Factor Copula Models for Replicated Spatial Data

Raphaël Huser ©2016
(King Abdullah University of Science and Technology, SA)

With Pavel Krupskii (KAUST)
and Marc Genton (KAUST)
Agenda

- KAUST
- Copula modeling
- Tail dependence and independence
- Factor copula models for spatial data
- Simulation study
- Application to temperature data
- Conclusion
KAUST

- 7-year old graduate university in Thuwal, Saudi Arabia;
- Located by the Red Sea;
- Highly multi-cultural (more than 100 nationalities on campus);
Current KAUST President is the former president of Caltech;
Currently, about 140 faculty, 150 research scientists, 400 postdocs, 850 students (academic population: $\sim 1600$; campus population: $\sim 6000$);
3 statistics groups (currently):
- M. G. Genton (Spatio-Temporal Statistics and Data Science)
- Y. Sun (Environmental Statistics)
- R. Huser (Extreme Statistics), http://extstat.kaust.edu.sa
Current KAUST President is the former president of Caltech;
Currently, about 140 faculty, 150 research scientists, 400 postdocs, 850 students (academic population: \( \sim 1600 \); campus population: \( \sim 6000 \));
3 statistics groups (currently):
- M. G. Genton (Spatio-Temporal Statistics and Data Science)
- Y. Sun (Environmental Statistics)
- R. Huser (Extreme Statistics), http://extstat.kaust.edu.sa
Two new faculty members in 2017:
Current KAUST President is the former president of Caltech;
Currently, about 140 faculty, 150 research scientists, 400 postdocs, 850 students (academic population: \(\sim\) 1600; campus population: \(\sim\) 6000);
3 statistics groups (currently):
- M. G. Genton (Spatio-Temporal Statistics and Data Science)
- Y. Sun (Environmental Statistics)
- R. Huser (Extreme Statistics), http://extstat.kaust.edu.sa
Two new faculty members in 2017:
- Harvard Rue (NUST, Norway)
Current KAUST President is the former president of Caltech;

Currently, about 140 faculty, 150 research scientists, 400 postdocs, 850 students (academic population: $\sim 1600$; campus population: $\sim 6000$);

3 statistics groups (currently):
- M. G. Genton (Spatio-Temporal Statistics and Data Science)
- Y. Sun (Environmental Statistics)
- R. Huser (Extreme Statistics), http://extstat.kaust.edu.sa

Two new faculty members in 2017:
- Harvard Rue (NUST, Norway)
- Hernando Ombao (UC Irvine, USA)
Current KAUST President is the former president of Caltech;
Currently, about 140 faculty, 150 research scientists, 400 postdocs, 850 students (academic population: $\sim 1600$; campus population: $\sim 6000$);
3 statistics groups (currently):
- M. G. Genton (Spatio-Temporal Statistics and Data Science)
- Y. Sun (Environmental Statistics)
- R. Huser (Extreme Statistics), http://extstat.kaust.edu.sa
Two new faculty members in 2017:
- Harvard Rue (NUST, Norway)
- Hernando Ombao (UC Irvine, USA)

We are looking for talented PhD students and postdocs.
Extremes are everywhere: Jeddah (SA), November, 2015

Saudi Arabia urges all Jeddah residents to remain indoors

Images being shared on social media showed al-Falak roundabout, one of many Jeddah’s traffic landmarks, being flooded. (via Twitter)
Copula modeling

- **Sklar’s Theorem**: Any continuous \( n \)-dimensional joint distribution \( G \) with margins \( G_1, \ldots, G_n \) may be uniquely represented as

\[
G(x) = C\{G_1(x_1), \ldots, G_n(x_n)\} \iff C(u) = G\{G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n)\},
\]

where \( C \) is the associated **copula** or **dependence function**.
Copula modeling

- **Sklar's Theorem:** Any continuous $n$-dimensional joint distribution $G$ with margins $G_1, \ldots, G_n$ may be uniquely represented as

$$G(x) = C\{G_1(x_1), \ldots, G_n(x_n)\} \iff C(u) = G\{G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n)\},$$

where $C$ is the associated copula or dependence function.

- A copula is a multivariate distribution with uniform $U(0, 1)$ margins.
Copula modeling

□ Sklar’s Theorem: Any continuous $n$-dimensional joint distribution $G$ with margins $G_1, \ldots, G_n$ may be uniquely represented as

$$G(x) = C\{G_1(x_1), \ldots, G_n(x_n)\} \iff C(u) = G\{G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n)\},$$

where $C$ is the associated copula or dependence function.

$\Rightarrow$ A copula is a multivariate distribution with uniform $U(0, 1)$ margins.

