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Partitions and the maximal excludant

Shane Chern

Abstract. For each nonempty integer partition π, we define the maximal excludant of π as the largest nonnegative integer smaller than the largest part of π that is not itself a part. Let $\sigma\mathrm{maex}(n)$ be the sum of maximal excludants over all partitions of n. We show that the generating function of $\sigma\mathrm{maex}(n)$ is closely related to a mock theta function studied by Andrews, Dyson and Hickerson, and Cohen, respectively. Further, we show that, as $n \to \infty$, $\sigma\mathrm{maex}(n)$ is asymptotic to the sum of largest parts over all partitions of n. Finally, the expectation of the difference of the largest part and the maximal excludant over all partitions of n is shown to converge to 1 as $n \to \infty$.

Keywords. Partition, maximal excludant, mock theta function, asymptotics.

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1. Introduction

In a recent paper [3], Andrews and Newman studied the *minimal excludant* of an integer partition π, which is the smallest positive integer that is not a part of π. Since a nonempty partition π is a finite sequence of positive integers, we may also study the *maximal excludant* of π, by which we mean the largest nonnegative integer smaller than the largest part of π that is not itself a part. For example, 5 has seven partitions: 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$, the maximal excludants of which are, respectively, 4, 3, 1, 2, 0, 0 and 0.

Let $\mathrm{mex}(\pi)$ and $\mathrm{maex}(\pi)$ denote, respectively, the minimal and maximal excludant of π. Andrews and Newman further investigated the function

$$\sigma\mathrm{mex}(n) := \sum_{\pi \vdash n} \mathrm{mex}(\pi)$$

in which the summation is over all partitions of n. They proved that the generating function of $\sigma\mathrm{mex}(n)$ satisfies

$$\sum_{n\geq 0} \sigma\mathrm{mex}(n) q^n = (-q;q)_\infty^2, \tag{1.1}$$

where we adopt the conventional q-Pochhammer symbol for $n \in \mathbb{N} \cup \{\infty\}$:

$$(A;q)_n := \prod_{k=0}^{n-1} (1 - Aq^k).$$

Likewise, we may define another function

$$\sigma\mathrm{maex}(n) := \sum_{\pi \vdash n} \mathrm{maex}(\pi)$$

where, again, the summation runs over all partitions of n. In this paper, we are to study the generating function of $\sigma\mathrm{maex}(n)$. As we shall see in Theorem 1.1, unlike the generating function of $\sigma\mathrm{mex}(n)$, which is modular, the generating function of $\sigma\mathrm{maex}(n)$ is closely related to a mock theta function studied in two side-by-side papers of Andrews, Dyson and Hickerson [2] and Cohen [4].

Theorem 1.1. *We have*

$$\sum_{n\geq 1}\sigma\mathrm{maex}(n)q^n = \sum_{k\geq 1}\frac{k}{(q;q)_{k-1}}\sum_{m\geq 1}q^{m(k+1)}(-q;q)_{m-1} \tag{1.2}$$

$$= \frac{1}{(q;q)_\infty}\left(\sum_{n\geq 1}\frac{q^n}{1-q^n}-\sum_{n\geq 1}q^n(q^2;q^2)_{n-1}\right) \tag{1.3}$$

$$= \frac{1}{(q;q)_\infty}\left(\sum_{n\geq 1}\frac{q^n}{1-q^n}+\sum_{n\geq 1}\frac{(-1)^n q^{n^2}}{(q;q^2)_n}\right). \tag{1.4}$$

Remark 1.1. Using a formula due to Andrews, Dyson and Hickerson [2], we may give an explicit expression of $\sigma\mathrm{maex}(n)$. This will be discussed in Section 2.

Now recall that if $L(\pi)$ denotes the largest part of a partition π and

$$\sigma L(n) = \sum_{\pi\vdash n} L(\pi)$$

denotes the sum of largest parts over all partitions of n, a standard result tells us that

$$\sum_{n\geq 1}\sigma L(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\frac{q^n}{1-q^n}. \tag{1.5}$$

In light of (1.3) and (1.5), we have the following corollary.

