1 Introduction

My main interest is in dynamical systems and ergodic theory. The bulk of the work I did during graduate school involved extending aspects of the *AbC* method (also known as the *Approximation by Conjugation* method or the *Anosov-Katok* method) to the realm of real-analytic diffeomorphisms on the torus. Very recently, following the landmark works of Belezney, Foreman, Hjorth, Rudolph and Weiss, I developed an interest in working on the interface of ergodic theory and foundations of mathematics.

For the rest of the introduction I will give a brief description of real-analytic diffeomorphisms on the torus and the AbC method which forms the foundations for my work.

1.1 Real-analytic diffeomorphisms on the torus

A function defined on a subset of the real line is called real-analytic at a point if the Taylor series can be defined at that point and it converges to the value of the function at the point. Equivalently, if we think of the natural inclusion $x \mapsto (x + i0)$ of the real line $\mathbb{R}$ in the complex plane $\mathbb{C}$, a function defined on a subset of the real line is real-analytic at a point if there exists an open neighborhood of this point in $\mathbb{C}$ where this function can be extended as a holomorphic function. We can extended this concept of real-analytic maps on $\mathbb{R}$ to maps on $\mathbb{R}^n$ and subsequently we can define further concepts like real-analytic manifolds, real-analytic maps and real-analytic diffeomorphisms on real-analytic manifolds in a canonical way.

Measure preserving real-analytic diffeomorphisms on the torus $\mathbb{T}^d$ which are homotopic to the identity has a certain description which makes it easier to work with them. Any real-analytic diffeomorphism on $\mathbb{T}^d$ homotopic to the identity admits a lift to a map $F : \mathbb{R}^d \to \mathbb{R}^d$ and this map has the form $F(x_1, \dots, x_d) = (x_1 + f_1(x_1, \dots, x_d), \dots, x_d + f_d(x_1, \dots, x_d))$, where $f_i : \mathbb{R}^d \to \mathbb{R}$ are $\mathbb{Z}^d$-periodic real-analytic functions.

Any real-analytic $\mathbb{Z}^d$-periodic function on $\mathbb{R}^d$ can be extended as a holomorphic function using the natural inclusion $(x_1, \dots, x_d) \mapsto (x_1 + i0, \dots, x_d + i0)$ of $\mathbb{R}^d$ in $\mathbb{C}^d$. For a fixed $\rho > 0$, let $\Omega_\rho := \{(z_1, \dots, z_d) \in \mathbb{C}^d : |\text{Im}(z_i)| < \rho\}$ and for a function f defined on this set, define $\|f\|_\rho := \sup_{\Omega_\rho} |f((z_1, \dots, z_d))|$. We define $C^\omega_\rho(\mathbb{T}^d)$ to be the space of all $\mathbb{Z}^d$-periodic real-analytic functions on $\mathbb{R}^d$ that extends to a holomorphic function on Ω_ρ and $\|f\|_\rho < \infty$. Now we define, $\text{Diff}^{\,\omega}_{\,\rho}(\mathbb{T}^d, \mu)$ to be the set of all measure preserving real-analytic diffeomorphisms of $\mathbb{T}^d$ homotopic to the identity, whose lift F to $\mathbb{R}^d$ satisfies $f_i \in C^\omega_\rho(\mathbb{T}^d)$. The metric in $\text{Diff}^{\,\omega}_{\,\rho}(\mathbb{T}^d, \mu)$ is defined by $d_\rho(F, G) = \max_{i=1,\dots,d}\{\inf_{n\in\mathbb{Z}} \|f_i - g_i + n\|_\rho\}$.

This completes the description of the topology necessary for our construction. From now on the word "diffeomorphism" will refer to a real-analytic diffeomorphism and the word "analytic topology" will refer to the topology of $\text{Diff}^{\,\omega}_{\,\rho}(\mathbb{T}^d, \mu)$ described above. See [4] for a more extensive treatment of these spaces.

1.2 The AbC method in ergodic theory

The *AbC* method, short for the *approximation by conjugation* method, also known as the *Anosov-Katok* method was introduced by D.V.Anosov and A.Katok in their landmark work titled 'New examples in smooth ergodic theory. Ergodic diffeomorphisms' (see [1]) nearly fifty years ago. For a relatively upto date description of this method and its usefulness, one might refer to [5].