□ Sklar’s Theorem enables separate treatment of marginal distributions and dependence structure.
Copula modeling

- Sklar’s Theorem: Any continuous $n$-dimensional joint distribution $G$ with margins $G_1, \ldots, G_n$ may be uniquely represented as

\[ G(x) = C \{ G_1(x_1), \ldots, G_n(x_n) \} \iff C(u) = G \{ G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n) \}, \]

where $C$ is the associated copula or dependence function.

$\Rightarrow$ A copula is a multivariate distribution with uniform $U(0, 1)$ margins.

- Sklar’s Theorem enables separate treatment of marginal distributions and dependence structure.

- If replicates are available, margins $G_1, \ldots, G_n$ are typically estimated non-parametrically using ranks.
Copula modeling

- **Sklar's Theorem**: Any continuous $n$-dimensional joint distribution $G$ with margins $G_1, \ldots, G_n$ may be uniquely represented as

$$ G(x) = C\{G_1(x_1), \ldots, G_n(x_n)\} \iff C(u) = G\{G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n)\}, $$

where $C$ is the associated copula or dependence function.

- A copula is a multivariate distribution with uniform $U(0, 1)$ margins.

- Sklar's Theorem enables separate treatment of marginal distributions and dependence structure.

- If replicates are available, margins $G_1, \ldots, G_n$ are typically estimated non-parametrically using ranks.

- Here, we focus on building flexible yet parsimonious copula models for spatial data with replicates.
Copula modeling

- **Sklar’s Theorem**: Any continuous $n$-dimensional joint distribution $G$ with margins $G_1, \ldots, G_n$ may be uniquely represented as

$$G(x) = C\{G_1(x_1), \ldots, G_n(x_n)\} \iff C(u) = G\{G_1^{-1}(u_1), \ldots, G_n^{-1}(u_n)\},$$

where $C$ is the associated copula or dependence function.

⇒ A copula is a multivariate distribution with uniform $U(0, 1)$ margins.

- Sklar’s Theorem enables separate treatment of marginal distributions and dependence structure.

- If replicates are available, margins $G_1, \ldots, G_n$ are typically estimated non-parametrically using ranks.

- Here, we focus on building flexible yet parsimonious copula models for spatial data with replicates.

- A main difference with previous talks (Thibaud, Wadsworth, etc.): We do not focus only on extremes, but we propose new spatial models for the full data range (i.e., extreme and non-extreme data), while keeping flexible lower and upper tails.
Tail dependence and independence

- Tail dependent coefficients are standard measures of tail dependence in the copula literature.
Tail dependence and independence

- **Tail dependent coefficients** are standard measures of tail dependence in the copula literature.

- Let \((U_1, U_2)\) be a random vector distributed according to the copula \(C(u_1, u_2)\), with survival copula \(\overline{C}(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2)\). For each \(q \in (0, 1)\), we define the coefficients \(\lambda_L(q)\) and \(\lambda_U(q)\) as

\[
\lambda_L(q) = \Pr(U_1 \leq q \mid U_2 \leq q) = \frac{C(q, q)}{q}, \\
\lambda_U(q) = \Pr(U_1 > 1 - q \mid U_2 > 1 - q) = \frac{\overline{C}(1 - q, 1 - q)}{q}.
\]
Tail dependence and independence

- **Tail dependent coefficients** are standard measures of tail dependence in the copula literature.

- Let \((U_1, U_2)\) be a random vector distributed according to the copula \(C(u_1, u_2)\), with survival copula \( \overline{C}(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2) \). For each \(q \in (0, 1)\), we define the coefficients \(\lambda_L(q)\) and \(\lambda_U(q)\) as

  \[
  \lambda_L(q) = \Pr(U_1 \leq q \mid U_2 \leq q) = C(q, q)/q,
  \]

  \[
  \lambda_U(q) = \Pr(U_1 > 1 - q \mid U_2 > 1 - q) = \overline{C}(1 - q, 1 - q)/q.
  \]

- The **lower and upper tail dependent coefficients** are respectively the limits

  \[
  \lambda_L = \lim_{q \to 0} \lambda_L(q), \quad \lambda_U = \lim_{q \to 0} \lambda_U(q).
  \]

  They represent the strength of dependence in the lower (respectively upper) tails for a given copula.
Tail dependence and independence

- Tail dependent coefficients are standard measures of tail dependence in the copula literature.

