A spatio-temporal model for extreme precipitation simulated by a climate model.

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Joint work with Anne-Catherine Favre, Claude Bélisle and Jean-François Angers

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Climate and Weather Extremes

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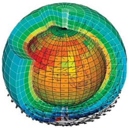
Introduction

Goal



With climate, *in situ* experimentations are impossible. Climate models are therefore the only tools for providing quantitative predictions of the coming climate.

The goal of the talk is to present a spatio-temporal statistical model especially suited for extreme precipitation simulated by a climate model. More specifically, the statistical model takes into account

- non-stationarity in transient time series;
- large spatial simulation domain;
- spatial dependence among grid points.

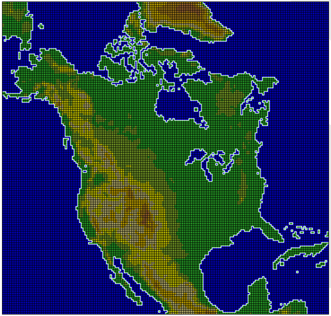
Data

The dataset consists in the daily precipitation outputs from a run of the Canadian Regional Climate Model (CRCM). The data were simulated and provided by OURANOS

- 12,570 land grid points;
- Daily precipitation series for the period $[1961, 2100]$ at every grid point.

Let $Y_{ik\ell}$ be the precipitation depth (mm) of day ℓ of year k at grid point i, where

- $1 \leq i \leq 12,570$;
- $1 \leq k \leq 140$;
- $1 \leq \ell \leq 365$.



Simulation domain

Non-Stationarity

Non-stationarity in the maxima series

Let M_{ik} be the annual maximum of year k at grid point i:

$$M_{ik} = \max_{1 \leq \ell \leq 365} Y_{ik\ell}.$$

Let M_i be the annual maxima series at grid point i:

$$M_i = \left(M_{ik} : 1 \leq k \leq 140\right).$$

For 2/3 of the grid points, the series of maxima M_i exhibits temporal non-stationarity (the grid points in red in the following figure).

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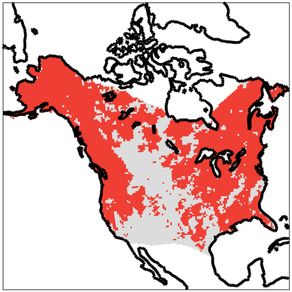
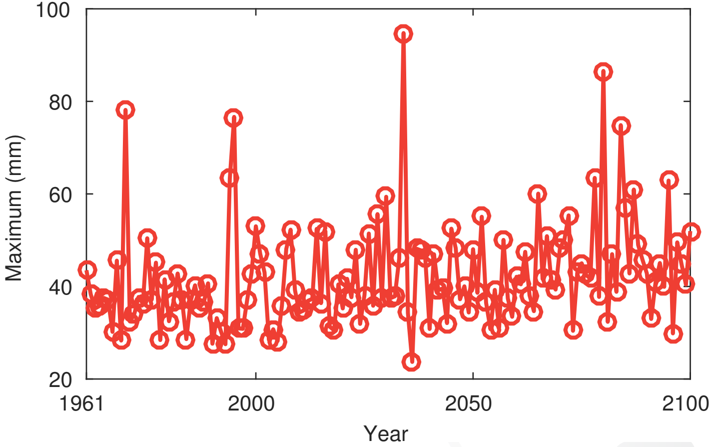
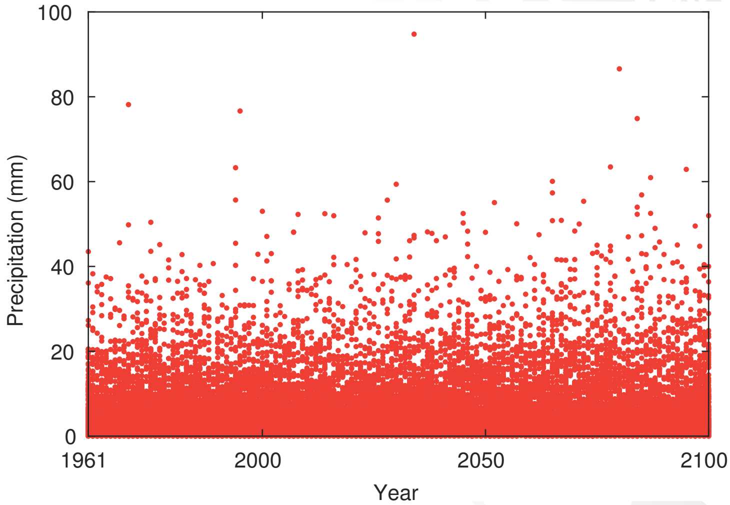


