To appear in *Bull. Korean Math. Soc.*

Further results on biases in integer partitions

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Abstract. Let $p_{a,b,m}(n)$ be the number of integer partitions of n with more parts congruent to a modulo m than parts congruent to b modulo m. We prove that $p_{a,b,m}(n) \geq p_{b,a,m}(n)$ whenever $1 \leq a < b \leq m$. We also propose some conjectures concerning series with nonnegative coefficients in their expansions.

Keywords. Integer partition, bias, generating function, nonnegativity.

2010MSC. 05A17, 11P81.

1. Introduction

In analogy to *Chebyshev's bias* [3] concerning the excess of the number of primes of the form $4k+3$ over the number of primes of the form $4k+1$, B. Kim, E. Kim and Lovejoy [5] introduced a phenomenon called *parity bias* for integer partitions.

Theorem 1.1 (B. Kim, E. Kim and Lovejoy)**.** *Let $p_o(n)$ (resp. $p_e(n)$) denote the number of integer partitions of n with more odd parts than even parts (resp. with more even parts than odd parts). Then*

$$p_o(n) \geq p_e(n).$$

This phenomenon is called "parity bias" for integer partitions.

Recently, B. Kim and E. Kim [4] went on to investigate this phenomenon in a more general setting. Let us first adopt their notation.

Definition 1.1. We denote by $p_{a,b,m}(n)$ the number of partitions of n with more parts congruent to a modulo m than parts congruent to b modulo m.

Making use of the above notation, we have $p_o(n) = p_{1,2,2}(n)$ and $p_e(n) = p_{2,1,2}(n)$ and therefore arrive at the inequality $p_{1,2,2}(n) \geq p_{2,1,2}(n)$ from Theorem 1.1. Similar phenomena shown in [4] also include inequalities as follows.

Theorem 1.2 (B. Kim and E. Kim)**.** *Let $m \geq 2$ be an integer. Then*

$$p_{1,m,m}(n) \geq p_{m,1,m}(n),$$
$$p_{1,m-1,m}(n) \geq p_{m-1,1,m}(n).$$

Our object here is to extend the above results for general $p_{a,b,m}(n)$.

Theorem 1.3. *Let $m \geq 2$ be an integer. For any two integers a and b with $1 \leq a < b \leq m$, we have*

$$p_{a,b,m}(n) \geq p_{b,a,m}(n). \tag{1.1}$$

We separate this theorem into two cases. First, we prove the case $(a,b) \neq (1,2)$ using q-series manipulations. Then we provide an injective proof for $(a,b) = (1,2)$.

2. Case $(a,b)\neq(1,2)$

Let us first recall the notation of q-Pochhammer symbols: for $n\in\mathbb{N}\cup\{\infty\}$,

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k),$$
$$(A_1,A_2,\ldots,A_m;q)_n := (A_1;q)_n(A_2;q)_n\cdots(A_m;q)_n.$$

Next, given an integer partition λ, we denote by $|\lambda|$ the sum of parts in λ and by $\sharp_{a,m}(\lambda)$ the number of parts in λ that are congruent to a modulo m. Let $\mathscr{P}$ be the set of integer partitions.

Our starting point is the following trivial trivariate generating function:

$$\sum_{\lambda\in\mathscr{P}} x^{\sharp_{a,m}(\lambda)}y^{\sharp_{b,m}(\lambda)}q^{|\lambda|} = \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty}\frac{1}{(xq^a,yq^b;q^m)_\infty}, \tag{2.1}$$

provided that $1\le a,b\le m$ and $a\neq b$.

We are then led to the following lemma.

Lemma 2.1. *Let $1\le a,b\le m$ and $a\neq b$. We have*

$$\sum_{n\ge0} p_{a,b,m}(n)q^n = \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty}\sum_{\substack{i,j\ge0\\ i>j}}\frac{q^{ai+bj}}{(q^m;q^m)_i(q^m;q^m)_j}. \tag{2.2}$$

Proof. Recall Euler's first identity [2, p. 19, (2.2.5)]:

$$\frac{1}{(z;q)_\infty} = \sum_{n\ge0}\frac{z^n}{(q;q)_n}. \tag{2.3}$$

Setting $y=x^{-1}$ in (2.1) yields

$$\begin{aligned}\sum_{\lambda\in\mathscr{P}} x^{\sharp_{a,m}(\lambda)-\sharp_{b,m}(\lambda)}q^{|\lambda|} &= \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty}\frac{1}{(xq^a,x^{-1}q^b;q^m)_\infty}\\ &= \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty}\sum_{i\ge0}\frac{x^iq^{ai}}{(q^m;q^m)_i}\sum_{j\ge0}\frac{x^{-j}q^{bj}}{(q^m;q^m)_j}\\ &\qquad\text{(by using (2.3) twice)}\\ &= \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty}\sum_{i,j\ge0}\frac{x^{i-j}q^{ai+bj}}{(q^m;q^m)_i(q^m;q^m)_j}.\end{aligned}$$

Noticing that $p_{a,b,m}(n)$ counts the number of partitions λ of n such that $\sharp_{a,m}(\lambda)>\sharp_{b,m}(\lambda)$, we must single out terms in the above with positive exponents in x and therefore terms with $i-j>0$. The desired result immediately follows. □

Now, we are in a position to prove Theorem 1.3 for $(a,b)\neq(1,2)$.

