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Weighted partition rank and crank moments. III. A list of Andrews–Beck type congruences modulo 5, 7, 11 and 13

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Abstract. Let $NT(r,k,n)$ count the total number of parts among partitions of n with rank congruent to r modulo k and let $M_\omega(r,k,n)$ count the total appearances of ones among partitions of n with crank congruent to r modulo k. We provide a list of over 70 congruences modulo 5, 7, 11 and 13 involving $NT(r,k,n)$ and $M_\omega(r,k,n)$, which are known as congruences of Andrews–Beck type. Some recent conjectures of Chan, Mao and Osburn are also included in this list.

Keywords. Partition, rank, crank, weighted moment, Andrews–Beck type congruence.

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1. Introduction

A *partition* of a natural number n is a weakly decreasing sequence of positive integers whose sum equals n. Let $p(n)$ count the number of partitions of n. One of the most fascinating properties of $p(n)$ is due to Ramanujan [1, Chapter 10], saying that

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

In order to give a combinatorial interpretation of Ramanujan's congruences, Dyson [9] defined a statistic for partitions which he called *rank* as the largest part minus the number of parts. He conjectured that this statistic could be utilized to show (1.1) and (1.2), which was later confirmed by Atkin and Swinnerton-Dyer [5]. Dyson also predicted the existence of another partition statistic which he named *crank*, with which all the three of Ramanujan's congruences (1.1), (1.2) and (1.3) could be combinatorially explained. Such a statistic was found some forty years later by Andrews and Garvan [3]. More specifically, for any partition λ, Andrews and Garvan defined its crank as

$$\operatorname{crank}(\lambda) := \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0, \end{cases}$$

where $\ell(\lambda)$, $\omega(\lambda)$ and $\mu(\lambda)$ denote the largest part in λ, the number of ones in λ and the number of parts in λ larger than $\omega(\lambda)$, respectively.

As usual, we denote by $N(m,n)$ the number of partitions of n with rank m and by $M(m,n)$ the number of partitions of n with crank m. We also define

$$N(r,k,n) := \sum_{\substack{m=-\infty \\ m\equiv r \pmod k}}^{\infty} N(m,n)$$

and

$$M(r,k,n) := \sum_{\substack{m=-\infty \\ m\equiv r \pmod k}}^{\infty} M(m,n).$$

Properties of these functions have been studied extensively. Recently, Andrews [2] investigated variations of these rank and crank counting functions which are attributed to George Beck.

Let $NT(m,n)$ count the total number of parts among partitions of n with rank m and let $M_\omega(m,n)$ count the total appearances of ones among partitions of n with crank m. We also define

$$NT(r,k,n) := \sum_{\substack{m=-\infty \\ m\equiv r \pmod k}}^{\infty} NT(m,n)$$

and

$$M_\omega(r,k,n) := \sum_{\substack{m=-\infty \\ m\equiv r \pmod k}}^{\infty} M_\omega(m,n).$$

One of the surprising properties conjectured by Beck and shown by Andrews [2] says that for $i = 1,\ 4$,

$$\begin{aligned} &NT(1,5,5n+i) + 2NT(2,5,5n+i) \\ &\quad - 2NT(3,5,5n+i) - NT(4,5,5n+i) \equiv 0 \pmod 5. \end{aligned}$$

More congruences of the same manner were given in the first paper of this series [7].

In Summer 2019, George Beck conveyed in a private communication the idea that congruences of such type are far more than known ones. Recently, Chan, Mao and Osburn [6] conjectured eight more congruences related to NT and M_ω, two of which have an unexpected modulus 13.

The objective of this paper is to provide a list of over 70 congruences for NT and M_ω modulo 5, 7, 11 and 13. Through a computer search, it is believed that this list is to some extent complete for these moduli (it should be noted that a handful of unlisted congruences could be generated by congruences in the main theorems; see remarks below each theorem).

Theorem 1.1. *Let*

$$NT[a_1,a_2](n) := \sum_{r=1}^{2} a_r\big(NT(r,5,n) - NT(5-r,5,n)\big)$$

and

$$M_\omega[a_1, a_2](n) := \sum_{r=1}^{2} a_r \big(M_\omega(r,5,n) - M_\omega(5-r,5,n) \big).$$

Then (i).

$$NT[1,2](5n+1) \equiv 0 \pmod 5, \tag{1.4-1}$$
$$NT[1,2](5n+4) \equiv 0 \pmod 5; \tag{1.4-2}$$

(ii).

$$M_\omega[1,2](5n) \equiv 0 \pmod 5, \tag{1.5-1}$$
$$M_\omega[1,2](5n+4) \equiv 0 \pmod 5; \tag{1.5-2}$$

(iii).

$$\begin{aligned} NT[0,1](5n) &\equiv M_\omega[0,1](5n) \equiv M_\omega[1,3](5n) \equiv M_\omega[2,0](5n) \\ &\equiv M_\omega[3,2](5n) \equiv M_\omega[4,4](5n) \pmod 5, \end{aligned} \tag{1.6-1}$$
$$NT[0,1](5n+1) \equiv M_\omega[0,1](5n+1) \pmod 5, \tag{1.6-2}$$
$$NT[1,0](5n+1) \equiv M_\omega[0,3](5n+1) \pmod 5, \tag{1.6-3}$$
$$NT[0,1](5n+2) \equiv M_\omega[2,0](5n+2) \pmod 5, \tag{1.6-4}$$
$$NT[1,0](5n+2) \equiv M_\omega[0,3](5n+2) \pmod 5, \tag{1.6-5}$$
$$NT[1,3](5n+3) \equiv M_\omega[1,3](5n+3) \pmod 5, \tag{1.6-6}$$
$$\begin{aligned} NT[0,1](5n+4) &\equiv M_\omega[0,1](5n+4) \equiv M_\omega[1,3](5n+4) \equiv M_\omega[2,0](5n+4) \\ &\equiv M_\omega[3,2](5n+4) \equiv M_\omega[4,4](5n+4) \pmod 5, \end{aligned} \tag{1.6-7}$$
$$\begin{aligned} NT[1,0](5n+4) &\equiv M_\omega[0,3](5n+4) \equiv M_\omega[1,0](5n+4) \equiv M_\omega[2,2](5n+4) \\ &\equiv M_\omega[3,4](5n+4) \equiv M_\omega[4,1](5n+4) \pmod 5. \end{aligned} \tag{1.6-8}$$

Remark 1.1. It should be pointed out that one may derive more congruences from (1.6-2) and (1.6-3). For example,

$$NT[1,1](5n+1) \equiv M_\omega[0,4](5n+1) \pmod 5,$$

which comes from

$$\begin{aligned} NT[1,1](5n+1) &\equiv NT[0,1](5n+1) + NT[1,0](5n+1) \\ &\equiv M_\omega[0,1](5n+1) + M_\omega[0,3](5n+1) \\ &\equiv M_\omega[0,4](5n+1) \pmod 5. \end{aligned}$$

Similarly, more congruences could be derived from (1.6-4) and (1.6-5), and from (1.6-7) and (1.6-8). Also, in (1.6-1), we have $M_\omega[0,1](5n) \equiv M_\omega[1,3](5n) \equiv \cdots \pmod 5$. This is a consequence of (1.5-1) by noticing that

$$M_\omega[1,3](5n) \equiv M_\omega[0,1](5n) + M_\omega[1,2](5n) \equiv M_\omega[0,1](5n) \pmod 5.$$

Similar arguments could be applied to (1.6-7) and (1.6-8) with the help of (1.5-2).