Pointwise Mann-Kendall stationarity test ($\alpha = 5\%$)

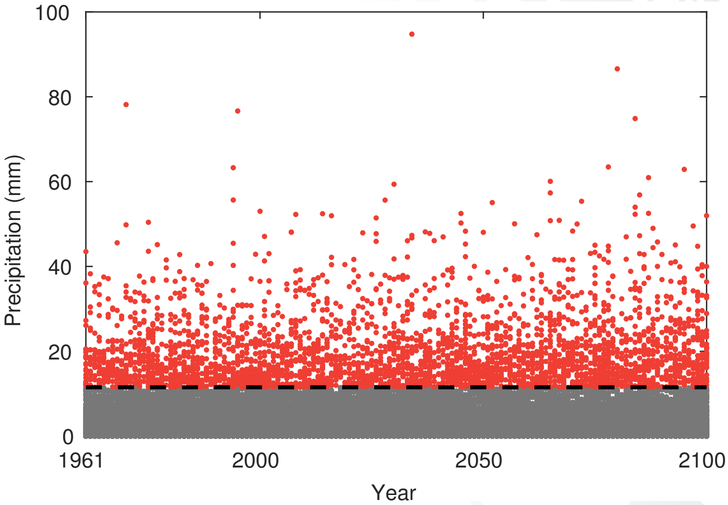
Maxima series - Montreal



Daily series - Montreal



Exceedances - Montreal



Preprocessing approach

Suppose that there exist sequences of constants ν_{ik} and τ_{ik} such that

$$\left(M'_{ik} = \frac{M_{ik} - \nu_{ik}}{\tau_{ik}} : 1 \leq k \leq 140 \right); \tag{1}$$

can be assumed identically distributed along index k for each grid point i. Therefore, the distribution of a given transformed maximum can be approximated by

$$M'_{ik} \overset{\mathcal{L}}{\approx} \mathcal{G}EV\left(\mu_i, \sigma_i, \xi_i\right). \tag{2}$$

The trend in the tail should be isolated from the trend in the bulk of the distribution.

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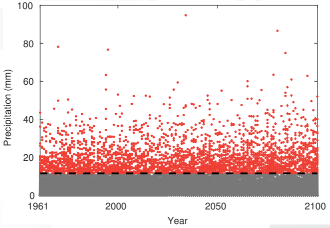
Preprocessing approach

Let Z_{ik} be the vector of precipitation exceedences over the threshold u_i of year k at grid point i:

$$Z_{ik} = \left(Y_{ikl} : Y_{ikl} > u_i, 1 \leq l \leq 140 \right);$$

and let

$$\nu_{ik} = \mathbb{E}(Z_{ik}) \text{ and } \tau_{ik}^2 = \mathbb{V}ar(Z_{ik}).$$



The threshold has to be chosen in order that the transformation:

$$M'_{ik} = \frac{M_{ik} - \nu_{ik}}{\tau_{ik}}; \quad (3)$$

removes the trend in the maxima series M'_i. Its definition does not rely on asymptotic convergence requirements as in the Peaks-Over-Threshold model.