We now give a very high level description of this method customized for the d dimensional torus. Let ϕ, be a measure preserving action of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ defined as follows:

$$\phi^t\big((x_1, x_2, \dots, x_d)\big) = (x_1 + t, x_2, \dots, x_d)$$

We construct a sequence of periodic measure preserving diffeomorphisms $\{T_n\}_{n\in\mathbb{N}}$ defined by the conjugacy $T_n := H_n^{-1} \circ \phi^{\alpha_n} \circ H_n$, where $\alpha_n \in \mathbb{Q} \cap [0, 1)$ and $H_n \in \text{Diff}^{\,\omega}_{\,\rho}(\mathbb{T}^2)$.

The diffeomorphisms H_n and the rationals α_n s are constructed inductively. Given H_n and α_n, we construct at the $n + 1$ th stage, a measure preserving diffeomorphism h_{n+1} which commutes with ϕ^{1/q_n} and define $H_{n+1} = h_{n+1} \circ H_n$. The rational α_{n+1} is defined at the end of the $n + 1$ th stage and we would require it to be very close to α_n in order to ensure the convergence of our construction in the real-analytic topology.

This construction is done in a way so that the limiting diffeomorphism satisfies certain dynamical properties. In order to achieve this, one often defines a finite version of the property and requires h_{n+1} to satisfy the same.

2 Present research

My present work involves applying the AbC method to category of real-analytic diffeomorphisms on the torus. In the smooth category, on can apply this method to arbitrary manifolds that admit an effective circle action. But in the real-analytic category this technology is much more difficult to apply and I developed some technology which helps to deal with the problems on the torus.

2.1 Block-slide type maps and their approximations

Here we talk a little about the main technicality behind all the constructions. If a reader is primarily interested in the final results obtained by these techniques, then its is safe to skip this subsection and move to subsection 2.2.

The main reason we are able to do real-analytic AbC construction on the torus is the fact that step functions can be 'well approximated' by real-analytic functions outside a set of small measure. This allows us to construct a large class of measure preserving diffeomorphisms on the torus. A *step function* on the unit interval is a finite linear combination of indicator function on subintervals. Define for $1 \leq i, j \leq d$ and $i \neq j$, the following piecewise continuous map on the d dimensional torus,

$$\mathfrak{h} : \mathbb{T}^d \to \mathbb{T}^d \qquad \text{defined by} \qquad \mathfrak{h}(x_1, \ldots, x_d) := (x_1, \ldots, x_i + s(x_j) \mod 1, \ldots, x_d)$$

where s is a step function on the unit interval. We refer to any finite composition of maps of the above kind as a *block-slide* type of map on the torus. The nomenclature is motivated from the fact that a finite composition of maps of the above kind has the effect of moving solid blocks on the torus (like a rubiks cube or a game of nine).

Next we note that step function can be well approximated by periodic function with enough analyticity.

Lemma. *Let k and N be two positive integer and $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in [0, 1)^k$. Consider a step function of the form*

$$\tilde{s}_{\alpha,N} : [0, 1) \to \mathbb{R} \quad \textit{defined by} \quad \tilde{s}_{\alpha,N}(x) = \sum_{i=0}^{kN-1} \tilde{\alpha}_i \chi_{[\frac{i}{kN}, \frac{i+1}{kN})}(x)$$

Here $\tilde{\alpha}_i := \alpha_j$ where $j := i \mod k$. Then, given any $\varepsilon > 0$ and $\delta > 0$, there exists a $\frac{1}{N}$-periodic function $s_{\alpha,N} : \mathbb{R} \to \mathbb{R}$ which can be extended to the whole complex plane as a holomorphic function and additionally it satisfies:

$$\sup_{x \in [0,1) \setminus F} |s_{\alpha,N}(x) - \tilde{s}_{\alpha,N}(x)| < \varepsilon, \qquad \sup_{x \in [0,1) \setminus F} |s'_{\alpha,N}(x)| < \varepsilon, \qquad s_{\alpha,N}(z + n) = s_{\alpha,N}(z) \quad \forall\, z \in \mathbb{C} \textit{ and } n \in \mathbb{Z}$$

Where $F = \cup_{i=0}^{kN-1} I_i \subset [0, 1)$ is a union of intervals centred around $\frac{i}{kN}$, $i = 1, \ldots, kN - 1$ and $I_0 = [0, \frac{\delta}{2kN}] \cup [1 - \frac{\delta}{2kN}, 1)$ and $\lambda(I_i) = \frac{\delta}{kN}\ \forall\, i$.