We notice that (1.6-1) and (1.6-7) imply [6, (4.10)], and (1.6-3) and (1.6-5) imply [6, (4.12)].

Theorem 1.2. *Let*

$$NT[a_1,a_2,a_3](n) := \sum_{r=1}^{3} a_r\big(NT(r,7,n) - NT(7-r,7,n)\big)$$

and

$$M_\omega[a_1,a_2,a_3](n) := \sum_{r=1}^{3} a_r\big(M_\omega(r,7,n) - M_\omega(7-r,7,n)\big).$$

Then (i).

$$NT[0,1,4](7n) \equiv 0 \pmod 7, \tag{1.7-1}$$
$$NT[0,1,4](7n+1) \equiv 0 \pmod 7, \tag{1.7-2}$$
$$NT[1,0,2](7n+1) \equiv 0 \pmod 7, \tag{1.7-3}$$
$$NT[1,0,2](7n+3) \equiv 0 \pmod 7, \tag{1.7-4}$$
$$NT[1,0,2](7n+4) \equiv 0 \pmod 7, \tag{1.7-5}$$
$$NT[0,1,4](7n+5) \equiv 0 \pmod 7, \tag{1.7-6}$$
$$NT[1,0,2](7n+5) \equiv 0 \pmod 7; \tag{1.7-7}$$

(ii).

$$M_\omega[0,1,4](7n) \equiv 0 \pmod 7, \tag{1.8-1}$$
$$M_\omega[1,0,2](7n) \equiv 0 \pmod 7, \tag{1.8-2}$$
$$M_\omega[0,1,4](7n+1) \equiv 0 \pmod 7, \tag{1.8-3}$$
$$M_\omega[1,0,2](7n+2) \equiv 0 \pmod 7, \tag{1.8-4}$$
$$M_\omega[1,3,0](7n+3) \equiv 0 \pmod 7, \tag{1.8-5}$$
$$M_\omega[0,1,4](7n+4) \equiv 0 \pmod 7, \tag{1.8-6}$$
$$M_\omega[0,1,4](7n+5) \equiv 0 \pmod 7, \tag{1.8-7}$$
$$M_\omega[1,0,2](7n+5) \equiv 0 \pmod 7, \tag{1.8-8}$$
$$M_\omega[1,0,2](7n+6) \equiv 0 \pmod 7. \tag{1.8-9}$$

Remark 1.2. Linear combinations of (1.7-2) and (1.7-3) imply more congruences. For example, (1.7-2) + (1.7-3) gives

$$NT[1,1,6](7n+1) \equiv 0 \pmod 7,$$

which is the $i = 1$ case of [2, Theorem 1.2]. More congruences could be derived from linear combinations of (1.7-6) and (1.7-7), of (1.8-1) and (1.8-2), and of (1.8-7) and (1.8-8).

We notice that [6, (4.15) and (4.16)] are shown in Part (ii).

Theorem 1.3. *Let*

$$NT[a_1,a_2,a_3,a_4,a_5](n) := \sum_{r=1}^{5} a_r\big(NT(r,11,n) - NT(11-r,11,n)\big)$$

and

$$M_\omega[a_1,a_2,a_3,a_4,a_5](n) := \sum_{r=1}^{5} a_r\big(M_\omega(r,11,n) - M_\omega(11-r,11,n)\big).$$

We also adopt the notation

$$M_\omega \begin{bmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1, & c_2, & c_3, & c_4, & c_5 \end{bmatrix} (n) := \begin{bmatrix} M_\omega[a_1,a_2,a_3,a_4,a_5](n) \\ M_\omega[b_1,b_2,b_3,b_4,b_5](n) \\ \vdots \\ M_\omega[c_1,c_2,c_3,c_4,c_5](n) \end{bmatrix}.$$

Then (i).

$$NT[0,1,4,10,9](11n) \equiv 0 \pmod{11}, \tag{1.9-1}$$

$$NT[1,8,5,9,4](11n+1) \equiv 0 \pmod{11}, \tag{1.9-2}$$

$$NT[1,3,7,3,3](11n+6) \equiv 0 \pmod{11}; \tag{1.9-3}$$

(ii).

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-1}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \end{bmatrix} (11n+1) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-2}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+2) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-3}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+3) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-4}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 5, & 0 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+4) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-5}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 5, & 0 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+5) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-6}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+6) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-7}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 8, & 0 \end{bmatrix} (11n+7) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-8}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 2, & 0 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 8, & 0 \end{bmatrix} (11n+8) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-9}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+9) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{1.10-10}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+10) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}. \tag{1.10-11}$$

Remark 1.3. Each of (1.10-1)–(1.10-11) may lead to more Andrews–Beck type congruences modulo 11 for M_ω.

We notice that (1.9-2) is [6, (4.6)] and (1.9-3) is [6, (4.5)].

Theorem 1.4. *Let*

$$NT[a_1,a_2,a_3,a_4,a_5,a_6](n) := \sum_{r=1}^{6} a_r\big(NT(r,13,n) - NT(13-r,13,n)\big)$$

and

$$M_\omega[a_1,a_2,a_3,a_4,a_5,a_6](n) := \sum_{r=1}^{6} a_r\big(M_\omega(r,13,n) - M_\omega(13-r,13,n)\big).$$

Then (i).

$$NT[0,1,4,12,10,3](13n) \equiv 0 \pmod{13}, \tag{1.11-1}$$
$$NT[1,1,6,0,0,3](13n+1) \equiv 0 \pmod{13}, \tag{1.11-2}$$
$$NT[0,0,1,9,6,8](13n+2) \equiv 0 \pmod{13}, \tag{1.11-3}$$
$$NT[1,0,3,9,1,11](13n+3) \equiv 0 \pmod{13}, \tag{1.11-4}$$
$$NT[1,5,8,7,12,12](13n+5) \equiv 0 \pmod{13}, \tag{1.11-5}$$
$$NT[1,2,8,0,7,11](13n+6) \equiv 0 \pmod{13}, \tag{1.11-6}$$
$$NT[1,12,8,7,10,7](13n+7) \equiv 0 \pmod{13}, \tag{1.11-7}$$
$$NT[1,6,11,8,0,0](13n+9) \equiv 0 \pmod{13}, \tag{1.11-8}$$
$$NT[1,9,4,5,10,7](13n+10) \equiv 0 \pmod{13}; \tag{1.11-9}$$

(ii).

$$M_\omega[1,2,3,4,5,6](13n) \equiv 0 \pmod{13}. \tag{1.12-1}$$

Remark 1.4. We notice that (1.11-2) is [6, (4.7)] and (1.11-4) is [6, (4.8)].