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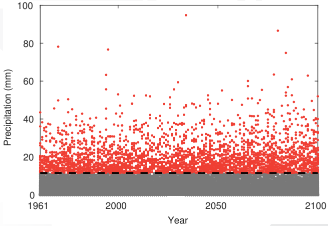

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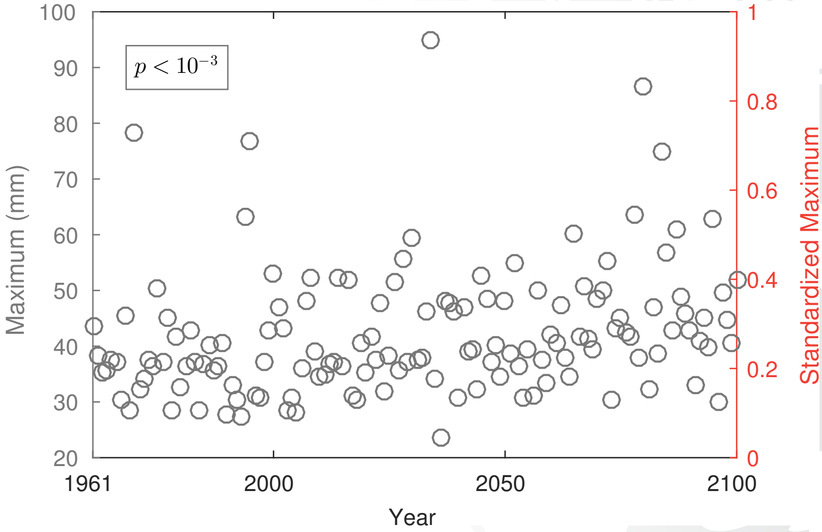

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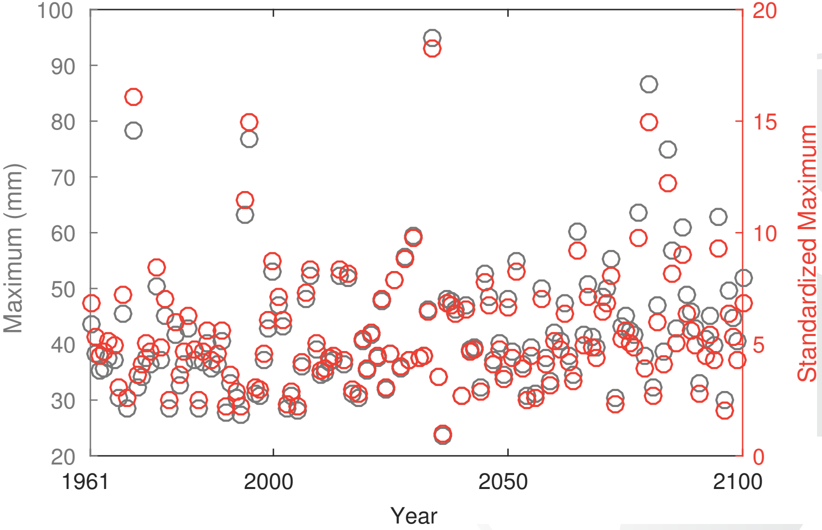

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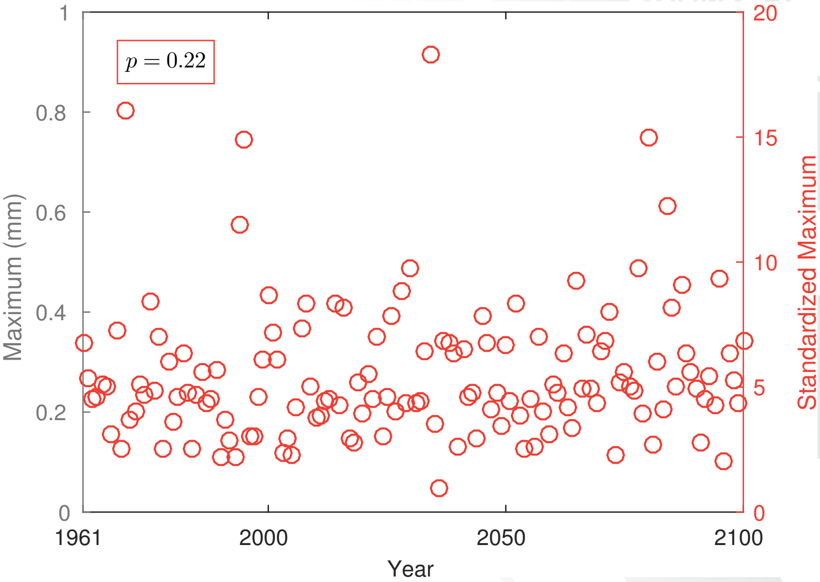
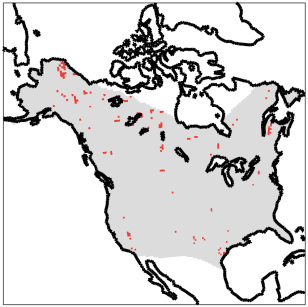