Proof of Theorem 1.3 for $(a,b)\neq(1,2)$. Recall that $1\le a<b\le m$. The following is a simple consequence of Lemma 2.1:

$$\sum_{n\ge0}\big(p_{a,b,m}(n)-p_{b,a,m}(n)\big)q^n$$

$$= \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty} \sum_{\substack{i,j\ge 0\\ i>j}} \left(\frac{q^{ai+bj}}{(q^m;q^m)_i(q^m;q^m)_j} - \frac{q^{bi+aj}}{(q^m;q^m)_i(q^m;q^m)_j} \right)$$
$$= \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty} \sum_{\substack{i,j\ge 0\\ i>j}} \frac{q^{ai+bj}(1-q^{a(j-i)+b(i-j)})}{(q^m;q^m)_i(q^m;q^m)_j}$$
$$= \frac{(q^a,q^b;q^m)_\infty}{(q;q)_\infty} \sum_{j\ge 0}\sum_{k\ge 1} \frac{q^{a(j+k)+bj}(1-q^{(b-a)k})}{(q^m;q^m)_j(q^m;q^m)_{j+k}}.$$

We then consider two subcases.

Subcase I. $a \neq 1$. Noticing that $(b-a)k$ is always a positive integer, we may factor $1-q^{(b-a)k}$ as $(1-q)(1+q+q^2+\cdots q^{(b-a)k-1})$. Thus,

$$\sum_{n\ge 0}\big(p_{a,b,m}(n)-p_{b,a,m}(n)\big)q^n$$
$$= \frac{(1-q)(q^a,q^b;q^m)_\infty}{(q;q)_\infty} \sum_{j\ge 0}\sum_{k\ge 1} \frac{q^{a(j+k)+bj}(1+q+q^2+\cdots q^{(b-a)k-1})}{(q^m;q^m)_j(q^m;q^m)_{j+k}}.$$

Apparently, the Taylor expansion of the double series in the above has nonnegative coefficients. For the infinite product in the above, we have, as $2 \le a < b \le m$,

$$\frac{(1-q)(q^a,q^b;q^m)_\infty}{(q;q)_\infty} = \frac{(q^a,q^b;q^m)_\infty}{(q^2;q)_\infty},$$

which also has nonnegative coefficients in its series expansion. We therefore conclude that $p_{a,b,m}(n) \ge p_{b,a,m}(n)$ for $a \neq 1$.

Subcase II. $a = 1$ and $b \neq 2$. We have

$$\sum_{n\ge 0}\big(p_{1,b,m}(n)-p_{b,1,m}(n)\big)q^n$$
$$= \frac{(q,q^b;q^m)_\infty}{(q;q)_\infty} \sum_{j\ge 0}\sum_{k\ge 1} \frac{q^{(j+k)+bj}(1-q^{(b-1)k})}{(q^m;q^m)_j(q^m;q^m)_{j+k}}.$$

Notice that $b > a = 1$. This time we should factor $1-q^{(b-1)k}$ as $(1-q^{b-1})(1+q^{b-1}+\cdots q^{(b-1)(k-1)})$. Thus,

$$\sum_{n\ge 0}\big(p_{1,b,m}(n)-p_{b,1,m}(n)\big)q^n$$
$$= \frac{(1-q^{b-1})(q,q^b;q^m)_\infty}{(q;q)_\infty} \sum_{j\ge 0}\sum_{k\ge 1} \frac{q^{(j+k)+bj}(1+q^{b-1}+\cdots q^{(b-1)(k-1)})}{(q^m;q^m)_j(q^m;q^m)_{j+k}}.$$

Similarly, the double series in the above can be expanded as a nonnegative series in q. Also, as $b \neq 2$, we have $1 < b-1 < b \le m$. This implies that the infinite product part in the above is also a nonnegative series in q. Therefore, $p_{1,b,m}(n) \ge p_{b,1,m}(n)$ for $b \neq 2$. □