2. Weighted and ordinary rank and crank moments

In the first two papers of this series [7, 8], I have connected the ordinary rank and crank moments

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n), \tag{2.1}$$

$$M_k(n) := \sum_{m=-\infty}^{\infty} m^k M(m,n), \tag{2.2}$$

with the so-called weighted rank and crank moments

$$N_k^\sharp(n) := \sum_{m=-\infty}^{\infty} m^k NT(m,n), \tag{2.3}$$

$$M_k^{\omega}(n) := \sum_{m=-\infty}^{\infty} m^k M_{\omega}(m,n). \tag{2.4}$$

Two of the main results in [8] read as follows.

Lemma 2.1. *We have*

$$N_{2k-1}^{\sharp}(n) = -\frac{1}{2}N_{2k}(n), \tag{2.5}$$

$$M_{2k-1}^{\omega}(n) = -\frac{1}{2}M_{2k}(n). \tag{2.6}$$

The weighted rank and crank moments will play an important role in the proof of our congruences. Thus, we first need to rewrite the left-hand sides of our congruences in terms of linear combinations of weighted rank and crank moments of odd order after reducing modulo 5, 7, 11 or 13.

Lemma 2.2. *Let p be an odd prime. Given any $(a_1, a_2, \ldots, a_{(p-1)/2}) \in (\mathbb{Z}/p\mathbb{Z})^{\frac{p-1}{2}}$, there always exists unique $(c_1, c_2, \ldots, c_{(p-1)/2}) \in (\mathbb{Z}/p\mathbb{Z})^{\frac{p-1}{2}}$ such that*

$$\sum_{s=1}^{\frac{p-1}{2}} c_s N_{2s-1}^{\sharp}(n) \equiv \sum_{r=1}^{\frac{p-1}{2}} a_r \big(NT(r,p,n) - NT(p-r,p,n)\big) \pmod p \tag{2.7}$$

and

$$\sum_{s=1}^{\frac{p-1}{2}} c_s M_{2s-1}^{\omega}(n) \equiv \sum_{r=1}^{\frac{p-1}{2}} a_r \big(M_{\omega}(r,p,n) - M_{\omega}(p-r,p,n)\big) \pmod p. \tag{2.8}$$

Proof. We only consider (2.7) while (2.8) can be shown analogously. By (2.3),

$$\begin{aligned}\sum_{s=1}^{\frac{p-1}{2}} c_s N_{2s-1}^{\sharp}(n) &= \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\frac{p-1}{2}} c_s m^{2s-1} NT(m,n)\\ &\equiv \sum_{r=1}^{\frac{p-1}{2}} \sum_{s=1}^{\frac{p-1}{2}} c_s r^{2s-1} \big(NT(r,p,n) - NT(p-r,p,n)\big) \pmod p.\end{aligned}$$

Therefore, (2.7) requires that for each $1 \le r \le (p-1)/2$,

$$\sum_{s=1}^{\frac{p-1}{2}} c_s r^{2s-1} \equiv a_r \pmod p,$$

or,

$$\begin{pmatrix} 1^1 & 1^3 & \cdots & 1^{p-2} \\ 2^1 & 2^3 & \cdots & 2^{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{p-1}{2}\right)^1 & \left(\frac{p-1}{2}\right)^3 & \cdots & \left(\frac{p-1}{2}\right)^{p-2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{(p-1)/2} \end{pmatrix} \equiv \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{(p-1)/2} \end{pmatrix} \pmod p. \tag{2.9}$$

We simply notice that the square matrix on the left-hand side of the above is invertible in $\mathbb{Z}/p\mathbb{Z}$. This is because the determinant of the square matrix can be expressed as $((p-1)/2)!$ times the determinant of a Vandermonde matrix. This fact then indicates the existence and uniqueness of $(c_1, c_2, \ldots, c_{(p-1)/2}) \in (\mathbb{Z}/p\mathbb{Z})^{\frac{p-1}{2}}$. □

Now proofs of Theorems 1.1–1.4 could be completed through the following steps.

Step 1. By Lemma 2.2, we find

$$F[a_1,\ldots,a_{(p-1)/2}](n)\equiv\sum_{s=1}^{\frac{p-1}{2}}c_sG_{2s-1}(n)\pmod p,$$

where F stands for NT or M_ω, and G stands for $N^\sharp$ or M^ω.

Step 2. By Lemma 2.1, we deduce

$$F[a_1,\ldots,a_{(p-1)/2}](n)\equiv-\frac{1}{2}\sum_{s=1}^{\frac{p-1}{2}}c_sH_{2s}(n)\pmod p,$$

where H stands for N or M.

Step 3. There are many relations between rank and crank moments $N_{2s}(n)$ and $M_{2s}(n)$ to be utilized. An important collection of these relations is presented in [4]. Below are some that will be used.

[4, (5.6)]:

$$N_4(n)=\frac{2}{3}\big(-3n-1\big)M_2(n)+\frac{8}{3}M_4(n)+\big(-12n+1\big)N_2(n),\tag{2.10}$$

[4, (5.7)]:

$$\begin{aligned}N_6(n)=&\frac{2}{33}\big(324n^2+69n-10\big)M_2(n)+\frac{20}{33}\big(-45n+4\big)M_4(n)\\&+\frac{18}{11}M_6(n)+\big(108n^2-24\,n+1\big)N_2(n),\end{aligned}\tag{2.11}$$

[4, (5.8)]:

$$\begin{aligned}N_8(n)=&\frac{2}{913}\big(-72972n^3-1728n^2+5667n-289\big)M_2(n)\\&+\frac{280}{913}\big(732n^2-195n+8\big)M_4(n)+\frac{84}{913}\big(-196n+15\big)M_6(n)\\&+\frac{1248}{913}M_8(n)+\big(-864n^3+360n^2-36n+1\big)N_2(n).\end{aligned}\tag{2.12}$$

Also, [4, (6.5)]:

$$\big(n+2\big)M_4(n)+\big(6n^2+4n+1\big)M_2(n)\equiv0\pmod 7,\tag{2.13}$$

[4, (6.6)]:

$$\big(n+5\big)^3M_4(n)\equiv\big(5n^4+10n^3+8n^2+8n+9\big)M_2(n)\pmod{11},\tag{2.14}$$