Threshold choice

We chose the 80th empirical quantiles at each grid point as the thresholds.

Then, the stationarity hypothesis of the preprocessed maxima series was rejected for only 1.5% of the grid points (the grid points in red in the following figure).



Pointwise Mann-Kendall stationarity test ($\alpha = 5\%$) for the preprocessed maxima series

Preprocessing approach

Benefits of the proposed preprocessing approach are:

- if there exist constants $a_{ik} > 0$ and b_{ik} such that

$$\frac{M_{ik} - b_{ik}}{a_{ik}} \xrightarrow{\mathcal{L}} \mathcal{G}EV(0, 1, \xi_i);$$

then there exists constants $a'_{ik} > 0$ and b'_{ik} such that

$$\frac{M'_{ik} - b'_{ik}}{a'_{ik}} \xrightarrow{\mathcal{L}} \mathcal{G}EV(0, 1, \xi_i).$$

- The model for the untransformed maxima is tractable:

$$M_{ik} \overset{\mathcal{L}}{\approx} \mathcal{G}EV\left(\tau_{ik}\mu_i + \nu_{ik}, \tau_{ik}\sigma_i, \xi_i\right). \tag{4}$$

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Spatial modeling

Spatial dependence structure

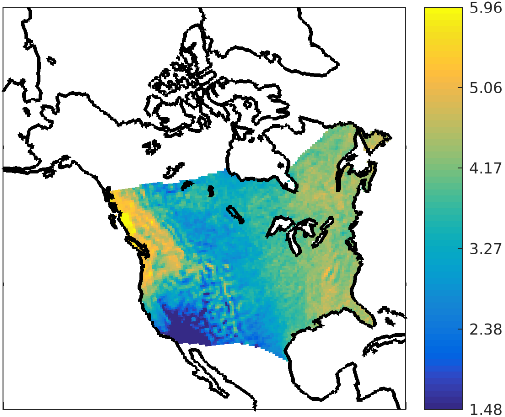
Following the idea of Cooley & Sain (2010) and Reich & Shaby (2012), the spatial dependence is taken into account by modeling spatial variation in the GEV parameters mainly for two reasons:

- such a latent variable approach is very flexible;
- the local properties of extremal distributions (such as return levels) are well reproduced (Davison *et al.*, 2012; Sebille *et al.*, 2016).

However such an approach can neither model nor predict an event occurring simultaneously at several grid points.

Spatial latent model

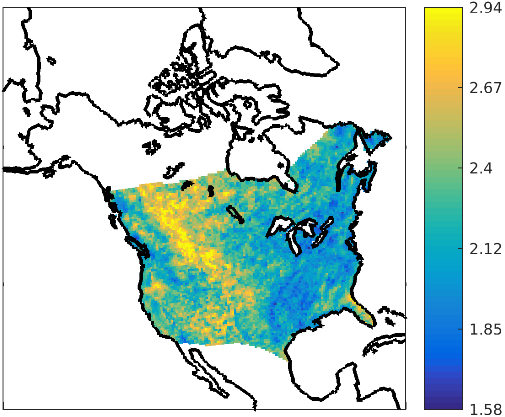
Local estimates of the GEV location parameter.



Local parameter estimates could definitely benefit from neighboring site values.

Spatial latent model

Local estimates of the GEV scale parameter.