This allows us to conclude that any block-slide type of map can be well approximated by measure preserving diffeomorphisms with enough analyticity. We note that these ideas are motivated from similar ones arising in [2] in the symplectic category.

Proposition. *Let $\mathfrak{h} : \mathbb{T}^d \to \mathbb{T}^d$ be a block-slide type of map which commutes with $\phi^{1/q}$ for some positive integer q. Then for any $\varepsilon > 0$ and $\delta > 0$ there exists a measure preserving real-analytic diffeomorphim $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to the identity such that the following conditions are satisfied:*

1. *Analyticity: The lift $H : \mathbb{R}^d \to \mathbb{R}^d$ of $h : \mathbb{T}^d \to \mathbb{T}^d$ can be exended to an analytic function $H : \mathbb{C}^d \to \mathbb{C}^d$.*
2. *Commuting property: $h \circ \phi^{1/q} = \phi^{1/q} \circ h$*
3. *Proximity: There exists a set $E \subset \mathbb{T}^d$ such that $\mu(E) < \delta$ and $\sup_{x \in \mathbb{T}^d \setminus E} \|h(x) - \mathfrak{h}(x)\| < \varepsilon$.*

2.2 Non-standard real-analytic realization of some circle rotations

The first result I obtained using the kind of approximation talked about in the previous subsection was a real-analytic non-standard realization theorem. Non standard realization results demonstrates that certain dynamical systems can live on a 'non native' manifold with enough regularity. Anosov and Katok in [1] showed that certain circle rotations live on manifolds admitting an effective circle action as smooth ergodic dynamical systems. I was able to prove a real-analytic counterpart of this result for the torus.

Theorem 1 (S.B. [9])**.** *For any $\rho > 0$ and any integer $d \geq 1$, there exists measure preserving real-analytic diffeomorphisms $T \in \text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^d\, \mu)$ which are measure theoretically isomorphic to some irrational rotations of the circle.*

This theorem can be improved further if we restrict our attention to the two dimensional torus. In some modern versions of the AbC method (see [5], [7]), at each stage of the conjugacy one can be careful with the combinatorics and ensure that control is maintained over every orbit of the ϕ action most of the time. This way one can obtain results like unique ergodicity. In the the analytic set-up this means the following theorem.

Theorem 2 (S.B. [9])**.** *For any $\rho > 0$, there exists measure preserving uniquely ergodic real-analytic diffeomorphisms $T \in \text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^2, \mu)$ which are measure theoretically isomorphic to some irrational rotations of the circle.*

In a joint work with P.Kunde (see [11]) we also obtain an estimated version of the real-analytic AbC method on the torus. In the smooth category one can show that all Liouvillian circle rotations can be realized ([7]), but, in the analytic category our results are much weaker.

2.3 Twisted version of real-analytic AbC method

As a natural next step we examine if one can obtain more complicated version of the AbC method. In particular one can ask if the 'untwisted' version of the AbC method can be done on the torus using these block-slide type maps. The answer is yes, we can do some untwisted constructions real-analytically on the torus. As a consequence we obtain (jointly with P.Kunde) the following result:

Theorem 3 (S.B., P.Kunde [11])**.** *For any $\rho > 0$ and any two integers, $h \geq 1$ and $d \geq 2$, there exists measure preserving ergodic real-analytic diffeomorphisms $T \in \text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^d)$ which is measure theoretically isomorphic to some ergodic translation of $\mathbb{T}^h$.*

And the obvious corollary follows,

Corollary 1. *For any $\rho > 0$ and any two integers $h > 0$ and $d \geq 2$, there exists an ergodic real-analytic diffeomorphism $T \in \text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^d)$ such that T has a discrete spectrum generated (over $\mathbb{Z}$) by h linearly independent eigenvalues.*

I would mention at this point that weak mixing measure preserving real-analytic AbC diffeomorphisms were first obtained by B.Fayad and M.Saprykina using Fubini type lemmas ([6]). Existence of such diffeomorphisms also follows from the general twisted construction we do here. In fact our techniques can be used to obtain more and P.Kunde was able to prove the existence of measure preserving weak mixing real-analytic AbC diffeomorphisms that preserves a measurable Riemannian metric (see [10]).