[4, (6.7)]:

$$M_6(n)\equiv2\big(n+7\big)M_4(n)-\big(n+8\big)^2M_2(n)\pmod{11},\tag{2.15}$$

[4, (6.8)]:

$$M_8(n)\equiv6\big(n^2+n+1\big)M_4(n)+2\big(n+5\big)\big(n^2+5n+10\big)M_2(n)\pmod{11}.\tag{2.16}$$

3. Modulo 5

We collect some auxiliary results. By the fact ([3, (1.1) and (1.13)] and [3, Theorem 1])

$$N(r,5,5n+4) = M(r,5,5n+4) = \frac{1}{5}p(5n+4)$$

for $0 \le r \le 4$, we have

$$N_2(5n+4) \equiv \sum_{r=0}^{4} r^2 N(r,5,5n+4) \equiv 0 \pmod{5}, \tag{3.1}$$

$$M_2(5n+4) \equiv \sum_{r=0}^{4} r^2 M(r,5,5n+4) \equiv 0 \pmod{5}. \tag{3.2}$$

We also deduce from the fact $M_2(n) = 2np(n)$ that

$$M_2(5n) \equiv 0 \pmod{5}. \tag{3.3}$$

Further, in (2.11), using the facts that $M_6(n) \equiv M_2(n) \pmod{5}$ and $N_6(n) \equiv N_2(n) \pmod{5}$ since $m^6 \equiv m^2 \pmod{5}$, we have

$$N_2(5n+1) \equiv 5M_2(5n+1) \equiv 0 \pmod{5}, \tag{3.4}$$

$$N_2(5n+2) \equiv 4M_2(5n+2) \pmod{5}. \tag{3.5}$$

3.1. Rank.

We prove (1.4-1) and (1.4-2).

Proof. Putting $p = 5$, $a_1 = 1$ and $a_2 = 2$ in (2.9) gives

$$\begin{pmatrix} 1 & 1 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix} \pmod{5}. \tag{3.6}$$

Therefore, $c_1 = 1$ and $c_2 = 0$. We then have

$$NT[1,2](n) \equiv N_1^{\sharp}(n) \pmod{5}.$$

By (2.5), we then have

$$NT[1,2](n) \equiv -\frac{1}{2}N_2(n) \pmod{5}. \tag{3.7}$$

(1.4-1) follows from (3.7) and (3.4).
(1.4-2) follows from (3.7) and (3.1). □

3.2. Crank.

We prove (1.5-1) and (1.5-2).

Proof. Noticing that from the solution of (3.6), we have

$$\begin{aligned} M_\omega[1,2](n) &\equiv M_1^{\omega}(n) \pmod{5} \\ &= -\frac{1}{2}M_2(n). \end{aligned} \tag{3.8}$$

(1.5-1) follows from (3.8) and (3.3).
(1.5-2) follows from (3.8) and (3.2). □

3.3. Hybrid. We prove (1.6-1)–(1.6-8). For (1.6-1), (1.6-7) and (1.6-8), as we have pointed out in Remark 1.1, it suffices to show the first congruence in each of them.

Proof. Akin to how we proceed with the previous cases, we obtain, by solving (2.9) and applying Lemma 2.1, that

$$NT[0,1](n) \equiv -\frac{1}{2}\big(4N_2(n)+N_4(n)\big) \pmod 5, \tag{3.9}$$

$$NT[1,0](n) \equiv -\frac{1}{2}\big(3N_2(n)+3N_4(n)\big) \pmod 5, \tag{3.10}$$

$$NT[1,3](n) \equiv -\frac{1}{2}N_4(n) \pmod 5, \tag{3.11}$$

and

$$M_\omega[0,1](n) \equiv -\frac{1}{2}\big(4M_2(n)+M_4(n)\big) \equiv 3M_2(n)+2M_4(n) \pmod 5, \tag{3.12}$$

$$M_\omega[0,3](n) \equiv -\frac{1}{2}\big(2M_2(n)+3M_4(n)\big) \equiv 4M_2(n)+M_4(n) \pmod 5, \tag{3.13}$$

$$M_\omega[2,0](n) \equiv -\frac{1}{2}\big(M_2(n)+M_4(n)\big) \equiv 2M_2(n)+2M_4(n) \pmod 5, \tag{3.14}$$

$$M_\omega[1,3](n) \equiv -\frac{1}{2}M_4(n) \equiv 2M_4(n) \pmod 5. \tag{3.15}$$

Also, substituting (2.10) into (3.9), (3.10) and (3.11), we have

$$NT[0,1](5n) \equiv 2M_2(5n)+2M_4(5n) \pmod 5, \tag{3.16}$$

$$NT[0,1](5n+1) \equiv 3M_2(5n+1)+2M_4(5n+1)+N_2(5n+1) \pmod 5, \tag{3.17}$$

$$NT[0,1](5n+2) \equiv 4M_2(5n+2)+2M_4(5n+2)+2N_2(5n+2) \pmod 5, \tag{3.18}$$

$$NT[0,1](5n+4) \equiv M_2(5n+4)+2M_4(5n+4)+4N_2(5n+4) \pmod 5, \tag{3.19}$$

$$NT[1,0](5n+1) \equiv 4M_2(5n+1)+M_4(5n+1) \pmod 5, \tag{3.20}$$

$$NT[1,0](5n+2) \equiv 2M_2(5n+2)+M_4(5n+2)+3N_2(5n+2) \pmod 5, \tag{3.21}$$

$$NT[1,0](5n+4) \equiv 3M_2(5n+4)+M_4(5n+4)+4N_2(5n+4) \pmod 5, \tag{3.22}$$

$$NT[1,3](5n+3) \equiv 2M_4(5n+3) \pmod 5. \tag{3.23}$$

(1.6-1) follows from (3.12), (3.16) and (3.3).
(1.6-2) follows from (3.12), (3.17) and (3.4).
(1.6-3) follows from (3.13) and (3.20).
(1.6-4) follows from (3.14), (3.18) and (3.5).
(1.6-5) follows from (3.13), (3.21) and (3.5).
(1.6-6) follows from (3.15) and (3.23).
(1.6-7) follows from (3.12), (3.19), (3.1) and (3.2).
(1.6-8) follows from (3.13), (3.22), (3.1) and (3.2). □