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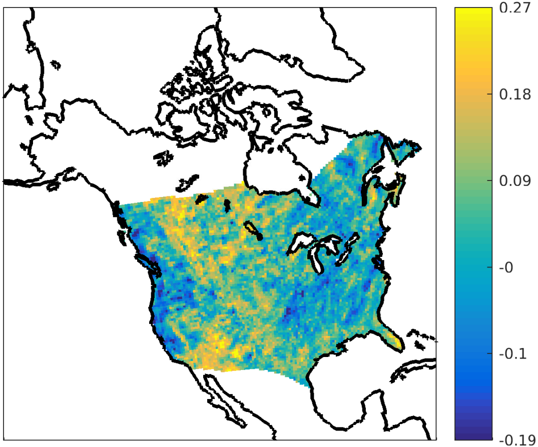
Local estimates of the GEV shape parameter.



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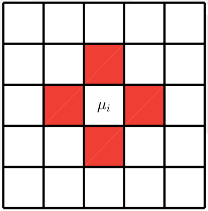
Spatial latent model

Because the random variables lie on a regular lattice, Gaussian Markov random fields are well appropriate.



Such a field inherits the Markov property. For the GEV location parameter, we have:

$$f_{[\boldsymbol{\mu}_i | \boldsymbol{\mu}_{-i} = \mu_{-i}]}(\mu_i) = f_{[\boldsymbol{\mu}_i | \boldsymbol{\mu}_{\delta_i} = \mu_{\delta_i}]}(\mu_i);$$

where δ_i is the set of neighbors of grid point i.

The precision matrix Q of the joint distribution of $\boldsymbol{\mu}$ is sparse because of the important following simplification:

$$\boldsymbol{\mu}_i \perp \boldsymbol{\mu}_j \mid \boldsymbol{\mu}_{-i,-j} \;\Leftrightarrow\; q_{ij} = 0; \tag{5}$$

where q_{ij} is the element (i, j) of the precision matrix Q.

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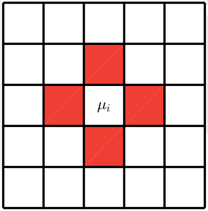

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Intrinsic Gaussian Markov random fields

A Gaussian Markov Random field is a multivariate normal vector $\boldsymbol{\mu} = (\boldsymbol{\mu}_i, i = 1, \ldots, n)^\top$ where the precision matrix Q fulfills the following property:

$$q_{ij} = 0 \text{ if } \boldsymbol{\mu}_i \perp \boldsymbol{\mu}_j \mid \boldsymbol{\mu}_{-i,-j}.$$

The marginal pairwise correlation that can be modeled by Gaussian Markov random fields is limited to 0.8 (Besag & Kooperberg, 1995).

An option is to use intrinsic Gaussian Markov random fields, where the precision matrix is not of full rank.

The rank deficiency controls the smoothness of the field. First-order iGMRFs better capture small-scale variations whereas second-order iGMRFs better model large-scale ones.

Intrinsic Gaussian Markov random fields

The most popular iGMRFs are defined with a scaled precision matrix:

$$Q = \kappa W;$$

where $0 < \kappa < \infty$ is a precision parameter that controls the smoothness of the field and W is a structure matrix known from the grid.

Let k be the rank deficiency of the precision matrix Q. The improper joint distribution is proportional to

$$f_{\boldsymbol{\mu}}(\mu) \propto \kappa^{\frac{n-k}{2}} \exp\left\{-\frac{\kappa}{2}\mu^{\top} W \ \mu\right\}.$$

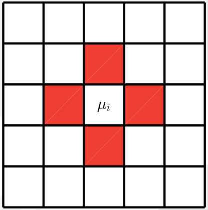
Under the Bayesian paradigm, iGMRFs generally yield proper posterior distributions when used as prior.