3. Case $(a,b)=(1,2)$

When $(a,b)=(1,2)$, it looks like a q-theoretic proof is painfully difficult. Therefore, we consider this case in a combinatorial manner. First, for $d\in\mathbb{Z}$, we define

$$\mathscr{P}_d(n)=\mathscr{P}_d^{(m)}(n):=\big\{\lambda\in\mathscr{P}:|\lambda|=n \text{ and } \sharp_{1,m}(\lambda)-\sharp_{2,m}(\lambda)=d\big\}.$$

Then

$$p_{1,2,m}(n)=\sum_{d\geq 1}\operatorname{card}\mathscr{P}_d(n), \tag{3.1}$$

$$p_{2,1,m}(n)=\sum_{d\geq 1}\operatorname{card}\mathscr{P}_{-d}(n). \tag{3.2}$$

Our object is to show the following inequalities, from which our desired result $p_{1,2,m}(n)\geq p_{2,1,m}(n)$ follows as a direct consequence if we make use of the above two relations.

Theorem 3.1. *Let $m\geq 3$ be an integer. For $k\geq 0$,*

$$\operatorname{card}\mathscr{P}_{-(km+1)}(n)\leq \operatorname{card}\mathscr{P}_{km+2}(n), \tag{3.3}$$

$$\operatorname{card}\mathscr{P}_{-(km+2)}(n)\leq \operatorname{card}\mathscr{P}_{km+1}(n), \tag{3.4}$$

$$\operatorname{card}\mathscr{P}_{-(km+r)}(n)\leq \operatorname{card}\mathscr{P}_{km+r}(n), \tag{3.5}$$

where $3\leq r\leq m$ in the third inequality.

Proof. We simply construct injections $\mathscr{P}_{-d}(n)\hookrightarrow\mathscr{P}_{d^*}(n)$ for $d=km+r>0$ with $1\leq r\leq m$ and

$$d^*=\begin{cases}km+2 & \text{if } r=1,\\ km+1 & \text{if } r=2,\\ km+r & \text{if } 3\leq r\leq m.\end{cases}$$

Given any partition λ, we start with the following process.

Process (I). We replace any part in λ that is congruent to 1 modulo m, say $um+1$, by $um+2$ and replace any part in λ that is congruent to 2 modulo m, say $vm+2$, by $vm+1$. The resulting partition is called λ^*.

Now, if $\lambda\in\mathscr{P}_{-d}(n)$, then $\sharp_{1,m}(\lambda)-\sharp_{2,m}(\lambda)=-d$. Also, trivially,

$$|\lambda^*|=|\lambda|-d=n-d.$$

Thus, to arrive at a partition of size n, we need to append some additional parts that sum to d. We have three subcases.

Subcase I. $3\leq r\leq m$. Recall that $d=km+r$. We append a part of size d to λ^* and call the new partition λ^{**}. Since $d\not\equiv 1,2 \pmod m$, we have

$$\begin{aligned}\sharp_{1,m}(\lambda^{**})-\sharp_{2,m}(\lambda^{**})&=\sharp_{1,m}(\lambda^{*})-\sharp_{2,m}(\lambda^{*})\\ &=\sharp_{2,m}(\lambda)-\sharp_{1,m}(\lambda) \qquad \text{(by Process (I))}\\ &=-(-d)\\ &=d^*.\end{aligned}$$

Thus, $\lambda^{**}\in\mathscr{P}_{d^*}(n)$.

Subcase II. $r = 1$. Recall that $d = km + 1$. We append a part of size 1 and a part of size km to λ^* and call the new partition λ^{**}. Notice that $km \equiv 0 \not\equiv 1, 2 \pmod{m}$ for $m \geq 3$. Thus,

$$\begin{aligned}
\sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \big(1 + \sharp_{1,m}(\lambda^*)\big) - \sharp_{2,m}(\lambda^*) \\
&= 1 + \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) && \text{(by Process (I))} \\
&= 1 - (-d) \\
&= km + 2 \\
&= d^*,
\end{aligned}$$

which implies that $\lambda^{**} \in \mathscr{P}_{d^*}(n)$.

Subcase III. $r = 2$. Recall that $d = km + 2$. We append a part of size 2 and a part of size km to λ^* and call the new partition λ^{**}. We also have $km \equiv 0 \not\equiv 1, 2 \pmod{m}$ for $m \geq 3$. Thus,

$$\begin{aligned}
\sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \sharp_{1,m}(\lambda^*) - \big(1 + \sharp_{2,m}(\lambda^*)\big) \\
&= -1 + \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) && \text{(by Process (I))} \\
&= -1 - (-d) \\
&= km + 1 \\
&= d^*,
\end{aligned}$$

and therefore, $\lambda^{**} \in \mathscr{P}_{d^*}(n)$.