4. Modulo 7

We collect some auxiliary results. By the fact ([3, (1.2) and (1.14)] and [3, Theorem 1])

$$N(r,7,7n+5) = M(r,7,7n+5) = \frac{1}{7}p(7n+5)$$

for $0 \le r \le 6$, we have

$$N_2(7n+5) \equiv \sum_{r=0}^{6} r^2 N(r,7,7n+5) \equiv 0 \pmod{7}, \tag{4.1}$$

$$N_4(7n+5) \equiv \sum_{r=0}^{6} r^4 N(r,7,7n+5) \equiv 0 \pmod{7}, \tag{4.2}$$

$$M_2(7n+5) \equiv \sum_{r=0}^{6} r^2 M(r,7,7n+5) \equiv 0 \pmod{7}, \tag{4.3}$$

$$M_4(7n+5) \equiv \sum_{r=0}^{6} r^4 M(r,7,7n+5) \equiv 0 \pmod{7}. \tag{4.4}$$

We also deduce from the fact $M_2(n) = 2np(n)$ that

$$M_2(7n) \equiv 0 \pmod{7}. \tag{4.5}$$

Further, in (2.12), using the facts that $M_8(n) \equiv M_2(n) \pmod{7}$ and $N_8(n) \equiv N_2(n) \pmod{7}$ since $m^8 \equiv m^2 \pmod{7}$, we have

$$N_2(7n+1) \equiv 7M_2(7n+1) \equiv 0 \pmod{7}, \tag{4.6}$$

$$N_2(7n+3) \equiv 2M_2(7n+3) \pmod{7}, \tag{4.7}$$

$$N_2(7n+4) \equiv 4M_2(7n+4) \pmod{7}. \tag{4.8}$$

Finally, by (2.13),

$$M_2(7n) \equiv 5M_4(7n) \pmod{7}, \tag{4.9}$$

$$M_2(7n+1) \equiv M_4(7n+1) \pmod{7}, \tag{4.10}$$

$$M_2(7n+2) \equiv 2M_4(7n+2) \pmod{7}, \tag{4.11}$$

$$M_2(7n+3) \equiv 4M_4(7n+3) \pmod{7}, \tag{4.12}$$

$$M_2(7n+4) \equiv M_4(7n+4) \pmod{7}, \tag{4.13}$$

$$M_2(7n+6) \equiv 2M_4(7n+6) \pmod{7}. \tag{4.14}$$

4.1. Rank. We first prove (1.7-1), (1.7-2) and (1.7-6).

Proof. Solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 \\ 2^1 & 2^3 & 2^5 \\ 3^1 & 3^3 & 3^5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \pmod{7},$$

we have $c_1 = 1$, $c_2 = 6$ and $c_3 = 0$. Hence,

$$\begin{aligned} NT[0,1,4](n) &\equiv N_1^\sharp(n) + 6N_3^\sharp(n) \pmod{7} \\ &= -\frac{1}{2}\big(N_2(n) + 6N_4(n)\big). \end{aligned}$$

By (4.1) and (4.2),

$$NT[0,1,4](7n+5) \equiv -\frac{1}{2}\big(N_2(7n+5) + 6N_4(7n+5)\big) \equiv 0 \pmod{7},$$

which gives (1.7-6).

Also, it follows from (2.10) that

$$NT[0,1,4](7n) \equiv 2M_2(7n) + 6M_4(7n) \pmod 7, \tag{4.15}$$

$$NT[0,1,4](7n+1) \equiv M_2(7n+1) + 6M_4(7n+1) + N_2(7n+1) \pmod 7. \tag{4.16}$$

(1.7-1) follows from (4.15), (4.5) and (4.9).
(1.7-2) follows from (4.16), (4.6) and (4.10). □

We next prove (1.7-3), (1.7-4), (1.7-5) and (1.7-7).

Proof. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 \\ 2^1 & 2^3 & 2^5 \\ 3^1 & 3^3 & 3^5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \pmod 7,$$

we have

$$\begin{aligned} NT[1,0,2](n) &\equiv 6N_1^\sharp(n) + 2N_3^\sharp(n) \pmod 7 \\ &= -\frac{1}{2}\big(6N_2(n) + 2N_4(n)\big). \end{aligned}$$

By (4.1) and (4.2),

$$NT[1,0,2](7n+5) \equiv -\frac{1}{2}\big(6N_2(7n+5) + 2N_4(7n+5)\big) \equiv 0 \pmod 7,$$

which proves (1.7-7).

Also, it follows from (2.10) that

$$NT[1,0,2](7n+1) \equiv 5M_2(7n+1) + 2M_4(7n+1) + N_2(7n+1) \pmod 7, \tag{4.17}$$

$$NT[1,0,2](7n+3) \equiv 2M_2(7n+3) + 2M_4(7n+3) + 4N_2(7n+3) \pmod 7, \tag{4.18}$$

$$NT[1,0,2](7n+4) \equiv 4M_2(7n+4) + 2M_4(7n+4) + 2N_2(7n+4) \pmod 7. \tag{4.19}$$

(1.7-3) follows from (4.17), (4.6) and (4.10).
(1.7-4) follows from (4.18), (4.7) and (4.12).
(1.7-5) follows from (4.19), (4.8) and (4.13). □

4.2. Crank. We first prove (1.8-1), (1.8-3), (1.8-6) and (1.8-7).

Proof. We have

$$\begin{aligned} M_\omega[0,1,4](n) &\equiv M_1^\omega(n) + 6M_3^\omega(n) \pmod 7 \\ &= -\frac{1}{2}\big(M_2(n) + 6M_4(n)\big). \end{aligned} \tag{4.20}$$

(1.8-1) follows from (4.20), (4.5) and (4.9).
(1.8-3) follows from (4.20) and (4.10).
(1.8-6) follows from (4.20) and (4.13).
(1.8-7) follows from (4.20), (4.3) and (4.4). □

We next prove (1.8-2), (1.8-4), (1.8-8) and (1.8-9).

Proof. We have

$$\begin{aligned} M_\omega[1,0,2](n) &\equiv 6M_1^\omega(n) + 2M_3^\omega(n) \pmod 7 \\ &= -\frac{1}{2}\big(6M_2(n) + 2M_4(n)\big). \end{aligned} \tag{4.21}$$

(1.8-2) follows from (4.21), (4.5) and (4.9).
(1.8-4) follows from (4.21) and (4.11).
(1.8-8) follows from (4.21), (4.3) and (4.4).
(1.8-9) follows from (4.21) and (4.14). □

Finally, we prove (1.8-5).