First-order intrinsic Gaussian Markov random fields

Let $n_i = \text{Card}(\delta_i)$, be the number of neighbors of grid point i.

$$f_{[\boldsymbol{\mu}_i|\boldsymbol{\mu}_{\delta_i}=\mu_{\delta_i}]}(\mu_i) = \mathcal{N}\left(\mu_i \,\middle|\, \frac{1}{n_i}\sum_{j\in\delta_i}\mu_j, \frac{1}{\kappa n_i}\right);$$

where $\kappa > 0$ is the precision parameter.



This model approximates a two-dimensional Brownian motion.

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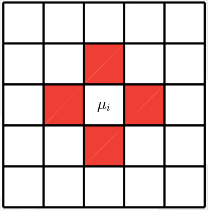

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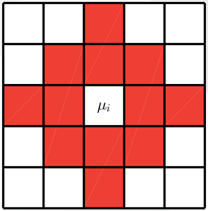
Second-order intrinsic Gaussian Markov random fields

$$\mathbb{E}(X_i|X_{-i}=x_{-i}) =$$

$$\frac{1}{20}\left(8\begin{matrix} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \otimes & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{matrix} - 2\begin{matrix} \circ & \circ & \circ & \circ & \circ \\ \circ & \bullet & \circ & \bullet & \circ \\ \circ & \circ & \otimes & \circ & \circ \\ \circ & \bullet & \circ & \bullet & \circ \\ \circ & \circ & \circ & \circ & \circ \end{matrix} - 1\begin{matrix} \circ & \circ & \bullet & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \bullet & \circ & \otimes & \circ & \bullet \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ & \circ \end{matrix} \right)$$

$$\mathbb{P}\text{rec}(X_i|X_{-i}=x_{-i}) = 20\kappa.$$

This model is an approximation to the thin plate spline, the two-dimensional extension of cubic splines.



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		-1/20		
	-2/20	8/20	-2/20	
-1/20	8/20	μ_i	8/20	-1/20
	-2/20	8/20	-2/20	
		-1/20		

Complete spatial model

The model is therefore:

$$f_{[M'_{ik}|(\mu_i,\phi_i,\xi_i)]}(m_{ik}) \overset{\mathcal{L}}{\approx} \mathcal{G}EV\left\{m'_{ik}|\mu_i, \exp\left(\phi_i\right), \xi_i\right\};$$

$$f_{(\boldsymbol{\mu},\boldsymbol{\phi},\boldsymbol{\xi})}(\mu,\phi,\xi) \propto$$
$$(\kappa_\mu\kappa_\phi\kappa_\xi)^{\frac{n-k}{2}} \exp\left\{-\frac{\kappa_\mu}{2}\mu^\top W\mu - \frac{\kappa_\phi}{2}\phi^\top W\phi - \frac{\kappa_\xi}{2}\xi^\top W\xi\right\};$$

- Three independent intrinsic Gaussian Markov random fields for the GEV parameter prior.
- Vague gamma hyperpriors for the precision parameters.

Also, iGMRFs are semi-informative:

- marginally non-informative, for example $\mathbb{E}(\boldsymbol{\mu}_i)$ is undefined and $\mathbb{V}\text{ar}(\boldsymbol{\mu}_i) = \infty$;
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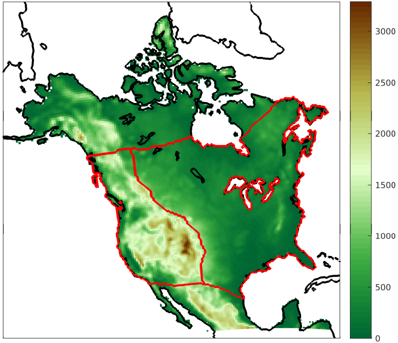
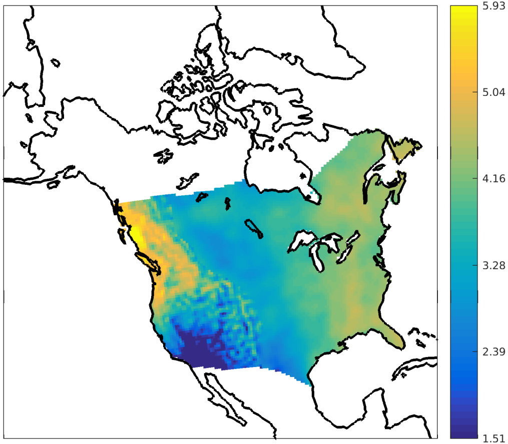
Results

Homogeneous regions



Chosen model

Spatial estimates of the GEV location parameter.



