Partitions, geometric progressions and a Putnam problem

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1. Introduction

1.1. A 1983 Putnam problem. Our starting point is the following problem in the 1983 Putnam Mathematical Competition [1]:

Problem (1983: B2). For positive integers $n$, let $C(n)$ be the number of representations of $n$ as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8, 4 + 4, 4 + 2 + 2, 4 + 2 + 1 + 1, \text{ and } 2 + 2 + 2 + 1 + 1.$$ 

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = \lfloor P(n) \rfloor$ for all positive integers $n$; here $\lfloor u \rfloor$ denotes the greatest integer less than or equal to $u$.

Before giving a solution to this Putnam problem, we would like to introduce some terminology in the theory of partitions, the study of which can be dated back to the time of Euler.

A partition of a positive integer $n$ is a nonincreasing sequence of positive integers (which are called the parts of this partition) whose sum equals $n$. Conventionally, we use $p(n)$ to count the number of partitions of $n$. We also set $p(0) = 1$. This is helpful when we deal with the generating function of $p(n)$:

$$\sum_{n \geq 0} p(n)q^n = \prod_{k \geq 1} \left(1 + q^k + q^{2k} + q^{3k} + \cdots \right) = \prod_{k \geq 1} \frac{1}{1 - q^k} \quad \text{with } |q| < 1.$$

Here in the first infinite product, the term $q^{m-k}$ can be treated in the sense that the part $k$ appears $m$ times.

We may also restrict the parts of a partition to a prescribed set, i.e. all parts must be chosen from a given set. In the 1983 Putnam problem, this prescribed set is the set of powers of 2, namely, $\{1, 2, 4, 8, \ldots \}$. Noting that each different part may appear at most three times, we have the following generating function identity

$$\sum_{n \geq 0} C(n)q^n = \prod_{k \geq 0} \left(1 + q^{2k} + q^{2 \cdot 2k} + q^{3 \cdot 2k}\right) = \prod_{k \geq 0} \frac{1 - q^{4 \cdot 2k}}{1 - q^{2k}} = \prod_{k \geq 0} \frac{1 - q^{k+2}}{1 - q^{2k}} = \frac{1}{(1 - q)(1 - q^2)}.$$
\[(1 + q + q^2 + q^3 + \cdots)(1 + q^2 + q^3 + q^4 + \cdots). \tag{1.1}\]

Now we only need to count the number of partitions of \(n\) where the parts are chosen from \(\{1, 2\}\) with no restrictions on the number of appearances of each different part. It is therefore not hard to see that \(C(n) = \lfloor \frac{n}{2} + 1 \rfloor\).

We remark that this approach is different to the official solution; see [1, p. 130].


**Theorem 1.1.** For positive integers \(n\), let \(C_m(n)\) be the number of representations of \(n\) as a sum of nonincreasing powers of \(m\), where no power can be used more than \(m^2 - 1\) times. Then

\[C_m(n) = \lfloor \frac{n}{m} \rfloor + 1. \tag{1.2}\]

Note that Rucci’s result can be shown by the same generating function approach. In fact, what Rucci proved states that (cf. [2, pp. 31–32])

\[
\sum_{n \geq 0} C_m(n)q^n = \frac{1}{(1-q)(1-q^m)} = (1 + q + q^2 + q^3 + \cdots)(1 + q^m + q^{2m} + q^{3m} + \cdots). \tag{1.3}
\]

which is parallel to (1.1). On the one hand, it is not hard to see that (1.3) can be paraphrased as

\[C_m(n) = \text{the number of partitions of } n \text{ wherein each part is either } 1 \text{ or } m. \tag{1.4}\]

This relation, unlike (1.2), has more combinatorial meaning. On the other hand, for other generalizations of the Putnam problem, we will have generating function identities similar to (1.3). However, unlike the last identity of (1.3) which contains merely two multiplicands, there might be more multiplicands. In such cases, the exact expressions of the number of representations would not be as neat as (1.2). Hence, in the rest of this note, we want to shift the reader’s attention from exact expressions such as (1.2) to combinatorial relations like (1.4).