Lastly, it is straightforward to verify that the map $\lambda \mapsto \lambda^{**}$ is injective. □

Proof of Theorem 1.3 for $(a, b) = (1, 2)$. For $m = 2$, see Theorem 1.1 due to B. Kim, E. Kim and Lovejoy. For $m \geq 3$, we have

$$\begin{aligned}
p_{2,1,m}(n) &= \sum_{d \geq 1} \operatorname{card} \mathscr{P}_{-d}(n) && \text{(by (3.2))} \\
&= \sum_{k \geq 0} \operatorname{card} \mathscr{P}_{-(km+1)}(n) + \sum_{k \geq 0} \operatorname{card} \mathscr{P}_{-(km+2)}(n) \\
&\quad + \sum_{3 \leq r \leq m} \sum_{k \geq 0} \operatorname{card} \mathscr{P}_{-(km+r)}(n) \\
&\leq \sum_{k \geq 0} \operatorname{card} \mathscr{P}_{km+2}(n) + \sum_{k \geq 0} \operatorname{card} \mathscr{P}_{km+1}(n) \\
&\quad + \sum_{3 \leq r \leq m} \sum_{k \geq 0} \operatorname{card} \mathscr{P}_{km+r}(n) && \text{(by Theorem 3.1)} \\
&= \sum_{d \geq 1} \operatorname{card} \mathscr{P}_{d}(n) \\
&= p_{1,2,m}(n). && \text{(by (3.1))}
\end{aligned}$$

This is exactly what we need. □

4. Closing remarks

Following Section 2, the case $(a,b)=(1,2)$ of Theorem 1.3 is equivalent to the nonnegativity of

$$\frac{(q,q^2;q^m)_\infty}{(q;q)_\infty}\sum_{j\ge 0}\sum_{k\ge 1}\frac{q^{3j+k}(1-q^k)}{(q^m;q^m)_j(q^m;q^m)_{j+k}}, \tag{4.1}$$

that is, its series expansion has nonnegative coefficients. Although we do not find a q-theoretic proof of this fact, our numerical calculations indicate the following conjecture.

Conjecture 4.1. For $m\ge 2$, the double series

$$\sum_{j\ge 0}\sum_{k\ge 1}\frac{q^{3j+k}(1-q^k)}{(q^m;q^m)_j(q^m;q^m)_{j+k}} \tag{4.2}$$

has nonnegative coefficients in its expansion.

Notice that

$$\sum_{j\ge 0}\sum_{k\ge 1}\frac{q^{3j+k}(1-q^k)}{(q^m;q^m)_j(q^m;q^m)_{j+k}}=\sum_{j\ge 0}\frac{q^{3j}}{(q^m;q^m)_j(q^m;q^m)_j}\sum_{k\ge 0}\frac{q^k(1-q^k)}{(q^{(j+1)m};q^m)_k}.$$

Regarding the inner series, we also have a more surprising conjecture.

Conjecture 4.2. For $m,s\ge 1$,

$$\sum_{k\ge 0}\frac{q^k(1-q^k)}{(q^s;q^m)_k} \tag{4.3}$$

has nonnegative coefficients in its expansion.

Here the case $s=m$ is to some extent easier.

Proof of Conjecture 4.2 for $s=m$. We have

$$\begin{aligned}\sum_{k\ge 0}\frac{q^k(1-q^k)}{(q^m;q^m)_k}&=\sum_{k\ge 0}\frac{q^k}{(q^m;q^m)_k}-\sum_{k\ge 0}\frac{q^{2k}}{(q^m;q^m)_k}\\&=\frac{1}{(q;q^m)_\infty}-\frac{1}{(q^2;q^m)_\infty}\qquad\text{(by (2.3))}\\&=\sum_{n\ge 0}\rho_{1,m}(n)q^n-\sum_{n\ge 0}\rho_{2,m}(n)q^n,\end{aligned}$$

where for $i=1$ or 2, we denote by $\rho_{i,m}(n)$ the number of partitions of n with parts of the form $km+i$ with $k\ge 0$.

Now we recall a result due to Andrews [1, Theorem 3]:

Let $S=\{a_i\}_{i\ge 1}$ and $T=\{b_i\}_{i\ge 1}$ be two strictly increasing sequences of positive integers such that $b_1=1$ and $a_i\ge b_i$ for all i. Then for any $n\ge 0$,

$$\rho_T(n)\ge\rho_S(n),$$

where $\rho_S(n)$ (resp. $\rho_T(n)$) denotes the number of partitions of n into parts taken from S (resp. T).

By the above theorem, we immediately have $\rho_{1,m}(n)\ge\rho_{2,m}(n)$ for all n. Thus, (4.3) is a nonnegative series in q when $s=m$. □

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