Proof. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 \\ 2^1 & 2^3 & 2^5 \\ 3^1 & 3^3 & 3^5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \pmod 7,$$

we have

$$\begin{aligned} M_\omega[1,3,0](n) &\equiv 2M_1^\omega(n) + 6M_3^\omega(n) \pmod 7 \\ &= -\frac{1}{2}\big(2M_2(n) + 6M_4(n)\big). \end{aligned} \tag{4.22}$$

(1.8-5) follows from (4.22) and (4.12). □

5. Modulo 11

We collect some auxiliary results. By the fact ([3, (1.15)] and [3, Theorem 1])

$$M(r, 11, 11n+6) = \frac{1}{11} p(11n+6)$$

for $0 \le r \le 10$, we have

$$M_2(11n+6) \equiv \sum_{r=0}^{10} r^2 M(r,11,11n+6) \equiv 0 \pmod{11}, \tag{5.1}$$

$$M_4(11n+6) \equiv \sum_{r=0}^{10} r^4 M(r,11,11n+6) \equiv 0 \pmod{11}, \tag{5.2}$$

$$M_6(11n+6) \equiv \sum_{r=0}^{10} r^6 M(r,11,11n+6) \equiv 0 \pmod{11}, \tag{5.3}$$

$$M_8(11n+6) \equiv \sum_{r=0}^{10} r^8 M(r,11,11n+6) \equiv 0 \pmod{11}. \tag{5.4}$$

We also deduce from the fact $M_2(n) = 2np(n)$ that

$$M_2(11n) \equiv 0 \pmod{11}. \tag{5.5}$$

Finally, by (2.14), (2.15) and (2.16),

$$M_2(11n) \equiv 9M_4(11n) \equiv 2M_6(11n) \equiv 8M_8(11n) \pmod{11}, \tag{5.6}$$

$$M_2(11n+1) \equiv M_4(11n+1) \equiv M_6(11n+1) \equiv M_8(11n+1) \pmod{11}, \tag{5.7}$$

$$M_2(11n+2) \equiv 3M_4(11n+2) \equiv 9M_6(11n+2) \equiv 5M_8(11n+2) \pmod{11}, \tag{5.8}$$

$$M_2(11n+3) \equiv 5M_4(11n+3) \equiv 3M_6(11n+3) \equiv 4M_8(11n+3) \pmod{11}, \tag{5.9}$$

$$M_2(11n+4) \equiv 8M_4(11n+4) \equiv 10M_6(11n+4) \equiv 9M_8(11n+4) \pmod{11}, \tag{5.10}$$

$$M_2(11n+5) \equiv 8M_4(11n+5) \equiv 10M_6(11n+5) \equiv 9M_8(11n+5) \pmod{11}, \tag{5.11}$$

$$M_2(11n+7) \equiv 7M_4(11n+7) \equiv 10M_6(11n+7) \equiv 5M_8(11n+7) \pmod{11}, \tag{5.12}$$

$$M_2(11n+8) \equiv 6M_4(11n+8) \equiv 6M_6(11n+8) \equiv M_8(11n+8) \pmod{11}, \tag{5.13}$$

$$M_2(11n+9) \equiv 9M_4(11n+9) \equiv 4M_6(11n+9) \equiv 3M_8(11n+9) \pmod{11}, \tag{5.14}$$

$$M_2(11n+10) \equiv 5M_4(11n+10) \equiv 3M_6(11n+10) \equiv 4M_8(11n+10) \pmod{11}. \tag{5.15}$$

5.1. Rank. We first prove (1.9-1).

Proof. Solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 4 \\ 10 \\ 9 \end{pmatrix} \pmod{11},$$

we have $c_1 = 9$, $c_2 = 2$ and $c_3 = c_4 = c_5 = 0$. Hence,

$$\begin{aligned} NT[0,1,4,10,9](n) &\equiv 9N_1^\sharp(n) + 2N_3^\sharp(n) \pmod{11} \\ &= -\frac{1}{2}\big(9N_2(n) + 2N_4(n)\big). \end{aligned}$$

It follows from (2.10) that

$$NT[0,1,4,10,9](11n) \equiv M_2(11n) + M_4(11n) \pmod{11}. \tag{5.16}$$

(1.9-1) then follows from (5.16), (5.5) and (5.6). □

We then prove (1.9-2).

Proof. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 8 \\ 5 \\ 9 \\ 4 \end{pmatrix} \pmod{11},$$

we have

$$NT[1,8,5,9,4](n) \equiv N_3^\sharp(n) \pmod{11}$$

$$= -\frac{1}{2} N_4(n).$$

It follows from (2.10) that

$$NT[1,8,5,9,4](11n+1) \equiv 5M_2(11n+1) + 6M_4(11n+1) \pmod{11}. \tag{5.17}$$

(1.9-2) then follows from (5.17) and (5.7). □

Finally, we prove (1.9-3).

Proof. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 3 \\ 7 \\ 3 \\ 3 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} NT[1,3,7,3,3](n) &\equiv 10N_1^\sharp(n) + 2N_3^\sharp(n) \pmod{11} \\ &= -\frac{1}{2}\big(10N_2(n) + 2N_4(n)\big). \end{aligned}$$

It follows from (2.10) that

$$NT[1,3,7,3,3](11n+6) \equiv 9M_2(11n+6) + M_4(11n+6) \pmod{11}. \tag{5.18}$$

(1.9-3) then follows from (5.18), (5.1) and (5.2). □

5.2. Crank. There are seven types of linear combinations among congruences in (1.10-1)–(1.10-11). We prove them separately.

Proof of congruences involving $M_\omega[0,0,0,1,8]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 8 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[0,0,0,1,8](n) &\equiv 4M_1^\omega(n) + 8M_3^\omega(n) + 4M_5^\omega(n) + 6M_7^\omega(n) \pmod{11} \\ &= -\frac{1}{2}\big(4M_2(n) + 8M_4(n) + 4M_6(n) + 6M_8(n)\big). \end{aligned} \tag{5.19}$$

Thus, $M_\omega[0,0,0,1,8](11n) \equiv 0 \pmod{11}$ follows from (5.19), (5.5) and (5.6).

$M_\omega[0,0,0,1,8](11n+1) \equiv 0 \pmod{11}$ follows from (5.19) and (5.7).

$M_\omega[0,0,0,1,8](11n+2) \equiv 0 \pmod{11}$ follows from (5.19) and (5.8).

$M_\omega[0,0,0,1,8](11n+3) \equiv 0 \pmod{11}$ follows from (5.19) and (5.9).

$M_\omega[0,0,0,1,8](11n+6) \equiv 0 \pmod{11}$ follows from (5.19), (5.1), (5.2), (5.3) and (5.4).

$M_\omega[0,0,0,1,8](11n+10) \equiv 0 \pmod{11}$ follows from (5.19) and (5.15). □

Proof of congruences involving $M_\omega[0,0,1,0,6]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 6 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[0,0,1,0,6](n) &\equiv 5M_1^\omega(n) + M_3^\omega(n) + 8M_5^\omega(n) + 8M_7^\omega(n) \pmod{11} \\ &= -\frac{1}{2}\big(5M_2(n) + M_4(n) + 8M_6(n) + 8M_8(n)\big). \end{aligned} \tag{5.20}$$

Thus, $M_\omega[0,0,1,0,6](11n) \equiv 0 \pmod{11}$ follows from (5.20), (5.5) and (5.6).