- Let \((U_1, U_2)\) be a random vector distributed according to the copula \(C(u_1, u_2)\), with survival copula \(\overline{C}(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2)\). For each \(q \in (0, 1)\), we define the coefficients \(\lambda_L(q)\) and \(\lambda_U(q)\) as:

\[
\begin{align*}
\lambda_L(q) &= \Pr(U_1 \leq q \mid U_2 \leq q) = \frac{C(q, q)}{q}, \\
\lambda_U(q) &= \Pr(U_1 > 1 - q \mid U_2 > 1 - q) = \frac{\overline{C}(1 - q, 1 - q)}{q}.
\end{align*}
\]

- The lower and upper tail dependent coefficients are respectively the limits

\[
\lambda_L = \lim_{q \to 0} \lambda_L(q), \quad \lambda_U = \lim_{q \to 0} \lambda_U(q).
\]

They represent the strength of dependence in the lower (respectively upper) tails for a given copula.

- If \(\lambda_L = 0\) (respectively \(\lambda_U = 0\)), the copula \(C\) is called lower (respectively upper) tail independent.
Gaussian copula

The Gaussian copula $C(u) = \Phi_{\Sigma}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\}$ is tractable, easily interpretable, but tail-independent and tail-symmetric.
The Gaussian copula $C(u) = \Phi_\Sigma \{ \Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n) \}$ is tractable, easily interpretable, but tail-independent and tail-symmetric.
Trans-Gaussian copula

Same as Gaussian copula...

Original scale

Unif(0,1) scale

N(0,1) scale
Trans-Gaussian copula

Same as Gaussian copula...

Lower tail dependence

Upper tail dependence

\[ \lambda_L(q) \]

\[ \lambda_U(q) \]
The Student-$t_{\nu}$ copula $C(u) = T_{\Sigma,\nu}\{T_{\nu}^{-1}(u_1), \ldots, T_{\nu}^{-1}(u_n)\}$ is tractable, easily interpretable, tail-dependent but tail-symmetric.
The Student-$t_\nu$ copula $C(u) = T_{\Sigma,\nu} \{ T_{\nu}^{-1}(u_1), \ldots, T_{\nu}^{-1}(u_n) \}$ is tractable, easily interpretable, tail-dependent but tail-symmetric.
Extreme-value copulas (Here, logistic model)

Extreme-value copulas are upper tail-dependent and tail-asymmetric but lower tail-independent, intractable in high dimensions and theoretically justified for extremes, not the full data range.
Extreme-value copulas (Here, logistic model)

Extreme-value copulas are upper tail-dependent and tail-asymmetric but lower tail-independent, intractable in high dimensions and theoretically justified for extremes, not the full data range.
Other copulas?

- Archimedean copulas: *tractable* but have *exchangeable dependence structure* and are *not spatial* by nature.
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
Other copulas?

- Archimedean copulas: **tractable** but have **exchangeable dependence structure** and are **not spatial** by nature.
- Vine copulas: **flexible** but **lack interpretability** and are **complex to fit**.
- **Factor copula models:**
Other copulas?

- Archimedean copulas: **tractable** but have **exchangeable dependence structure** and are **not spatial** by nature.
- Vine copulas: **flexible** but lack **interpretability** and are **complex to fit**.
- **Factor copula models**: combine **flexibility**
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
- **Factor copula models**: combine flexibility, parsimony
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
- **Factor copula models:** combine flexibility, parsimony, interpretability
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
- **Factor copula models**: combine flexibility, parsimony, interpretability, tractability
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
- **Factor copula models:** combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor)
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
- **Factor copula models:** combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
Other copulas?

- Archimedean copulas: tractable but have exchangeable dependence structure and are not spatial by nature.
- Vine copulas: flexible but lack interpretability and are complex to fit.
- **Factor copula models:** combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
Other copulas?

- **Archimedeian copulas**: tractable but have exchangeable dependence structure and are not spatial by nature.
- **Vine copulas**: flexible but lack interpretability and estimation is intensive.
- **Factor copula models**: combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.

![Graphs of Lower tail dependence and Upper tail dependence](image)
Other copulas?

- **Archimedean copulas**: tractable but have exchangeable dependence structure and are not spatial by nature.
- **Vine copulas**: flexible but lack interpretability and estimation is intensive.
- **Factor copula models**: combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
Other copulas?

- **Archimedean copulas:** tractable but have exchangeable dependence structure and are not spatial by nature.
- **Vine copulas:** flexible but lack interpretability and estimation is intensive.
- **Factor copula models:** combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
Other copulas?

- Archimedean copulas: **tractable** but have **exchangeable dependence structure** and are **not spatial** by nature.
- Vine copulas: **flexible** but lack interpretability and estimation is intensive.
- **Factor copula models**: combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
Other copulas?