Motivated by Rucci’s generalization, Flowers et al. [3] recently considered a new partition function \(b_{m,j}(n)\), which counts the number of partitions of \(n\), where each part is a power of \(m\) and each different part appears at most \(m^j - 1\) times. Let \(b_{m,j}(n)\) enumerate the number of partitions of \(n\) with each part being chosen from \(\{1, m, m^2, \ldots, m^{j-1}\}\). Flowers et al. proved (both analytically and combinatorially) that

**Theorem 1.2.** For positive integers \(n\),

\[b_{m,j}(n) = b_{m,j}(n). \tag{1.5}\]

We observe the following things from the result of Flowers et al.

1. The parts of partitions counted by \(b_{m,j}(n)\) are chosen from the union of \(j\) pairwise disjoint geometric progressions

\[
\{1, m^j, m^{2j}, m^{3j}, \ldots\}, \\
\{m, m^{j+1}, m^{2j+1}, m^{3j+1}, \ldots\}, \\
\ldots.
\]
2. The above \( j \) geometric progressions have the same common ratio \( m^j \). Furthermore, each different part of partitions counted by \( b_{m,j}(n) \) appears at most \( m^j - 1 \) times, which is 1 less than the common ratio \( m^j \).

3. The parts of partitions counted by \( b_{m,j}(n) \) are chosen from \( \{1, m, m^2, \ldots, m^{j-1}\} \), which is the set of initial values of the above geometric progressions.

2. A new generalization

2.1. Main result. From the result of Flowers et al., we have a new generalization.

Let 
\[
\begin{align*}
\{a_1, a_1 t, a_1 t^2, a_1 t^3, \ldots\}, \\
\{a_2, a_2 t, a_2 t^2, a_2 t^3, \ldots\}, \\
\ldots \\
\{a_H, a_H t, a_H t^2, a_H t^3, \ldots\}
\end{align*}
\]

be a family \( \mathcal{F} \) of \( H \) pairwise disjoint geometric progressions with the same common ratio \( t \in \mathbb{Z} \) with \( t \geq 2 \). Here we may take \( H = \infty \), in which case there are infinitely many geometric progressions in \( \mathcal{F} \). We also require that all the initial values \( a_1, a_2, \ldots, a_H \) are distinct positive integers.

Let \( p_\sharp(n) \) count the number of partitions of \( n \) where each part is chosen from the union of the \( H \) geometric progressions and each different part appears at most \( t - 1 \) times.

Let \( p_\flat(n) \) count the number of partitions of \( n \) where each part is chosen from \( \{a_1, a_2, \ldots, a_H\} \), the set of initial values of the geometric progressions.

**Theorem 2.1.** For positive integers \( n \),
\[
p_\sharp(n) = p_\flat(n). \tag{2.1}
\]

**Remark 2.1.** Our result reduces to the result of Flowers et al. by taking \( H = j \), \( a_h = m^{h-1} \) for \( 1 \leq h \leq j \), and \( t = m^j \).

2.2. Generating functions. We first prove our main result by showing that the two partition functions share the same generating function.

We have
\[
\sum_{n \geq 0} p_\sharp(n) q^n = \prod_{h=1}^{H} \prod_{k \geq 0} \left( 1 + q^{a_h t^k} + q^{2 a_h t^k} + \cdots + q^{(t-1) a_h t^k} \right)
\]
\[
= \prod_{h=1}^{H} \prod_{k \geq 0} \frac{1 - q^{t a_h t^k}}{1 - q^{a_h t^k}} = \prod_{h=1}^{H} \prod_{k \geq 0} \frac{1 - q^{a_h t^{k+1}}}{1 - q^{a_h t^k}}
\]
\[
= \prod_{h=1}^{H} \left( \frac{1 - q^{a_h t}}{1 - q^{a_h t^2}} \frac{1 - q^{a_h t^3}}{1 - q^{a_h t^2}} \frac{1 - q^{a_h t^4}}{1 - q^{a_h t^3}} \cdots \right)
\]
\[
= \prod_{h=1}^{H} \frac{1}{1 - q^{a_h}},
\]
which is also the generating function of \( p_\flat(n) \). Hence we arrive at the desired result.
2.3. A bijection. Let $P_2(n)$ (resp. $P_5(n)$) denote the set of partitions counted by $p_2(n)$ (resp. $p_5(n)$). We now construct a bijection between $P_2(n)$ and $P_5(n)$ to make our result more transparent.