2.4 Coding untwisted AbC diffeomorphisms and the anti-classification problem

This work is motivated from a series of pioneering work done by Beleżney, Foreman, Hjorth, Rudolph and Weiss on the interface of ergodic theory and foundations of mathematics. They were able to show that the conjugacy problem in abstract ergodic theory is non Borel. Later Foreman and Weiss found a method to code a 'large' class smooth diffeomorphisms constructed on $\mathbb{T}^2$ or the annulus or the disk by an untwisted version of the AbC method into some symbolic systems known as *uniform circular systems*. This in particular shows that the measure isomorphism relation among pairs (S, T) of measure preserving diffeomorphisms of M is not a Borel set with respect to the C^∞ topology.

We were able to show that the constructions we do in the real-analytic category on $\mathbb{T}^2$ are robust enough to construct a large family of untwisted AbC diffeomorphisms measure theoretically isomorphic to uniform circular systems. Loosely this can be summarized into the following theorem:

Theorem 4 (S.B. [12])**.** *Let T be an ergodic transformation on a standard measure space. Then the following are equivalent:*

1. *T is measure theoretically isomorphic to a real-analytic (untwisted) AbC diffeomorphism (satisfying some requirements).*
2. *T is isomorphic to a uniform circular system (with 'fast' growing parameters).*

This along with some additional works of Foreman and Weiss would imply an anti-classification result for measure preserving real-analytic diffeomorphisms. More precisely,

Theorem 5. *The measure-isomorphism relation among pairs $(S, T) \in \text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^2, \mu) \times \text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^2, \mu)$ is not a Borel set with respect to the $\text{Diff}^{\,\omega}_{\rho}(\mathbb{T}^2, \mu)$ topology.*

3 Future work

There is a whole array of directions in which my present theme of work may be continued. I give a list of some problems with varying levels of difficulties that I am planning to think about in the future.

3.1 Real-analytic AbC diffeomorphisms on odd dimensional spheres

The success with real-analytic constructions on the torus made me highly interested in exploring how one can generalize such results obtained to other manifolds. As of date, odd dimensional spheres forms the only other class of manifolds where a real-analytic AbC method has been carried out with some success. Recently B.Fayad and A.Katok (see [5]) showed that there exists measure preserving real-analytic uniquely ergodic AbC diffeomorphisms on odd dimensional spheres.

For odd spheres like $\mathbb{S}^3$ we note that it sits naturally inside $\mathbb{C}^2$ and hence after borrowing coordinates from $\mathbb{C}^2$ we define $\mathbb{S}^3 := \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ and the underlying circle action is the Hopf fibration which is free and effective:

$$\phi^t : \mathbb{S}^3 \to \mathbb{S}^3, \qquad \text{defined by} \qquad \phi^t(z_1, z_2) = e^{2\pi i t}(z_1, z_2)$$

$\mathbb{S}^3$ naturally comes with a large collection of well behaved real-analytic flows. For example one can easily check that the collection of the following three flows:

$$\begin{aligned}&\xi_1^t : (z_1, z_2) \mapsto (e^{2\pi i t} z_1, z_2), \qquad \xi_2^t : (z_1, z_2) \mapsto (z_1, e^{2\pi i t} z_2),\\ &\tau^t : (z_1, z_2) \mapsto (1/2)((e^{2\pi i t} + 1)z_1 + (e^{2\pi i t} - 1)z_2, (e^{2\pi i t} - 1)z_1 + (e^{2\pi i t} + 1)z_2))\end{aligned}$$

has properties like transitivity (any point on the sphere can be moved to any other point by riding orbits of these flows for finite time) and adequate analyticity. Fayad and Katok constructed the conjugating diffeomorphisms for the AbC method using a finite composition of diffeomorphisms of the following type:

$$g_1(z) = \xi_1^{A\psi_1(z)}, \qquad g_2(z) = \xi_2^{A\psi_2(z)}, \qquad g_3(z) = \xi_3^{A\psi_3(z)}$$

where $\psi_1(z) := \mathrm{Re}(z_1^{q_n})$, $\psi_2(z) := \mathrm{Re}(z_2^{q_n})$ and $\psi_3(z) := \mathrm{Re}((z_1 - z_2)^{q_n})$. They were able to show that one can produce a finite version of unique ergodicity using the above diffeomorphisms and hence the resulting AbC diffeomorphisms they constructed were uniquely ergodic.

In an on-going project (with P.Kunde) we are able to show that we can construct an even larger collection of diffeomorphisms from the above flows if we consider a larger class of real-analytic functions ψ_i. For example we may consider the case of g_1 defined above and replace ψ_1 with any piecewise continuous function $\tilde{\psi}_1(z_2)$. This function can be chosen with care so that the piecewise continuous map $\tilde{g}_1(z) := \xi_1^{\tilde{\psi}_1(z)}$ spreads any 'small' segment of the ϕ orbit of 'most' points uniformly along the ξ_1 flow. Then we can approximate $\tilde{\psi}_1$ by a periodic real-analytic function $\psi_1(z_2)$ which extends holomorphically to $\mathbb{C}^2$ as an entire function. This way the function $g_1(z) := \xi_1^{\psi_1(z)}$ is extendible as an entire analytic function from $\mathbb{C}^4$ to $\mathbb{C}^4$ and when restricted to $\mathbb{S}^3$, it has almost the same combinatorial properties as that of $\tilde{g}_1$. More precisely one can show the following lemma

Lemma. *Given any $\varepsilon > 0$, $\delta > 0$, integers q and $\mathfrak{q}$ (we require $q|\mathfrak{q}$), there exists a real-analytic diffeomorphism $g_1 \in \mathrm{Diff}^{\,\omega}_{\infty}(\mathbb{S}^3, \mu)$ and a rational number α such that with $\Phi = g_1 \circ \phi^\alpha \circ g_1^{-1}$, the following distribution condition is satisfied:*

$$\mu_x(\{I_{(i,\mathfrak{q})} \cap \phi^{-1}(\xi_1^B I_{(i,\mathfrak{q})})\}) \in \big((1-\varepsilon)V(B),\ (1+\varepsilon)V(B)\big)$$

For any $x \in \mathbb{S}^3 \setminus \{(z_1, z_2) : |z_2| < \delta\}$, $I_{(i,\mathfrak{q})} = I_{(i,\mathfrak{q})}(x) := \{\phi^t(x) : t \in [i_1/\mathfrak{q}, (i+1)\mathfrak{q})\}$ and any interval $B \in [0, 1)$. Here μ_x is the conditional measure on the Hopf fibres.

We hope that the above conjugacy when used in combination with an appropriate shear $g_n^{(\mathfrak{s})}$ from ξ_2 and τ may be able to achieve the following type of combinatorics at the n th stage of the AbC method:

$$\mu_x(I_{i,n} \cap \Psi_n^{-1}(B)) \in \big((1-\varepsilon)\mu_x(I_{i,n})\mu(B),\ (1+\varepsilon)\mu_y(I_{i,n})\mu(B)\big)$$

here $x \in \mathbb{S}^3 \setminus \{(z_1, z_2) : |z_2| < \delta\}$, $I_{i,n} = I_{i,n}(x) := \{\phi^t x : t \in [i/q_n, (i+1)/q_n))\}$, B an open ball in $\mathbb{S}^3$ and $\psi_n := g_n^{(\mathfrak{s})} \circ \Phi_n$.

If such a distribution condition is satisfied, we are able to show that the conjugacy converges to a diffeomorphism which is weak mixing. The proof is obtained after some modification of the same proof done by M. Saprykina and B. Fayad for the torus in [6].

Difficulties and work that needs to be done: It seems that all technical tool that are necessary to make a weak-mixing version of the real-analytic AbC method to work is present. One needs to be very careful with the combinatorics to make everything add up and produce a result.