According to the deviance information criterion, the model with the second-order iGMRF prior is better.

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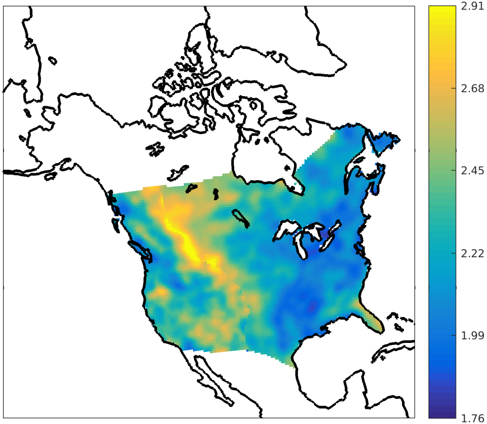

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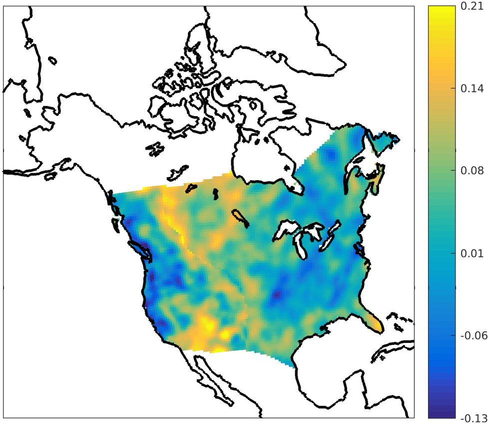

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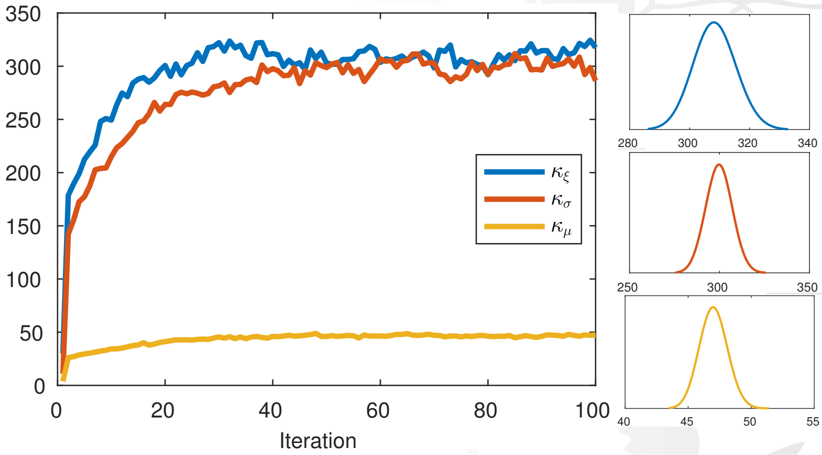
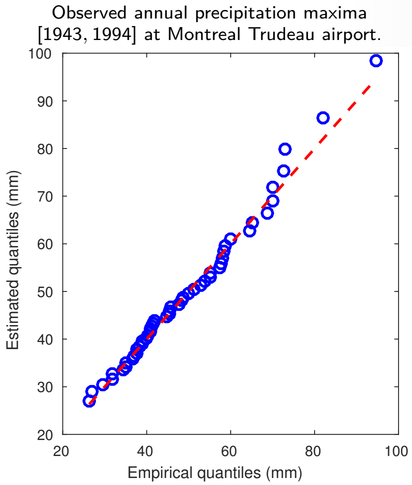
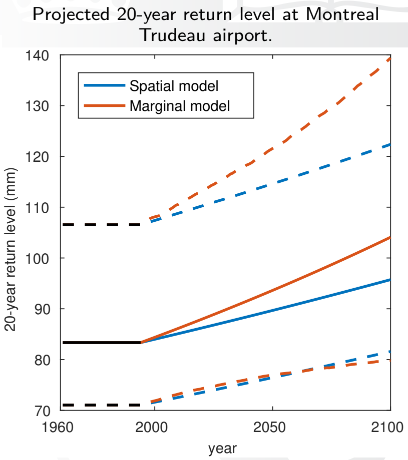
Model fit