- **Archimedean copulas**: tractable but have exchangeable dependence structure and are not spatial by nature.
- **Vine copulas**: flexible but lack interpretability and estimation is intensive.
- **Factor copula models**: combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.

![Lower tail dependence](chart1.png)

![Upper tail dependence](chart2.png)
Factor copulas for spatial data

Let $Z(s), s \in S$, be a standard Gaussian process, and $V_0 \sim F_{V_0}$ be an independent common factor, which does not depend on location $s$. 
Factor copulas for spatial data

- Let $Z(s), s \in S$, be a standard Gaussian process, and $V_0 \sim F_{V_0}$ be an independent common factor, which does not depend on location $s$.
- We consider the following Gaussian location mixture process:

$$W(s) = Z(s) + V_0, \quad s \in S.$$
Factor copulas for spatial data

- Let \( Z(s), s \in S \), be a standard Gaussian process, and \( V_0 \sim F_{V_0} \) be an independent common factor, which does not depend on location \( s \).

- We consider the following Gaussian location mixture process:

\[
W(s) = Z(s) + V_0, \quad s \in S.
\]

- When observed at \( n \) spatial locations \( s_1, \ldots, s_n \in S \), one can write

\[
W_j = Z_j + V_0, \quad j = 1, \ldots, n, \quad (Z_1, \ldots, Z_n)^T \sim \mathcal{N}_n(0, \Sigma_Z) \perp \perp V_0 \sim F_{V_0}.
\]
Factor copulas for spatial data

- Let $Z(s), s \in \mathcal{S}$, be a standard Gaussian process, and $V_0 \sim F_{V_0}$ be an independent common factor, which does not depend on location $s$.
- We consider the following Gaussian location mixture process:

$$W(s) = Z(s) + V_0, \quad s \in \mathcal{S}.$$  

- When observed at $n$ spatial locations $s_1, \ldots, s_n \in \mathcal{S}$, one can write

$$W_j = Z_j + V_0, \quad j = 1, \ldots, n,$$

$$(Z_1, \ldots, Z_n)^T \sim \mathcal{N}_n(0, \Sigma_Z) \perp \perp V_0 \sim F_{V_0}.$$

- One has $\Sigma_W = (\Sigma_Z + \sigma_0^2)/(1 + \sigma_0^2)$, where $\sigma_0^2 = \text{var}(V_0) < \infty$, so the random factor $V_0$ increases spatial correlation uniformly.
  $\Rightarrow$ May not be realistic in a large region.
Factor copulas for spatial data

- Let $Z(s)$, $s \in \mathcal{S}$, be a standard Gaussian process, and $V_0 \sim F_{V_0}$ be an independent common factor, which does not depend on location $s$.
- We consider the following Gaussian location mixture process:

$$W(s) = Z(s) + V_0, \quad s \in \mathcal{S}.$$ 

- When observed at $n$ spatial locations $s_1, \ldots, s_n \in \mathcal{S}$, one can write

$$W_j = Z_j + V_0, \quad j = 1, \ldots, n, \quad (Z_1, \ldots, Z_n)^T \sim \mathcal{N}_n(0, \Sigma_Z) \perp \perp V_0 \sim F_{V_0}.$$

- One has $\Sigma_W = (\Sigma_Z + \sigma_0^2)/(1 + \sigma_0^2)$, where $\sigma_0^2 = \text{var}(V_0) < \infty$, so the random factor $V_0$ increases spatial correlation uniformly.

  $\Rightarrow$ May not be realistic in a large region.

- $V_0$ may be interpreted as an underlying factor affecting all spatial locations simultaneously.
In dimension $n$, the distribution, density, copula and copula density of $(W_1, \ldots, W_n)^T$ are respectively

\[
F_n^W (w_1, \ldots, w_n) = \int_{-\infty}^{\infty} \Phi_\Sigma (w_1 - v_0, \ldots, w_n - v_0) dF_{V_0} (v_0),
\]

\[
f_n^W (w_1, \ldots, w_n) = \int_{-\infty}^{\infty} \phi_\Sigma (w_1 - v_0, \ldots, w_n - v_0) dF_{V_0} (v_0),
\]

\[
C_n^W (u_1, \ldots, u_n) = F_n^W \left\{ (F_1^W)^{-1} (u_1), \ldots, (F_1^W)^{-1} (u_n) \right\}
\]

\[
c_n^W (u_1, \ldots, u_n) = \frac{f_n^W \left\{ (F_1^W)^{-1} (u_1), \ldots, (F_1^W)^{-1} (u_n) \right\}}{\prod_{j=1}^{n} f_1 \left\{ (F_1^W)^{-1} (u_j) \right\}}
\]
In dimension $n$, the distribution, density, copula and copula density of $(W_1, \ldots, W_n)^T$ are respectively

