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# Note on partitions with even parts below odd parts

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**Abstract**. We undertake an investigation of integer partitions in which all even parts are smaller than odd parts with additional restrictions, which were first studied by Andrews. In particular, we focus on the relation between two disjoint subsets of this partition set separated by the residue classes of the largest even part modulo 4. We also provide an overpartition analog of this relation.

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## 1. Introduction

The study of integer partitions in which all even parts are smaller than odd parts with some additional restrictions was first considered by Andrews [1]. These partitions later attracted extensive research interests, with follow-ups by Andrews [2], Bringmann and Jennings-Shaffer [3] and the author [4]. In particular, Andrews defined the following partition set in [1].

**Definition 1.1.** We denote by $\mathcal{EO}^*$ the set of partitions with no even parts such that each different part appears an even number of times (in which we tacitly assume that 0 is the largest even part) or partitions with all even parts smaller than odd parts such that only the largest even part appears an odd number of times.

For example, 6 has four partitions in $\mathcal{EO}^*$, namely, $1+1+1+1+1+1$, $2+2+2$, $3+3$ and 6. Andrews showed that $\mathcal{EO}^*$ satisfies the generating function identity

$$\sum_{\pi\in\mathcal{EO}^*} q^{|\pi|} = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^2}$$

where $|\pi|$ denotes the sum of all parts in $\pi$. Throughout, we adopt the standard $q$-Pochhammer symbol for $n \in \mathbb{N}\cup\{\infty\}$,

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k).$$

One suggestion in [1, Problem 4] is to "undertake a more extensive investigation of the properties of $\mathcal{EO}^*$." Our first objective in this note is about two disjoint subsets of $\mathcal{EO}^*$ distinguished by the residue classes of the largest even part modulo 4.

**Theorem 1.1.** *Let* $\mathrm{eo}_0^*(n)$ *and* $\mathrm{eo}_2^*(n)$ *denote the number of partitions of* $n$ *in* $\mathcal{EO}^*$ *with largest even part congruent to* 0 *and* 2 *modulo* 4*, respectively. Then*

$$\mathrm{eo}_0^*(n) \begin{cases} = \mathrm{eo}_2^*(n) & \text{if } n \text{ is not divisible by } 4; \\ > \mathrm{eo}_2^*(n) & \text{if } n \text{ is divisible by } 4. \end{cases}$$

There are many natural extensions of integer partitions, among which the most important one is the overpartition [5]. An *overpartition* is an integer partition in which the first occurrence of each distinct part may be overlined. We have the following overpartition analog of $\mathcal{EO}^*$.

**Definition 1.2.** We denote by $\overline{\mathcal{EO}}^*$ the set of overpartitions with no even parts such that each different part appears an even number of times (in which we tacitly assume that 0 is the largest even part) or overpartitions with all even parts smaller than odd parts such that only the largest even part appears an odd number of times.

For example, 6 has eight partitions in $\overline{\mathcal{EO}}^*$, namely, $1+1+1+1+1+1$, $\overline{1}+1+1+1+1+1$, $2+2+2$, $\overline{2}+2+2$, $3+3$, $\overline{3}+3$, $6$ and $\overline{6}$. Our next objective is to give an overpartition analog of Theorem 1.1.

**Theorem 1.2.** *Let* $\overline{\mathrm{eo}}_0^*(n)$ *and* $\overline{\mathrm{eo}}_2^*(n)$ *denote the number of overpartitions of* $n$ *in* $\overline{\mathcal{EO}}^*$ *with largest even part congruent to* 0 *and* 2 *modulo* 4*, respectively. Then*

$$\overline{\mathrm{eo}}_0^*(n) \begin{cases} = \overline{\mathrm{eo}}_2^*(n) & \text{if } n \text{ is not divisible by } 4; \\ > \overline{\mathrm{eo}}_2^*(n) & \text{if } n \text{ is divisible by } 4. \end{cases}$$

## 2. Proof of Theorem 1.1

We first establish a generating function identity for $\mathrm{eo}_0^*(n) - \mathrm{eo}_2^*(n)$.

