicity of $u', u''$. To see that $\succ$ is transitive, let $z \succ y, y \succ x$. If $x \succ \left(\frac{1}{3}, \frac{2}{3}\right)$, then obviously $z \succ x$. If $x^1 + x^2 \geq \frac{1}{3}$ and $x^1 < \frac{1}{3}$, then $y \succ x$ means $u''(x) > u''(y)$ and, by (b), $y^2 > \frac{1}{3}$; therefore $z \succ y$ implies $u''(z) > u''(y)$ and we have $z \succ x$. If $x^1 + x^2 \geq \frac{1}{3}$ and $x^2 < \frac{1}{3}$, a symmetric argument applies. If $x^1 + x^2 < \frac{1}{3}$ and $y^1 + y^2 < \frac{1}{3}$, then
$z > x$ by (a) and the definition. If $x^1 + x^2 < \frac{4}{3}$ and $y^1 + y^2 \geq \frac{4}{3}$, then, by (b), $z^1 + z^2 > \frac{4}{3}$, i.e., $z > x$.

Take $\bar{p} = (\frac{1}{2}, \frac{1}{2})$, $\bar{w} = 1$ and suppose that $\eta: \text{Int } A \times [0, \infty) \to \Omega$ is an u.h.c. connected-valued subcorrespondence of $\mathcal{H}$. For $t \in [0, 1)$ take $p(t) = ((t+1)/2, (1-t)/2)$, $w(t) = 1 - t$. For every $x \in \Omega$ if $x^1 < \frac{4}{3}$ and $x^2 < 2$, then $(0, 2) > (x^1, x^2)$. Therefore, for every $t \in [0, 1)$, if $x \in \Omega$ and $0 < x^1 < \frac{4}{3}$, then $x \notin \mathcal{H}^r(p(t), w(t))$ and for $i$ close enough to 1, $\mathcal{H}^r(p(t), w(t)) = \{0, 2\}$. Hence $\eta(p(t), w(t)) = \{0, 2\}$ for every $t \in [0, 1)$, and so $(0, 2) \in \eta(\bar{p}, \bar{w})$. By a symmetric argument $(2, 0) \in \eta(\bar{p}, \bar{w})$ and therefore $[(0, 2), (2, 0)] \subseteq \eta(\bar{p}, \bar{w})$. But this is impossible since, for example, $(\frac{1}{2}, \frac{4}{3}) \notin \mathcal{H}^r(\bar{p}, \bar{w})$. Hence no such $\eta$ exists.

The relation $>$ satisfies Peleg's (1970) spaciousness condition for the existence of a continuous utility function; one is represented in fig. 3. It is immediate that the horizontal and vertical segments of fig. 3 should appear in any indifference map of a utility for $>$, hence no quasi-concave utility exists [examples of open relations having quasi-concave but no continuous utility have been given by Schmeidler (1969) and Peleg (1970)].

References