$$
F_n^W(w_1, \ldots, w_n) = \int_{-\infty}^{\infty} \Phi_{\Sigma}(w_1 - v_0, \ldots, w_n - v_0) dF_{V_0}(v_0),
$$

$$
f_n^W(w_1, \ldots, w_n) = \int_{-\infty}^{\infty} \phi_{\Sigma}(w_1 - v_0, \ldots, w_n - v_0) dF_{V_0}(v_0),
$$

$$
C_n^W(u_1, \ldots, u_n) = F_n^W \left\{ (F_1^W)^{-1}(u_1), \ldots, (F_1^W)^{-1}(u_n) \right\}
$$

$$
c_n^W(u_1, \ldots, u_n) = \frac{f_n^W \left\{ (F_1^W)^{-1}(u_1), \ldots, (F_1^W)^{-1}(u_n) \right\}}{\prod_{j=1}^{n} f_1 \left\{ (F_1^W)^{-1}(u_j) \right\}}
$$

For some choices of random factor distribution $F_{V_0}$ (e.g., exponential factors), the integrals above can be calculated in closed form.

⇒ Almost as fast as computing a Gaussian density.
Distribution, density and copula

- In dimension $n$, the distribution, density, copula and copula density of $(W_1, \ldots, W_n)^T$ are respectively

\[
F_n^W(w_1, \ldots, w_n) = \int_{-\infty}^{\infty} \Phi_{\Sigma}(w_1 - v_0, \ldots, w_n - v_0) dF_{V_0}(v_0),
\]

\[
f_n^W(w_1, \ldots, w_n) = \int_{-\infty}^{\infty} \phi_{\Sigma}(w_1 - v_0, \ldots, w_n - v_0) dF_{V_0}(v_0),
\]

\[
C_n^W(u_1, \ldots, u_n) = F_n^W \left\{ (F_1^W)^{-1}(u_1), \ldots, (F_1^W)^{-1}(u_n) \right\}
\]

\[
c_n^W(u_1, \ldots, u_n) = \frac{f_n^W \left\{ (F_1^W)^{-1}(u_1), \ldots, (F_1^W)^{-1}(u_n) \right\}}{\prod_{j=1}^{n} f_1 \left\{ (F_1^W)^{-1}(u_j) \right\}}
\]

- For some choices of random factor distribution $F_{V_0}$ (e.g., exponential factors), the integrals above can be calculated in closed form.
  ⇒ Almost as fast as computing a Gaussian density.

- In general, these unidimensional integrals can be efficiently and accurately approximated using Monte Carlo methods or finite integration.
The distribution $F_{V_0}$ determines the tail behavior of factor copula models:
Tail behavior of factor copulas

- The distribution $F_{V_0}$ determines the tail behavior of factor copula models:
  - If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
Tail behavior of factor copulas

- The distribution $F_{V_0}$ determines the tail behavior of factor copula models:
  - If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
  - If $V_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
The distribution $F_{V_0}$ determines the tail behavior of factor copula models:

- If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
- If $V_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
- If $\text{var}(V_0) >> 0$, $W(s)$ is dominated by $V_0$ $\Rightarrow$ Perfectly dependent.
The distribution $F_{V_0}$ determines the tail behavior of factor copula models:

- If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
- If $V_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
- If $\text{var}(V_0) >> 0$, $W(s)$ is dominated by $V_0$ $\Rightarrow$ Perfectly dependent.
- What happens between these extreme cases?
Tail behavior of factor copulas

- The distribution $F_{V_0}$ determines the tail behavior of factor copula models:
  - If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
  - If $V_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
  - If $\text{var}(V_0) >> 0$, $W(s)$ is dominated by $V_0 \Rightarrow$ Perfectly dependent.
  - What happens between these extreme cases?

Proposition 1: Let $1 - F_{V_0}(v_0) \sim Kv_0^\beta \exp(-\theta v_0^\alpha)$, $v_0 \rightarrow \infty$, where $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\theta > 0$, $K > 0$. Let $\rho = \Sigma_{Z,1,2} < 1$. One has the following cases:
  - If $0 < \alpha < 1$ or $\alpha = 0, \beta < 0$: Perfect upper tail dependence, $\lambda_U = 1$.
  - If $\alpha = 1$: Upper tail dependence with $\lambda_U = 2\Phi \left[ -\theta \left\{ \frac{(1 - \rho)}{2} \right\}^{1/2} \right]$.
  - If $\alpha > 1$: Tail independence, $\lambda_U = 0$. 
Tail behavior of factor copulas

- The distribution $F_{V_0}$ determines the tail behavior of factor copula models:
  - If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
  - If $V_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
  - If $\text{var}(V_0) \gg 0$, $W(s)$ is dominated by $V_0 \Rightarrow$ Perfectly dependent.
  - What happens between these extreme cases?