Corollary 1.2. *We have*

$$\sum_{n\geq 1}\big(\sigma L(n)-\sigma\mathrm{maex}(n)\big)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}q^n(q^2;q^2)_{n-1}. \tag{1.6}$$

It was shown by Kessler and Livingston [6] that $\sigma L(n)$ satisfies the asymptotic formula

$$\sigma L(n) \sim \frac{\log(6n)-2\log\pi+2\gamma}{4\pi\sqrt{2n}}e^{\pi\sqrt{\frac{2n}{3}}} \tag{1.7}$$

where γ is the Euler–Mascheroni constant.

Now we shall show asymptotic relations as follows.

Theorem 1.3. *We have, as $n\to\infty$,*

$$\sigma L(n)-\sigma\mathrm{maex}(n) \sim \frac{1}{4\sqrt{3}n}e^{\pi\sqrt{\frac{2n}{3}}}, \tag{1.8}$$

and, a fortiori,

$$\sigma\mathrm{maex}(n) \sim \sigma L(n). \tag{1.9}$$

Further, if $\mathbb{E}_n$ denotes the expectation of the difference of the largest part and the maximal excludant over all partitions of n, then

$$\lim_{n\to\infty}\mathbb{E}_n = 1. \tag{1.10}$$

Remark 1.2. Notice that for any nonempty partition π, we always have $L(\pi) - \operatorname{maex}(\pi) \geq 1$. Hence, for all $n \geq 1$, we have $\mathbb{E}_n \geq 1$. Further, for all $n \geq 3$, it is always able to find a partition π of n with $L(\pi) - \operatorname{maex}(\pi) > 1$ (if n is odd, then such a partition could be $((n+1)/2, (n-1)/2)$; if n is even, then such a partition could be $(n/2, (n-2)/2, 1)$). This implies that $\mathbb{E}_n > 1$ for $n \geq 3$.

2. A formula of $\sigma\mathrm{maex}(n)$

2.1. A mock theta function. In his paper [4], Cohen observed the following identity

$$\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = -\sum_{n\geq 1} q^n (q^2;q^2)_{n-1}. \tag{2.1}$$

Hence, (1.3) and (1.4) are equivalent.

It is worth mentioning that Cohen's identity (2.1) can be generalized to a trivariate identity as follows.

Proposition 2.1. *We have*

$$\sum_{n\geq 1} \frac{x^n q^{n^2}}{(yq;q^2)_n} = \sum_{n\geq 1} xy^{n-1} q^n (-xq^2/y;q^2)_{n-1}. \tag{2.2}$$

Taking $x = -1$ and $y = 1$ in (2.2) recovers (2.1). Further, this identity can be treated as a companion to [1, p. 29, Example 6]:

$$\sum_{n\geq 0} \frac{x^n q^{n^2}}{(y;q^2)_{n+1}} = \sum_{n\geq 0} y^n (-xq/y;q^2)_n.$$

Proof of Proposition 2.1. Both sides of (2.2) can be treated as the generating function of partitions in which the largest part appears only once and all the remaining distinct parts appear exactly twice. Here, the exponent of x represents the number of distinct parts in this partition and the exponent of y represents the largest part minus the number of distinct parts. □

Remark 2.1. Let us denote

$$\sigma^*(q) := 2\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = -2\sum_{n\geq 1} q^n (q^2;q^2)_{n-1}. \tag{2.3}$$

It is also necessary to introduce its companion

$$\sigma(q) := \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(-q;q)_n} = 1 - \sum_{n\geq 1} (-1)^n q^n (q;q)_{n-1}. \tag{2.4}$$

The two q-hypergeometric functions are of substantial research interest along the following lines. First, Andrews, Dyson and Hickerson [2] showed that the coefficients in the expansions of $\sigma(q)$ and $\sigma^*(q)$ are very small. In fact, these coefficients are related with the arithmetic of the field $\mathbb{Q}(\sqrt{6})$. Second, let us define a sequence $\{T(n)\}_{n\in 24\mathbb{Z}+1}$ by

$$q\sigma(q^{24}) = \sum_{n\geq 0} T(n) q^n \quad \text{and} \quad q^{-1}\sigma^*(q^{24}) = \sum_{n<0} T(n) q^{-n}. \tag{2.5}$$