$M_\omega[0,0,1,0,6](11n+1) \equiv 0 \pmod{11}$ follows from (5.20) and (5.7).

$M_\omega[0,0,1,0,6](11n+2) \equiv 0 \pmod{11}$ follows from (5.20) and (5.8).

$M_\omega[0,0,1,0,6](11n+4) \equiv 0 \pmod{11}$ follows from (5.20) and (5.10).

$M_\omega[0,0,1,0,6](11n+5) \equiv 0 \pmod{11}$ follows from (5.20) and (5.11).

$M_\omega[0,0,1,0,6](11n+6) \equiv 0 \pmod{11}$ follows from (5.20), (5.1), (5.2), (5.3) and (5.4).

$M_\omega[0,0,1,0,6](11n+7) \equiv 0 \pmod{11}$ follows from (5.20) and (5.12).

$M_\omega[0,0,1,0,6](11n+9) \equiv 0 \pmod{11}$ follows from (5.20) and (5.14). □

Proof of congruences involving $M_\omega[0,0,1,2,0]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[0,0,1,2,0](n) &\equiv 2M_1^\omega(n) + 6M_3^\omega(n) + 5M_5^\omega(n) + 9M_7^\omega(n) \pmod{11} \\ &= -\frac{1}{2}\big(2M_2(n) + 6M_4(n) + 5M_6(n) + 9M_8(n)\big). \end{aligned} \tag{5.21}$$

Thus, $M_\omega[0,0,1,0,6](11n+8) \equiv 0 \pmod{11}$ follows from (5.21) and (5.13). □

Proof of congruences involving $M_\omega[0,1,0,0,4]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 4 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[0,1,0,0,4](n) &\equiv 4M_1^\omega(n) + 6M_3^\omega(n) + 5M_5^\omega(n) + 7M_7^\omega(n) \pmod{11} \\ &= -\frac{1}{2}\big(4M_2(n) + 6M_4(n) + 5M_6(n) + 7M_8(n)\big). \end{aligned} \tag{5.22}$$

Thus, $M_\omega[0,1,0,0,4](11n) \equiv 0 \pmod{11}$ follows from (5.22), (5.5) and (5.6).

$M_\omega[0,1,0,0,4](11n+1) \equiv 0 \pmod{11}$ follows from (5.22) and (5.7).

$M_\omega[0,1,0,0,4](11n+3) \equiv 0 \pmod{11}$ follows from (5.22) and (5.9).

$M_\omega[0,1,0,0,4](11n+6) \equiv 0 \pmod{11}$ follows from (5.22), (5.1), (5.2), (5.3) and (5.4).

$M_\omega[0,1,0,0,4](11n+7) \equiv 0 \pmod{11}$ follows from (5.22) and (5.12).

$M_\omega[0,1,0,0,4](11n+8) \equiv 0 \pmod{11}$ follows from (5.22) and (5.13).

$M_\omega[0,1,0,0,4](11n+9) \equiv 0 \pmod{11}$ follows from (5.22) and (5.14).

$M_\omega[0,1,0,0,4](11n+10) \equiv 0 \pmod{11}$ follows from (5.22) and (5.15). $\square$

Proof of congruences involving $M_\omega[0,1,0,5,0]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \\ 0 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[0,1,0,5,0](n) &\equiv 2M_1^\omega(n) + 2M_3^\omega(n) + 3M_5^\omega(n) + 4M_7^\omega(n) \pmod{11} \\ &= -\frac{1}{2}\big(2M_2(n) + 2M_4(n) + 3M_6(n) + 4M_8(n)\big). \end{aligned} \tag{5.23}$$

Thus, $M_\omega[0,1,0,5,0](11n+4) \equiv 0 \pmod{11}$ follows from (5.23) and (5.10).

$M_\omega[0,1,0,5,0](11n+5) \equiv 0 \pmod{11}$ follows from (5.23) and (5.11). $\square$

Proof of congruences involving $M_\omega[1,0,0,0,2]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9 \\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9 \\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9 \\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9 \\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[1,0,0,0,2](n) &\equiv 6M_1^\omega(n) + 8M_3^\omega(n) + 5M_5^\omega(n) + 4M_7^\omega(n) \pmod{11} \\ &= -\frac{1}{2}\big(6M_2(n) + 8M_4(n) + 5M_6(n) + 4M_8(n)\big). \end{aligned} \tag{5.24}$$

Thus, $M_\omega[1,0,0,0,2](11n) \equiv 0 \pmod{11}$ follows from (5.24), (5.5) and (5.6).

$M_\omega[1,0,0,0,2](11n+2) \equiv 0 \pmod{11}$ follows from (5.24) and (5.8).

$M_\omega[1,0,0,0,2](11n+3) \equiv 0 \pmod{11}$ follows from (5.24) and (5.9).

$M_\omega[1,0,0,0,2](11n+4) \equiv 0 \pmod{11}$ follows from (5.24) and (5.10).

$M_\omega[1,0,0,0,2](11n+5) \equiv 0 \pmod{11}$ follows from (5.24) and (5.11).

$M_\omega[1,0,0,0,2](11n+6) \equiv 0 \pmod{11}$ follows from (5.24), (5.1), (5.2), (5.3) and (5.4).

$M_\omega[1,0,0,0,2](11n+9) \equiv 0 \pmod{11}$ follows from (5.24) and (5.14).

$M_\omega[1,0,0,0,2](11n+10) \equiv 0 \pmod{11}$ follows from (5.24) and (5.15). $\square$

Proof of congruences involving $M_\omega[1,0,0,8,0]$. By solving

$$\begin{pmatrix} 1^1 & 1^3 & 1^5 & 1^7 & 1^9\\ 2^1 & 2^3 & 2^5 & 2^7 & 2^9\\ 3^1 & 3^3 & 3^5 & 3^7 & 3^9\\ 4^1 & 4^3 & 4^5 & 4^7 & 4^9\\ 5^1 & 5^3 & 5^5 & 5^7 & 5^9 \end{pmatrix}\begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\\ c_5\end{pmatrix} \equiv \begin{pmatrix} 1\\ 0\\ 0\\ 8\\ 0\end{pmatrix} \pmod{11},$$

we have

$$\begin{aligned} M_\omega[1,0,0,8,0](n) &\equiv 5M_1^\omega(n)+6M_3^\omega(n)+4M_5^\omega(n)+8M_7^\omega(n) \pmod{11}\\ &= -\frac{1}{2}\big(5M_2(n)+6M_4(n)+4M_6(n)+8M_8(n)\big). \end{aligned} \tag{5.25}$$

Thus, $M_\omega[1,0,0,8,0](11n+7)\equiv 0 \pmod{11}$ follows from (5.25) and (5.12).