350
300
250
200
150
100
50
0
0
20
40
60
80
100
Iteration
κ_ξ
κ_σ
κ_μ
280
300
320
340
250
300
350
40
45
50
55

Application - Projected return level





Conclusion

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The statistical model developed was well suited for climate model outputs, specifically for:

- transient time series;
- data that lie on a regular grid but could also be adapted to irregular locations (Lindgren *et al.*, 2011, Paciorek, 2013).

The model's simplicity, intuitive interpretation and uncertainty description along with its fast adjustment make it very appealing.

Nevertheless, the model could be enhanced

- by integrating several climate simulations for a better description of future climate uncertainty;

We are also investigating the application of max-stable hierarchical models (Shaby & Reich, 2012).

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Appendix

Gaussian Markov random fields

For a stationary field in space, the conditional distributions have to take the following form:

$$f_{[\boldsymbol{\mu}_i|\boldsymbol{\mu}_{\delta_i}=\mu_{\delta_i}]}(\mu_i) = \mathcal{N}\left\{\mu_i \,\middle|\, \eta_i + \rho\sum_{j\in\delta_i}(\mu_j-\eta_j), \zeta^2\right\}; \quad (6)$$

where $0 \leq \rho \leq 1$ and $\zeta^2 > 0$.

It can be shown that marginal bivariate correlation coefficients between neighbors are necessarily less than 0.8: (Besag & Kooperberg, 1995)

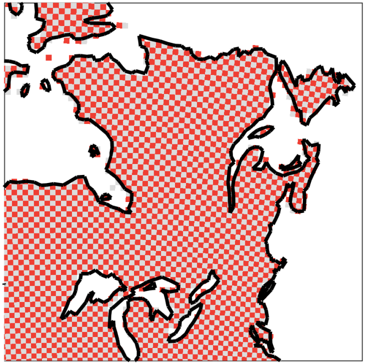
$$\mathbb{C}or(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \leq 0.8, \text{ for } j \in \delta_i. \quad (7)$$

The spatial correlation that can be modeled is therefore limited.

Model fit

A chain of length 6000 was generated where the first 1000 iterations were discarded as the burn-in period. It took less than 40 minutes of computation time on a 2.53 GHz processor.

- algorithms for sparse matrix;
- parallel MCMC.



Modeling the dependence between the GEV paremeters

We could consider a multivariate intrinsic Gaussian Markov random field as follows:

$$f_{(\boldsymbol{\mu},\boldsymbol{\phi},\boldsymbol{\xi})}(\mu,\phi,\xi) \propto |\Gamma|^{*} \exp\left\{ \left(\mu^{\top},\ \phi^{\top},\ \xi^{\top}\right) \times \Gamma \times \begin{pmatrix} \mu \\ \phi \\ \xi \end{pmatrix} \right\},$$

Under a separability assumption, Benerjee *et al.* (2004) proposed to model the precision matrix Γ as follows:

$$\Gamma = \begin{pmatrix} \kappa_{\mu} & \gamma_{\mu\phi} & \gamma_{\mu\xi} \\ \gamma_{\mu\phi} & \kappa_{\phi} & \gamma_{\phi\xi} \\ \gamma_{\mu\xi} & \gamma_{\phi\xi} & \kappa_{\xi} \end{pmatrix} \otimes W,$$

To ensure parameter identifiability, Cooley & Sain (2010) and Economou *et al.* (2014) fixed the precision of the Gaussian Markov fields modeling the spatial dependence between each GEV parameters.

Application - Postprocessing

Postprocessing of the annual maxima series at the grid point containing Montreal Trudeau airport.