Proposition 1: Let $1 - F_{V_0}(v_0) \sim K v_0^\beta \exp(-\theta v_0^\alpha)$, $v_0 \to \infty$, where $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\theta > 0$, $K > 0$. Let $\rho = \Sigma_{Z,1,2} < 1$. One has the following cases:
  - If $0 < \alpha < 1$ or $\alpha = 0$, $\beta < 0$: Perfect upper tail dependence, $\lambda_U = 1$.
  - If $\alpha = 1$: Upper tail dependence with $\lambda_U = 2\Phi \left[ -\theta \{ (1 - \rho)/2 \}^{1/2} \right]$.
  - If $\alpha > 1$: Tail independence, $\lambda_U = 0$.

A similar result holds for the lower tail.
The distribution $F_{V_0}$ determines the tail behavior of factor copula models:

- If $V_0 = v_0 \in \mathbb{R}$ almost surely, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
- If $V_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, $W(s)$ is Gaussian $\Rightarrow$ Tail-independent.
- If $\text{var}(V_0) >> 0$, $W(s)$ is dominated by $V_0$ $\Rightarrow$ Perfectly dependent.
- What happens between these extreme cases?

**Proposition 1:** Let $1 - F_{V_0}(v_0) \sim Kv_0^\beta \exp(-\theta v_0^\alpha)$, $v_0 \to \infty$, where $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\theta > 0$, $K > 0$. Let $\rho = \Sigma_{Z,1,2} < 1$. One has the following cases:

- If $0 < \alpha < 1$ or $\alpha = 0, \beta < 0$: Perfect upper tail dependence, $\lambda_U = 1$.
- If $\alpha = 1$: Upper tail dependence with $\lambda_U = 2\Phi \left[ -\theta \left\{ (1 - \rho)/2 \right\}^{1/2} \right]$.
- If $\alpha > 1$: Tail independence, $\lambda_U = 0$.

**Proposition 2:** When $\alpha = 1$, the limiting extreme-value copula is the Hüsler–Reiss model.

A similar result holds for the lower tail.
Specific models

- Weibull factor with $F_{V_0}(v_0) = 1 - \exp(-\theta v_0^\alpha), v_0 > 0$: this model interpolates from the (tail-independent) Gaussian dependence structure ($\alpha \to \infty$) to perfect upper tail dependence ($\alpha < 1$), including non-trivial upper tail dependence ($\alpha = 1$).
Specific models

- Weibull factor with $F_{V_0}(v_0) = 1 - \exp(-\theta v_0^\alpha)$, $v_0 > 0$: this model interpolates from the (tail-independent) Gaussian dependence structure ($\alpha \to \infty$) to perfect upper tail dependence ($\alpha < 1$), including non-trivial upper tail dependence ($\alpha = 1$).

- Exponential factor with $F_{V_0}(v_0) = 1 - \exp(-\theta v_0)$, $v_0 > 0$: upper tail dependence; density available in closed-form.
Specific models

- **Weibull factor** with \( F_{V_0}(v_0) = 1 - \exp(-\theta v_0^\alpha), \ v_0 > 0 \): this model interpolates from the (tail-independent) Gaussian dependence structure (\( \alpha \to \infty \)) to perfect upper tail dependence (\( \alpha < 1 \)), including non-trivial upper tail dependence (\( \alpha = 1 \)).

- **Exponential factor** with \( F_{V_0}(v_0) = 1 - \exp(-\theta v_0), \ v_0 > 0 \): upper tail dependence; density available in closed-form.

- \( V_0 = V_1 - V_2 \), with \( V_1 \sim \text{Exp}(\theta_1) \perp \perp V_2 \sim \text{Exp}(\theta_2) \): upper and lower tail dependence; tail asymmetry; density available in closed-form.
Specific models

- **Weibull factor** with \( F_{V_0}(v_0) = 1 - \exp(-\theta v_0^\alpha), \, v_0 > 0 \): this model interpolates from the (tail-independent) Gaussian dependence structure (\( \alpha \to \infty \)) to perfect upper tail dependence (\( \alpha < 1 \)), including non-trivial upper tail dependence (\( \alpha = 1 \)).

- **Exponential factor** with \( F_{V_0}(v_0) = 1 - \exp(-\theta v_0), \, v_0 > 0 \): upper tail dependence; density available in closed-form.