$M_\omega[1,0,0,8,0](11n+8)\equiv 0 \pmod{11}$ follows from (5.25) and (5.13). □

6. Modulo 13

For the proofs below, we need

$$M_2(13n)\equiv 0 \pmod{13}, \tag{6.1}$$

which is deduced from the relation $M_2(n)=2np(n)$.

Also, reducing modulo 13 in [4, (5.10)], and then using the facts that $M_{14}(n)\equiv M_2(n) \pmod{13}$ and $N_{14}(n)\equiv N_2(n) \pmod{13}$ since $m^{14}\equiv m^2 \pmod{13}$, we have

$$N_2(13n+2)\equiv 10M_2(13n+2) \pmod{13}, \tag{6.2}$$

$$N_2(13n+3)\equiv 12M_2(13n+3) \pmod{13}, \tag{6.3}$$

$$N_2(13n+5)\equiv 11M_2(13n+5) \pmod{13}, \tag{6.4}$$

$$N_2(13n+6)\equiv M_2(13n+6) \pmod{13}, \tag{6.5}$$

$$N_2(13n+7)\equiv 5M_2(13n+7) \pmod{13}, \tag{6.6}$$

$$N_2(13n+9)\equiv 6M_2(13n+9) \pmod{13}, \tag{6.7}$$

$$N_2(13n+10)\equiv 7M_2(13n+10) \pmod{13}. \tag{6.8}$$

6.1. Rank.
We prove (1.11-1)–(1.11-9).

Proof. By solving (2.9), we have

$$\begin{aligned} NT[0,1,4,12,10,3](n) &\equiv -\frac{1}{2}\big(7N_2(n)+6N_4(n)+7N_6(n)+6N_8(n)\big) \pmod{13},\\ NT[1,1,6,0,0,3](n) &\equiv -\frac{1}{2}\big(5N_2(n)+11N_4(n)+2N_6(n)+9N_8(n)\big) \pmod{13},\\ NT[0,0,1,9,6,8](n) &\equiv -\frac{1}{2}\big(11N_2(n)+6N_4(n)+9N_8(n)\big) \pmod{13},\\ NT[1,0,3,9,1,11](n) &\equiv -\frac{1}{2}\big(7N_4(n)+11N_6(n)+9N_8(n)\big) \pmod{13},\\ NT[1,5,8,7,12,12](n) &\equiv -\frac{1}{2}\big(7N_2(n)+5N_4(n)+9N_6(n)+6N_8(n)\big) \pmod{13},\\ NT[1,2,8,0,7,11](n) &\equiv -\frac{1}{2}\big(5N_2(n)+3N_4(n)+4N_6(n)+2N_8(n)\big) \pmod{13}, \end{aligned}$$

$$NT[1,12,8,7,10,7](n) \equiv -\frac{1}{2}\big(12N_2(n)+5N_4(n)+9N_6(n)+N_8(n)\big) \pmod{13},$$
$$NT[1,6,11,8,0,0](n) \equiv -\frac{1}{2}\big(11N_2(n)+10N_4(n)+9N_6(n)+10N_8(n)\big) \pmod{13},$$
$$NT[1,9,4,5,10,7](n) \equiv -\frac{1}{2}\big(10N_2(n)+11N_6(n)+6N_8(n)\big) \pmod{13}.$$

Substituting (2.10), (2.11) and (2.12) into the above and then reducing modulo 13, we have

$$NT[0,1,4,12,10,3](13n) \equiv 5M_2(13n) \pmod{13}, \tag{6.9}$$
$$NT[1,1,6,0,0,3](13n+1) \equiv 0 \pmod{13}, \tag{6.10}$$
$$NT[0,0,1,9,6,8](13n+2) \equiv 11M_2(13n+2)+8N_2(13n+2) \pmod{13}, \tag{6.11}$$
$$NT[1,0,3,9,1,11](13n+3) \equiv 4M_2(13n+3)+4N_2(13n+3) \pmod{13}, \tag{6.12}$$
$$NT[1,5,8,7,12,12](13n+5) \equiv 3M_2(13n+5)+8N_2(13n+5) \pmod{13}, \tag{6.13}$$
$$NT[1,2,8,0,7,11](13n+6) \equiv 11M_2(13n+6)+2N_2(13n+6) \pmod{13}, \tag{6.14}$$
$$NT[1,12,8,7,10,7](13n+7) \equiv 6M_2(13n+7)+4N_2(13n+7) \pmod{13}, \tag{6.15}$$
$$NT[1,6,11,8,0,0](13n+9) \equiv M_2(13n+9)+2N_2(13n+9) \pmod{13}, \tag{6.16}$$
$$NT[1,9,4,5,10,7](13n+10) \equiv 8M_2(13n+10)+10N_2(13n+10) \pmod{13}. \tag{6.17}$$

(1.11-1) follows from (6.9) and (6.1).
(1.11-2) is (6.10).
(1.11-3) follows from (6.11) and (6.2).
(1.11-4) follows from (6.12) and (6.3).
(1.11-5) follows from (6.13) and (6.4).
(1.11-6) follows from (6.14) and (6.5).
(1.11-7) follows from (6.15) and (6.6).
(1.11-8) follows from (6.16) and (6.7).
(1.11-9) follows from (6.17) and (6.8). □

6.2. Crank. We prove (1.12-1).

Proof. By solving (2.9), we have

$$M_\omega[1,2,3,4,5,6](n) \equiv -\frac{1}{2}M_2(n) \pmod{13}. \tag{6.18}$$

(1.12-1) follows from (6.18) and (6.1). □

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References

1. G. E. Andrews, *The theory of partitions*, Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.

2. G. E. Andrews, The Ramanujan–Dyson identities and George Beck's congruence conjectures, *Int. J. Number Theory* **17** (2021), no. 2, 239–249.
3. G. E. Andrews and F. G. Garvan, Dyson's crank of a partition, *Bull. Amer. Math. Soc. (N.S.)* **18** (1988), no. 2, 167–171.
4. A. O. L. Atkin and F. G. Garvan, Relations between the ranks and cranks of partitions, *Ramanujan J.* **7** (2003), no. 1-3, 343–366.
5. A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc. (3)* **4** (1954), 84–106.
6. S. H. Chan, R. Mao, and R. Osburn, Variations of Andrews–Beck type congruences, *J. Math. Anal. Appl.* **495** (2021), no. 2, 124771, 14 pp.
7. S. Chern, Weighted partition rank and crank moments. I. Andrews–Beck type congruences, *Proceedings of the Conference in Honor of Bruce Berndt*, to appear.
8. S. Chern, Weighted partition rank and crank moments. II. Odd-order moments, *Ramanujan J.*, in press.
9. F. J. Dyson, Some guesses in the theory of partitions, *Eureka* **8** (1944), 10–15.

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