Cohen [4] proved that the function (in which $K_0(x)$ is the Bessel function)

$$\phi_0(\tau) := y^{1/2} \sum_{n\in 24\mathbb{Z}+1} T(n) K_0(2\pi|n|y/24) e^{2\pi i n x/24} \qquad (\tau = x+iy \in \mathbb{H})$$

is a Maass wave form on the congruence group $\Gamma_0(2)$. This, in turn, explains the modularity nature of the identity

$$q\sigma(q^{24}) = \sum_{\substack{a,b\in\mathbb{Z}\\ a>6|b|}} \left(\frac{12}{a}\right)(-1)^b q^{a^2-24b^2}.$$

Third, by noticing the following relation due to Cohen [4]:

$$\sigma(q) = -\sigma^*(q^{-1}) \tag{2.6}$$

whenever q is a root of unity (here the definitions of $\sigma(q)$ and $\sigma^*(q)$ at roots of unity are valid since the second summations in both (2.3) and (2.4) are finite), Zagier [8] is able to construct a quantum modular form $f:\mathbb{Q}\to\mathbb{C}$ by

$$f(x) := q^{1/24}\sigma(q) = -q^{1/24}\sigma^*(q^{-1})$$

where $q = e^{2\pi i x}$.

2.2. A formula of Andrews, Dyson and Hickerson. Let us define $T(n)$ ($n \in 6\mathbb{Z}+1$) by the excess of the number of inequivalent solutions of the Pell's equation

$$u^2 - 6v^2 = n$$

with $u+3v \equiv \pm 1 \pmod{12}$ over the number of them with $u+3v\equiv \pm 5 \pmod{12}$. By investigating the arithmetic in $\mathbb{Q}(\sqrt{6})$, Andrews, Dyson and Hickerson [2] showed that if n has the prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

where each $p_i \equiv 1 \pmod 6$ or p_i is the negative of a prime $\equiv 5 \pmod 6$, then

$$T(n) = T(p_1^{e_1}) T(p_2^{e_2}) \cdots T(p_r^{e_r})$$

where

$$T(p^e) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd,}\\ 1 & \text{if } p \equiv 13, 19 \pmod{24} \text{ and } e \text{ is even,}\\ (-1)^{e/2} & \text{if } p\equiv 7 \pmod{24} \text{ and } e \text{ is even,}\\ e+1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p)=2,\\ (-1)^e(e+1) & \text{if } p\equiv 1 \pmod{24} \text{ and } T(p) = -2. \end{cases}$$

Andrews, Dyson and Hickerson further proved that if we restrict $T(n)$ to $n \in 24\mathbb{Z}+1$, then they coincide with the coefficients defined in (2.5). Hence, if we write

$$\sigma^*(q) = 2\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = 2\sum_{n\geq 1} S^*(n) q^n,$$

then

$$S^*(n) = \frac{1}{2} T(1-24n).$$

Further, we have

$$\sum_{n\geq 1} d(n) q^n = \sum_{n\geq 1} \frac{q^n}{1-q^n}$$

where $d(n) = \sum_{d|n} 1$ enumerates the number of positive divisors of n.

Consequently, (1.4) gives us the following formula of $\sigma\mathrm{maex}(n)$.

Theorem 2.2. *For $n \ge 1$, we have*

$$\sigma\mathrm{maex}(n) = \sum_{k=1}^{n} p(n-k)\left(d(k) + \frac{1}{2}T(1-24k)\right) \tag{2.7}$$

where $p(n)$ denotes the number of partitions of n.

3. Proof of Theorem 1.1

The equivalence of (1.3) and (1.4) has already been shown in Section 2. It suffices to prove (1.2) and (1.3).