**Theorem 2.1.** *We have*

$$\sum_{n\ge 0} \big( \mathrm{eo}_0^*(n) - \mathrm{eo}_2^*(n)\big) q^n = \frac{(-q^4;q^4)_\infty}{(q^4;q^8)_\infty}. \tag{2.1}$$

Our proof of Theorem 2.1 relies on the $q$-binomial theorem [6, Equation (II.3)].

**Lemma 2.2** ($q$-Binomial theorem)**.** *We have*

$$\sum_{n\ge 0} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.2}$$

*Proof of Theorem 2.1.* Let $k$ be a nonnegative integer. We first notice that the generating function for partitions in $\mathcal{EO}^*$ with largest even part equal to $2k$ is given by

$$\frac{q^{2k}}{(q^4;q^4)_k} \cdot \frac{1}{(q^{4k+2};q^4)_\infty},$$

where the first multiplicand comes from all even parts and the second multiplicand comes from all odd parts. It follows that

$$\sum_{n\ge 0} \big( \mathrm{eo}_0^*(n) - \mathrm{eo}_2^*(n)\big) q^n = \sum_{k\ge 0} (-1)^k \frac{q^{2k}}{(q^4;q^4)_k (q^{4k+2};q^4)_\infty}$$

$$= \frac{1}{(q^2;q^4)_\infty} \sum_{k\geq 0} \frac{(q^2;q^4)_k(-q^2)^k}{(q^4;q^4)_k}.$$

Finally, applying the $q$-binomial theorem (2.2) with $a \to q^2$, $z \to -q^2$ and $q \to q^4$ yields

$$\begin{aligned}\sum_{n\geq 0} \big(\mathrm{eo}_0^*(n) - \mathrm{eo}_2^*(n)\big)q^n &= \frac{1}{(q^2;q^4)_\infty} \frac{(-q^4;q^4)_\infty}{(-q^2;q^4)_\infty}\\ &= \frac{(-q^4;q^4)_\infty}{(q^4;q^8)_\infty}.\end{aligned}$$

We therefore arrive at Theorem 2.1. □

*Proof of Theorem 1.1.* We simply notice that the right-hand side of (2.1) is a series of $q^4$ with positive coefficients. Therefore, $\mathrm{eo}_0^*(n) - \mathrm{eo}_2^*(n)$ is positive when $n$ is a multiple of 4 and zero otherwise. This proves Theorem 1.1. □

## 3. Proof of Theorem 1.2

For $\overline{\mathcal{EO}}^*$, we will establish a bivariate generating function identity. Let $o(\pi)$ count the number of overlined parts in $\pi$ for any $\pi \in \overline{\mathcal{EO}}^*$. We also assign a weight $w(\pi)$ to each $\pi$ by

$$w(\pi) = \begin{cases} 1 & \text{if the largest even part of } \pi \text{ is divisible by 4;} \\ -1 & \text{if the largest even part of } \pi \text{ is not divisible by 4.} \end{cases}$$

**Theorem 3.1.** *We have*

$$\sum_{\pi\in\overline{\mathcal{EO}}^*} w(\pi) z^{o(\pi)} q^{|\pi|} = \frac{(-q^4;q^4)_\infty(-zq^4;q^8)_\infty^2}{(q^4;q^8)_\infty}. \tag{3.1}$$

For its proof, we require the Bailey–Daum sum, also known as the $q$-Kummer sum [6, Equation (II.9)].