- **V_0 = V_1 - V_2**, with \( V_1 \sim \text{Exp}(\theta_1) \perp \perp V_2 \sim \text{Exp}(\theta_2) \): upper and lower tail dependence; tail asymmetry; density available in closed-form.

- **Pareto factor** with \( F_{V_0}(v_0) = 1 - (v_0/v_*)^\beta, \, v_0 > v_*, \, \beta < 0 \): Perfect upper tail dependence.
Specific models

Exponential factor

Pareto factor

Weibull factor

\[ \lambda(q) \]

\[ \lambda(q) \]

\[ A(q) \]
Margins may be estimated **non-parametrically** or **parametrically** (provided a good parametric model may be found).
Inference

- Margins may be estimated non-parametrically or parametrically (provided a good parametric model may be found).
- A parametric copula family can be fitted using maximum likelihood inference (which is very efficient for exponential factor models).
Inference

- Margins may be estimated **non-parametrically** or **parametrically** (provided a good parametric model may be found).
- A parametric copula family can be fitted using **maximum likelihood inference** (which is very efficient for exponential factor models).
- Margins and the copula can be estimated in **two steps or one step**.
Inference

- Margins may be estimated non-parametrically or parametrically (provided a good parametric model may be found).
- A parametric copula family can be fitted using maximum likelihood inference (which is very efficient for exponential factor models).
- Margins and the copula can be estimated in two steps or one step.
- Pros and cons:
Inference

- Margins may be estimated non-parametrically or parametrically (provided a good parametric model may be found).
- A parametric copula family can be fitted using maximum likelihood inference (which is very efficient for exponential factor models).
- Margins and the copula can be estimated in two steps or one step.
- Pros and cons:
  - Non-parametric marginal estimation provides a robust approach when a parametric model is difficult to choose. However, there is a slight asymptotically-vanishing bias for finite samples.
Inference

- Margins may be estimated non-parametrically or parametrically (provided a good parametric model may be found).
- A parametric copula family can be fitted using maximum likelihood inference (which is very efficient for exponential factor models).
- Margins and the copula can be estimated in two steps or one step.
- Pros and cons:
  - Non-parametric marginal estimation provides a robust approach when a parametric model is difficult to choose. However, there is a slight asymptotically-vanishing bias for finite samples.
  - The one-step estimation procedure provide a more realistic assessment of the global uncertainty for copula parameters, but optimization is more difficult and may sometimes be unstable.
We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9$, 25, 100, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp}(\theta_1) \perp \perp V_2 \sim \text{Exp}(\theta_2)$. 
Simulation study

- We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9, 25, 100$, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp}(\theta_1) \perp V_2 \sim \text{Exp}(\theta_2)$.
- The underlying correlation function was $\rho(h) = \exp(-\theta Z h^\alpha)$. 
Simulation study

- We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9, 25, 100$, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp} (\theta_1) \perp \perp V_2 \sim \text{Exp} (\theta_2)$.
- The underlying correlation function was $\rho(h) = \exp(-\theta_Z h^\alpha)$.
- $N = 500, 1000, 2000$ independent replicates were simulated.
We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9, 25, 100$, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp}(\theta_1) \perp \perp V_2 \sim \text{Exp}(\theta_2)$.

The underlying correlation function was $\rho(h) = \exp(-\theta_Z h^\alpha)$.

$N = 500, 1000, 2000$ independent replicates were simulated.

500 independent experiments were performed to estimate the copula parameters $(\theta_1, \theta_2, \theta_Z, \alpha)$, and their bias, standard errors, and RMSEs.
Simulation study

- We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9, 25, 100$, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp}(\theta_1) \perp V_2 \sim \text{Exp}(\theta_2)$.
- The underlying correlation function was $\rho(h) = \exp(-\theta_Z h^\alpha)$.
- $N = 500, 1000, 2000$ independent replicates were simulated.
- 500 independent experiments were performed to estimate the copula parameters $(\theta_1, \theta_2, \theta_Z, \alpha)$, and their bias, standard errors, and RMSEs.
- Different parameter values were used.
Simulation study

- We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9$, 25, 100, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp}(\theta_1) \perp \perp V_2 \sim \text{Exp}(\theta_2)$.

- The underlying correlation function was $\rho(h) = \exp(-\theta_Z h^\alpha)$.

- $N = 500, 1000, 2000$ independent replicates were simulated.

- 500 independent experiments were performed to estimate the copula parameters $(\theta_1, \theta_2, \theta_Z, \alpha)$, and their bias, standard errors, and RMSEs.

- Different parameter values were used.