3.2 Real-analytic AbC method on real-analytic principal circle bundles

A natural next step and a more difficult problem would be to demonstrate that the AbC method can be generalized to some larger class of real-analytic manifolds.

Motivated from the Hopf fibration for the odd dimensional spheres, one may consider the class of real-analytic principle circle bundles where the action of $\mathbb{T}^1$ is free and effective. Now I outline what may be possible to do on such manifolds. Note that for each point on the quotient (orbit) space N, there exists $\dim(N)$ linearly independent divergence free smooth vector fields on N. Indeed, for one can work with local charts and construct rotational vector fields and extend them to the whole manifold.

In the next step, one may obtain divergence free perturbations of such vector fields which are real-analytic (see [3] for a method to do this). In fact the complexification of such vector field can be made to extend to the whole complex domain. Then one can lift this vector field to the original manifold M using the real-analytic Riemannian metric. The resulting vector field commutes with the $\mathbb{T}^1$ action and the flows of such vector fields are real-analytic. So when used along with the compactness of M, we obtain:

Proposition. *There exists a finite collection of real-analytic divergence free vector fields $\{X_1, \ldots, X_L\}$ on M which can be integrated to real-analytic flows $\{\xi_1, \ldots, \xi_L\}$ commuting with the $\mathbb{T}^1$ action such that given any $x, y \in M$, there exists $(t_1, \ldots, t_L) \in [-1, 1]^L$ such that $y = \xi_1^{t_1} \circ \ldots \circ \xi_L^{t_L}(x)$.*

Difficulties and work that needs to be done: The flows constructed above are real-analytic but they do not extend to the whole complex domain. As a next step one needs to ensure that the complexification extends to the whole complex domain. If this turns out to be possible then one can produce examples or real-analytic volume preserving minimal or uniquely ergodic diffeomorphisms on real-analytic principal circle bundles.

3.3 Real-analytic AbC method on real-analytic manifolds with non-free circle action

A basic questions in real-analytic ergodic theory would be to figure if there exist a zero entropy measure preserving real-analytic ergodic diffeomorphims on the two dimensional disk or the two dimensional sphere.

The AbC method in the smooth category has been used very successfully to produce similar examples. It may be possible to be able to modify this method adequately to produce examples of real-analytic ergodic diffeomorphisms on the disk.

One notes that there exists extremely well behaved real-analytic Hamiltonian vector fields on the disk. For example, one may consider the Hamiltonian function

$$H(x, y) := xy(1 - x^2 - y^2)$$

on $\mathbb{R}^2$ and note that the corresponding flow takes a point arbitrarily close to the origin to a point arbitrarily close to the boundary of the disk. This can be considered a form of transitivity and analysis of similar Hamiltonians could potentially serve as a pathfinder for the problem.

Difficulties and work that needs to be done: This is a very hard problem and we note that as one tries to take points closer and closer to the origin to points closer and closer to the boundary, one needs to try to ride the flow for longer and longer times resulting in loss of analyticity. Thus one needs to come up with alternate flows or use multiple flows to circumnavigate this issue.

3.4 AbC diffeomorphisms and foundations of mathematics

Recent series of works due to Beleznay, Foreman, Hjorth, Rudolph and Weiss (see [13], [14], [15]) on the interface of logic and dynamical systems brings out another aspect of this theory and has been of much interest to me. This line of thinking is very new and even basic questions remain unanswered.

It has been shown that the 'untwisted' AbC method (both in the smooth (see [14]) and real-analytic category can be successfully coded as uniform circular systems. The next natural problem would be to come up with a symbolic system coding some 'twisted' AbC systems. For achieving such a feat, on the AbC method side of the story, one needs to ensure that there is explicit description of the combinatorics at each stage. This should be easy to do. On the other hand on the symbolic systems side, one needs to find a successful replacement of the circular operator constructed for uniform circular systems which mimics the twisted AbC combinatorics and find the correct construction sequence. This may serve as a pathway in coding weakly mixing AbC transformations.

I believe a lot of these questions arising from this series of work are very interesting, need to be answered, and can be answered using existing technology.

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