Given a partition with maximal excludant k, it can be split into two components: the first component is a partition with parts not exceeding $k-1$ and the second component is a gap-free partition (i.e. a partition in which the difference between any consecutive parts is at most 1) with smallest part $k+1$. Further, by considering the conjugate, there is a bijection between gap-free partitions with smallest part $k+1$ and partitions in which the largest part repeats $k+1$ times and all remaining parts are distinct. Hence, if $g(k,n)$ counts the number of partitions of n with maximal excludant k, we have the generating function identity

$$\begin{aligned} G(z,q) :&= \sum_{n\ge 1}\sum_{k\ge 1} g(k,n) z^k q^n \\ &= \sum_{k\ge 1} \frac{z^k}{(q;q)_{k-1}} \sum_{m\ge 1} q^{m(k+1)}(-q;q)_{m-1}. \end{aligned} \tag{3.1}$$

Now applying the operator $[\partial/\partial z]_{z=1}$ directly to $G(z,q)$ implies (1.2). Next, we prove (1.3). Recall Euler's first summation [1, Eq. (2.2.5)]:

$$\sum_{k\ge 0} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}.$$

In light of (3.1), we have

$$\begin{aligned} G(z,q) &= \sum_{m\ge 1} zq^{2m}(-q;q)_{m-1} \sum_{k\ge 0} \frac{(zq^m)^k}{(q;q)_k} \\ &= \sum_{m\ge 1} \frac{zq^{2m}(q^2;q^2)_{m-1}}{(q;q)_{m-1}(zq^m;q)_\infty}. \end{aligned}$$

Notice that

$$\begin{aligned} \left[\frac{\partial}{\partial z}\frac{z}{(zq^m;q)_\infty}\right]_{z=1} &= \left[\frac{z}{(zq^m;q)_\infty}\frac{\partial}{\partial z}\log\frac{z}{(zq^m;q)_\infty}\right]_{z=1} \\ &= \frac{1}{(q^m;q)_\infty}\left[\frac{\partial}{\partial z}\left(\log z - \sum_{n\ge m}\log(1-zq^n)\right)\right]_{z=1} \\ &= \frac{1}{(q^m;q)_\infty}\left(1 + \sum_{n\ge m}\frac{q^n}{1-q^n}\right). \end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n\geq 1}\sigma\mathrm{maex}(n)q^n &= \left[\frac{\partial}{\partial z}G(z,q)\right]_{z=1}\\ &= \sum_{m\geq 1}\frac{q^{2m}(q^2;q^2)_{m-1}}{(q;q)_{m-1}}\left[\frac{\partial}{\partial z}\frac{z}{(zq^m;q)_\infty}\right]_{z=1}\\ &= \frac{1}{(q;q)_\infty}\sum_{m\geq 1}q^{2m}(q^2;q^2)_{m-1}\left(1+\sum_{n\geq m}\frac{q^n}{1-q^n}\right).\end{aligned}$$

An easy combinatorial argument implies that, for all $n\geq 1$,

$$\sum_{m=1}^{n}q^m(q;q)_{m-1}=1-(q;q)_n.$$

It follows that

$$\begin{aligned}\sum_{n\geq 1}\sigma\mathrm{maex}(n)q^n &= \frac{1}{(q;q)_\infty}\left(1-(q^2;q^2)_\infty+\sum_{n\geq 1}\frac{q^n}{1-q^n}\big(1-(q^2;q^2)_n\big)\right)\\ &= \frac{1}{(q;q)_\infty}\left(1-(q^2;q^2)_\infty+\sum_{n\geq 1}\frac{q^n}{1-q^n}-\sum_{n\geq 1}q^n(1+q^n)(q^2;q^2)_{n-1}\right)\\ &= \frac{1}{(q;q)_\infty}\left(1-(q^2;q^2)_\infty+\sum_{n\geq 1}\frac{q^n}{1-q^n}-\big(1-(q^2;q^2)_\infty\big)-\sum_{n\geq 1}q^n(q^2;q^2)_{n-1}\right)\\ &= \frac{1}{(q;q)_\infty}\left(\sum_{n\geq 1}\frac{q^n}{1-q^n}-\sum_{n\geq 1}q^n(q^2;q^2)_{n-1}\right).\end{aligned}$$

This completes the proof of (1.3).