- Four different estimation procedures were tested (known margins, non-parametric marginal estimation, one-step and two step parametric estimation).
Simulation study

- We simulated data on a $3 \times 3$, $5 \times 5$ and $10 \times 10$ uniform grid in $[0, 1]^2$ (so $n = 9, 25, 100$, respectively) from the factor copula model with $V_0 = V_1 - V_2$ and $V_1 \sim \text{Exp}(\theta_1) \perp \perp V_2 \sim \text{Exp}(\theta_2)$.

- The underlying correlation function was $\rho(h) = \exp(-\theta_Z h^\alpha)$.

- $N = 500, 1000, 2000$ independent replicates were simulated.

- $500$ independent experiments were performed to estimate the copula parameters $(\theta_1, \theta_2, \theta_Z, \alpha)$, and their bias, standard errors, and RMSEs.

- Different parameter values were used.

- Four different estimation procedures were tested (known margins, non-parametric marginal estimation, one-step and two step parametric estimation).

- Other factor copula models were also investigated.
Simulation study

Exponential factors, \( n = 100 \) fixed, \( N = 500, 1000, 2000 \), one-step approach.
Simulation study

Exponential factors, $n = 9, 25, 100$, $N = 2000$ fixed, one-step approach.

- Boxplots for $\theta_1$
- Boxplots for $\theta_2$
- Boxplots for $\theta_Z$
- Boxplots for $\alpha$
We looked at daily mean temperature data at ten monitoring stations in a small region of Switzerland. Altitudes vary from 316 to 611 meters.
Application

- We looked at daily mean temperature data at ten monitoring stations in a small region of Switzerland. Altitudes vary from 316 to 611 meters.
- To avoid complex modeling of non-stationarity, we restrict our attention to May to September 2011 (153 days in total).
We looked at daily mean temperature data at ten monitoring stations in a small region of Switzerland. Altitudes vary from 316 to 611 meters.

To avoid complex modeling of non-stationarity, we restrict our attention to May to September 2011 (153 days in total).

We first fitted the marginal model:

\[ M_{t,j} = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \beta_1 M_{t-1,j} + \varepsilon_{t,j} + \gamma_1 \varepsilon_{t-1,j}, \quad \varepsilon_{t,j} \overset{iid}{\sim} \text{Skew}-t(\nu, \delta), \]

where \( M_{t,j} \) denotes the mean temperature measured at the \( j \)th station on day \( t \).
We looked at daily mean temperature data at ten monitoring stations in a small region of Switzerland. Altitudes vary from 316 to 611 meters.

To avoid complex modeling of non-stationarity, we restrict our attention to May to September 2011 (153 days in total).

We first fitted the marginal model:

$$ M_{t,j} = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \beta_1 M_{t-1,j} + \varepsilon_{t,j} + \gamma_1 \varepsilon_{t-1,j}, \quad \varepsilon_{t,j} \overset{iid}{\sim} \text{Skew}-t(\nu, \delta), $$

where $M_{t,j}$ denotes the mean temperature measured at the $j$th station on day $t$.

After transformation the residuals to the uniform scale, we then fitted

1) the Gaussian copula;
2) the Student-$t$ copula;
3) the common factor model with $V_0 = V_1 - V_2$, $V_1, V_2$ independent and
   a) $V_j \sim \text{Pareto}(\theta_j, 4)$;
   b) $V_j \sim \text{Exp}(\theta_j)$. 
Application

Scatter plots of normal scores for different pairs of stations.

- distance = 13km
- distance = 84km
- distance = 148km
Application

Scatter plots of normal scores for different pairs of stations.

Likelihood values:
1307 (Model 1), 1341 (Model 2), 1325 (Model 3a), 1346 (Model 3b).
Application

Predicted 5%, 50% and 95% quantiles for model 1 (bottom) and model 3b (top), conditional on observed values on August 1, 2011.
We have proposed factor copula models for spatial data with replicates and explored their tail dependence properties.
We have proposed factor copula models for spatial data with replicates and explored their tail dependence properties.

These models remedy some of the drawbacks of classical geostatistical spatial models.
Conclusion

- We have proposed factor copula models for spatial data with replicates and explored their tail dependence properties.
- These models remedy some of the drawbacks of classical geostatistical spatial models.
- Factor copula models combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
Conclusion

- We have proposed factor copula models for spatial data with replicates and explored their tail dependence properties.
- These models remedy some of the drawbacks of classical geostatistical spatial models.
- Factor copula models combine flexibility, parsimony, interpretability, tractability, can capture tail dependence and tail asymmetry (depending on the latent random factor), and are based on classical geostatistical models.
- However, they may not be valid for large regions, as complete uncorrelatedness cannot be captured at large distances.