**Lemma 3.2** (Bailey–Daum sum)**.** *We have*

$$\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n}{(q;q)_n(aq/b;q)_n} \left(-\frac{q}{b}\right)^n = \frac{(-q;q)_\infty(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty}{(-q/b;q)_\infty(aq/b;q)_\infty}. \tag{3.2}$$

*Proof of Theorem 3.1.* First, it is a simple observation that the generating function for overpartitions in $\overline{\mathcal{EO}}^*$ with no even parts is

$$\frac{(-zq^2;q^4)_\infty}{(q^2;q^4)_\infty}.$$

Let $k$ be a positive integer. We then notice that the generating function for overpartitions in $\overline{\mathcal{EO}}^*$ with largest even part equal to $2k$ is given by

$$\frac{(1+z)q^{2k}(-zq^4;q^4)_{k-1}}{(q^4;q^4)_k} \cdot \frac{(-zq^{4k+2};q^4)_\infty}{(q^{4k+2};q^4)_\infty},$$

where, again, the first multiplicand comes from all even parts and the second multiplicand comes from all odd parts. Hence,

$$\sum_{\pi\in\overline{\mathcal{EO}}^*} w(\pi) z^{o(\pi)} q^{|\pi|}$$

$$= \frac{(-zq^2;q^4)_\infty}{(q^2;q^4)_\infty} + \sum_{k\geq 1}(-1)^k \frac{(1+z)q^{2k}(-zq^4;q^4)_{k-1}(-zq^{4k+2};q^4)_\infty}{(q^4;q^4)_k(q^{4k+2};q^4)_\infty}$$
$$= \frac{(-zq^2;q^4)_\infty}{(q^2;q^4)_\infty}\sum_{k\geq 0}\frac{(-z;q^4)_k(q^2;q^4)_k(-q^2)^k}{(q^4;q^4)_k(-zq^2;q^4)_k}.$$

It follows by the Bailey–Daum sum (3.2) with $a\to -z$, $b\to q^2$ and $q\to q^4$ that

$$\sum_{\pi\in\overline{\mathcal{EO}}^*} w(\pi)z^{o(\pi)}q^{|\pi|} = \frac{(-zq^2;q^4)_\infty}{(q^2;q^4)_\infty}\frac{(-q^4;q^4)_\infty(-zq^4;q^8)^2_\infty}{(-q^2;q^4)_\infty(-zq^2;q^4)_\infty}$$
$$= \frac{(-q^4;q^4)_\infty(-zq^4;q^8)^2_\infty}{(q^4;q^8)_\infty}.$$

Thus, Theorem 3.1 holds. □

*Proof of Theorem 1.2.* Taking $z=1$ in (3.1), we have

$$\sum_{n\geq 0}\big(\overline{\mathrm{eo}}_0^*(n)-\overline{\mathrm{eo}}_2^*(n)\big)q^n = \sum_{\pi\in\overline{\mathcal{EO}}^*} w(\pi)q^{|\pi|} = \frac{(-q^4;q^4)_\infty(-q^4;q^8)^2_\infty}{(q^4;q^8)_\infty}.$$

This is an overpartition analog of (2.1). We as well notice that the right-hand side of the above identity is a series of $q^4$ with positive coefficients. Theorem 1.2 therefore follows. □

*Remark* 3.1. We further take $z=0$ in (3.1). It is noticed that the left-hand side becomes

$$\sum_{\substack{\pi\in\overline{\mathcal{EO}}^*\\ o(\pi)=0}} w(\pi)q^{|\pi|},$$

which is simply the generating function of $\mathrm{eo}_0^*(n)-\mathrm{eo}_2^*(n)$. On the other hand, the term $(-zq^4;q^8)^2_\infty$ in the numerator of the right-hand side vanishes. Therefore, Theorem 3.1 reduces to Theorem 2.1 when $z=0$.

## 4. Final remark

It would be interesting to see combinatorial proofs of Theorems 1.1 and 1.2. Especially, for any nonegative integer $n$, we have $\mathrm{eo}_0^*(4n+2)=\mathrm{eo}_2^*(4n+2)$ and $\overline{\mathrm{eo}}_0^*(4n+2)=\overline{\mathrm{eo}}_2^*(4n+2)$. This indicates the existence of bijections between partitions enumerated by $\mathrm{eo}_0^*(4n+2)$ and $\mathrm{eo}_2^*(4n+2)$, as well as overpartitions enumerated by $\overline{\mathrm{eo}}_0^*(4n+2)$ and $\overline{\mathrm{eo}}_2^*(4n+2)$. It is appealing to find explicit constructions of such bijections.

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