Here the only fact we shall use is that every natural number $m$ can be uniquely represented in base $t$, i.e. we can uniquely write

$$m = m_0 t^0 + m_1 t^1 + m_2 t^2 + \cdots + m_k t^k$$

with $0 \leq m_0, m_1, \ldots, m_k \leq t-1$ and $m_k \neq 0$. For $m = 0$, we write $0 = 0 \cdot t^0$.

Given a partition in $P_2(n)$, say $n = f_1 a_1 + f_2 a_2 + \cdots + f_H a_H$ with $f_h$ counting the frequency of $a_h$ for $1 \leq h \leq H$, we represent the numbers $f_h$ in base $t$:

$$f_h = f_{h,0} t^0 + f_{h,1} t^1 + f_{h,2} t^2 + \cdots + f_{h,k_h} t^{k_h} \quad \text{ (for all $h$)}.$$

Then we have

$$n = \sum_{h=1}^H f_h a_h = \sum_{h=1}^H a_h \sum_{i=0}^{k_h} f_{h,i} t^i = \sum_{h=1}^H \sum_{i=0}^{k_h} f_{h,i} (a_h t^i),$$

yielding a partition in $P_2(n)$ since $0 \leq f_{h,i} \leq t-1$ for all $h$ and $i$, the consequence of base $t$ representations.

Conversely, for any partition in $P_2(n)$, say

$$n = \sum_{h=1}^H \sum_{i \geq 0} f_{h,i} (a_h t^i),$$

we have

$$n = \sum_{h=1}^H \left( \sum_{i \geq 0} f_{h,i} t^i \right) a_h,$$

which is a partition in $P_5(n)$.

We therefore obtain a bijection between $P_2(n)$ and $P_5(n)$.

**Example 2.1.** Suppose that the family $F$ contains three geometric progressions with initial values $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and common ratio $t = 4$, then the partitions in $P_5(8)$ and $P_2(8)$ are listed in Table 1, where the two partitions in the same row correspond to each other by the bijection mentioned above. Here $f \ast (i)$ means that the part $i$ appears $f$ times. For $1 \ast (3) + 5 \ast (1) \in P_2(8)$, we have $5 = 1 \times 4^0 + 1 \times 4^1$, and hence $5 \ast (1) = 1 \ast (1) + 1 \ast (4)$, which gives the corresponding partition in $P_5(8)$.

2.4. Closing remarks. Finally, we show how our result may recover a classical result due to Euler (cf. [4, Corollary 1.2]).

Let $a_h = 2h - 1$ for $h \geq 1$ and $t = 2$. The family $F$ becomes

$$\{1 \cdot 2^0, 1 \cdot 2^1, 1 \cdot 2^2, 1 \cdot 2^3, \ldots \},$$

$$\{3 \cdot 2^0, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, \ldots \},$$

$$\{5 \cdot 2^0, 5 \cdot 2^1, 5 \cdot 2^2, 5 \cdot 2^3, \ldots \},$$

$$\ldots .$$

We notice that these geometric progressions are pairwise disjoint and their union is the set of positive integers. Consequently, $p_2(n)$ counts the number of partitions where each different part appears at most once, whereas $p_5(n)$ counts the number of partitions into odd parts. It follows from our result that
Theorem 2.2 (Euler). The number of partitions of $n$ into distinct parts equals the number of partitions of $n$ into odd parts.

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References


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