Correction to: Note on sums involving the Euler function

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In my paper 'Note on sums involving the Euler function' [3], the estimate of the auxiliary function (with $\delta = 0$ or 1),

$$\mathfrak{S}_\delta^*(x,N) := \sum_{N<n\le 2N} \frac{\phi(n)}{n}\psi\left(\frac{x}{n+\delta}\right),$$

relies on a result due to Huxley [4], which is recorded as Theorem 6.40 in Bordellès' book *Arithmetic Tales* [1]. It was recently pointed out by Bordellès that there is a typo in his book: the assumption '$T \ge M$' is mistakenly written as '$T \ge 1$'. Hence, the corrected statement of [1, Theorem 6.40] (which is Lemma 2.1 of my paper) should read as follows.

Lemma 2.1*. *Let $r \ge 5$, $M \ge 1$ be integers and suppose $f \in \mathcal{C}^r[M,2M]$ is such that there exist real numbers $T \ge M$ and $1 \le c_0 \le \cdots \le c_r$ such that, for all $x \in [M,2M]$ and all $j \in \{0,\ldots,r\}$,*

$$\frac{T}{M^j} \le |f^{(j)}(x)| \le c_j\frac{T}{M^j}.$$

Then

$$\sum_{M<n\le 2M} \psi\big(f(n)\big) \ll (MT)^{131/416}(\log MT)^{18627/8320}.$$

This change, in consequence, affects my result significantly by creating a flaw in [3, Proposition 2.2]. In [3], I seek to apply Lemma 2.1* to [3, Equation (2.1)] which states

$$\mathfrak{S}_\delta^*(x,N) = \sum_{k\le 2N}\frac{\mu(k)}{k}\sum_{N/k<\ell\le 2N/k}\psi\left(\frac{x}{k\ell+\delta}\right).$$

It turns out that, with the correct assumption '$T \ge M$', the inner summation cannot be covered by Lemma 2.1* when $k \ll N^2/x$. Such k's exist when $N \gg \sqrt{x}$.

Since [3, Theorems 1.1 and 1.2] rely closely on Proposition 2.2, the proofs of the two theorems are therefore invalid. It is also worth mentioning that the reason why Huxley's result is suitable for the Dirichlet divisor problem $\sum_{n\le x}\tau(n)$ is that the Dirichlet hyperbola principle allows us to shorten the summation to the range $n \le \sqrt{x}$. Such an argument does not work for my problems.

My Theorems 1.1 and 1.2 were motivated by [2]. In particular, Theorem 1.2 was intended to serve as a partial answer to [2, Question 2.2]: *Is it true that*

$$\sum_{n\le x}\phi\left(\left[\frac{x}{n}\right]\right) = \frac{x\log x}{\zeta(2)} + o(x\log x) \quad \text{as } x\to\infty? \tag{1}$$

Recently, a stronger result was proved by Zhai [5]. In fact, it was shown in [5, Theorem 2] that the error term in (1) could be further refined as $O(x(\log x)^{2/3}(\log\log x)^{1/3})$.

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References

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