4. Proof of Theorem 1.3

We first show that the nonnegative sequence $\{\sigma L(n)-\sigma\mathrm{maex}(n)\}_{n\geq 1}$ is weakly increasing. To see this, we construct an injective map $\phi_n:\mathcal{P}_n\hookrightarrow\mathcal{P}_{n+1}$ (where $\mathcal{P}_n$ denotes the set of partitions of n) for each $n\geq 1$ by

$$\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)\mapsto(\pi_1,\pi_2,\ldots,\pi_\ell,1).$$

Then $L(\phi_n(\pi))=L(\pi)$ and

$$\mathrm{maex}(\phi_n(\pi))=\begin{cases}\mathrm{maex}(\pi) & \text{if } \mathrm{maex}(\pi)\neq 1,\\ 0 & \text{if } \mathrm{maex}(\pi)=1.\end{cases}$$

It follows that

$$\begin{aligned}\sigma L(n+1)-\sigma\mathrm{maex}(n+1) &= \sum_{\lambda\vdash n+1}\big(L(\lambda)-\mathrm{maex}(\lambda)\big)\\ &\geq \sum_{\pi\vdash n}\big(L(\phi_n(\pi))-\mathrm{maex}(\phi_n(\pi))\big)\\ &\geq \sum_{\pi\vdash n}\big(L(\pi)-\mathrm{maex}(\pi)\big)\end{aligned}$$

$$= \sigma L(n) - \sigma \mathrm{maex}(n).$$

It turns out that we may apply Ingham's Tauberian theorem to obtain the asymptotic behavior of $\sigma L(n) - \sigma \mathrm{maex}(n)$.

Theorem 4.1 (Ingham [5]). *Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a power series with weakly increasing nonnegative coefficients and radius of convergence equal to 1. If there are constants $A > 0$ and $\lambda, \alpha \in \mathbb{R}$ such that*

$$f\left(e^{-t}\right) \sim \lambda t^{\alpha} e^{\frac{A}{t}}$$

as $t \to 0^+$, then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

Recall from (1.6) and (2.3) that

$$\begin{aligned}\sum_{n\geq 1} \big(\sigma L(n) - \sigma \mathrm{maex}(n)\big) q^n &= \frac{1}{(q;q)_\infty} \sum_{n\geq 1} q^n (q^2;q^2)_{n-1} \\ &= -\frac{\sigma^*(q)}{2(q;q)_\infty}.\end{aligned}$$

Now the modular inversion formula for Dedekind's eta function (p. 121, Proposition 14 of [7]) implies that, as $t \to 0^+$,

$$\frac{1}{(e^{-t};e^{-t})_\infty} \sim \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t}}. \tag{4.1}$$

On the other hand, Zagier [8] showed that if we take $q = \xi e^{-t}$ with ξ a root of unity, then the identity (2.6) remains true as an identity in $\mathbb{Q}[\xi][[t]]$. Taking $\xi = 1$, Zagier further obtained the expansion

$$-\sigma^*(e^{-t}) = 2 + 2t + 5t^2 + \frac{55}{3}t^3 + \frac{1073}{12}t^4 + \frac{32671}{60}t^5 + \frac{286333}{72}t^6 + \cdots \tag{4.2}$$

as $t \to 0$. Hence, as $t \to 0^+$,

$$\left[\sum_{n\geq 1} \big(\sigma L(n) - \sigma \mathrm{maex}(n)\big) q^n\right]_{q=e^{-t}} \sim \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t}}.$$

Finally, (1.8) follows from Ingham's Tauberian theorem. Further, (1.9) can be deduced by comparing (1.8) with (1.7). Also, we know that the number of partitions of n satisfies

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \mathbb{E}_n = \lim_{n\to\infty} \frac{\sigma L(n) - \sigma \mathrm{maex}(n)}{p(n)} = 1.$$

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Department of Mathematics, Penn State University, University Park, PA 16802, USA
E-mail address: `shanechern@psu.edu`