Two Decompositions of Dependence for Multivariate Extremes

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Non-extreme multivariate analysis: linear algebra and covariance matrix

- PCA: $\boldsymbol{X}$ a p-dim'l random vector w/ cov matrix $\Sigma_{\boldsymbol{X}}$.
  - Spectral decomposition $\Sigma_{\boldsymbol{X}} = UDU^T$.
  - U an ordered orthonormal basis.
  - PCA/EOF: $\boldsymbol{Y} := U^T\boldsymbol{X}$, $\Sigma_{\boldsymbol{Y}} = D$.
  - Eigenvectors often interpreted, 'modes of variability'.
- $\boldsymbol{Z}_q$ q-dimensional random vector with cov mtx I.
  - A a $p \times q$ matrix, $\boldsymbol{X} := A\boldsymbol{Z}_q$: $\Sigma_{\boldsymbol{X}} = AA^T$.

- In extremes, covariance matrix not used to summarize dependence.
- Extremal dependence often modeled via framework of *multivariate regular variation*.

Outline

1. Inner product space via transformation
2. Regular variation and transformed linear operations
3. Tail pairwise dependence matrix
4. Decomposition 1: Eigen decomposition
5. Decomposition 2: Completely positive decomposition
6. Application: Extreme precipitation in Switzerland

Vector space via transformation

- $\boldsymbol{x} \in \mathbb{R}^p$
- t: 'transform', monotone function $\mathbb{R} \mapsto \mathbb{V}$, componentwise
- Example: $t(\boldsymbol{x}) = \exp(\boldsymbol{x})$
- $\boldsymbol{v} \in \mathbb{V}^p$
- $\boldsymbol{v}_1 \oplus \boldsymbol{v}_2 := t\Big(t^{-1}(\boldsymbol{v}_1) + t^{-1}(\boldsymbol{v}_2)\Big)$
- $c \circ \boldsymbol{v} = t(ct^{-1}(\boldsymbol{v}))$ for $c \in \mathbb{R}$
- $\boldsymbol{0} \in \mathbb{V}^p := t(\boldsymbol{0})$
- $-\boldsymbol{v} := t(-t^{-1}(\boldsymbol{v}))$

Claim: $\mathbb{V}^p$ is a vector space.

Associativity of vector addition

$$
\begin{aligned}
(\boldsymbol{v}_1 \oplus \boldsymbol{v}_2) \oplus \boldsymbol{v}_3 &= \left[t\left(t^{-1}\left(t\left(t^{-1}(v_{1j}) + t^{-1}(v_{2j})\right)\right) + t^{-1}(v_{3j})\right)\right]_{j=1,\ldots} \\
&= \left[t\left(t^{-1}(v_{1j}) + t^{-1}(v_{2j}) + t^{-1}(v_{3j})\right)\right]_{j=1,\ldots,p} \\
&= \left[t\left(t^{-1}(v_{1j}) + t^{-1}\left(t\left(t^{-1}(v_{2j}) + t^{-1}(v_{3j})\right)\right)\right)\right]_{j=1,\ldots} \\
&= \boldsymbol{v}_1 \oplus (\boldsymbol{v}_2 \oplus \boldsymbol{v}_3).
\end{aligned}
$$

Associativity of scalar multiplication

$$
\begin{aligned}
c_1 \circ (c_2 \circ \boldsymbol{v}) &= [c_1 \circ (t\left(c_2 t^{-1}(v_j)\right)]_{j=1,\ldots,p} \\
&= [t\left(c_1 t^{-1}(t\left(c_2 t^{-1}(v_j)\right)\right)]_{j=1,\ldots,p} \\
&= [t\left(c_1 c_2 t^{-1}(v_j)\right)]_{j=1,\ldots,p} \\
&= (c_1 c_2) \circ \boldsymbol{v}.
\end{aligned}
$$

Linear combinations, matrix/vector multiplication

$$c_1 \circ \boldsymbol{v}_1 \oplus \ldots \oplus c_q \circ \boldsymbol{v}_q = \left[t\Big(\sum_{j=1}^{q} c_j t^{-1}(v_{ji}) \Big) \right]_{i=1,\ldots,p}.$$

Linear indep: $c_1 \circ \boldsymbol{v}_1 \oplus \ldots \oplus c_p \circ \boldsymbol{v}_p = \mathbf{0} \Rightarrow c_1 = \ldots = c_p = 0$

Basis for $\mathbb{V}^p$: Any linearly indep set of p vectors in $\mathbb{V}^p$.

A a $p \times q$ matrix in $\mathbb{R}^{p\times q}$

$$\begin{aligned} A \circ \boldsymbol{v}_1 &:= [a_{i1} \circ v_{11} \oplus \ldots \oplus a_{ip} \circ v_{1p}]_{i=1,\ldots,p} \\ &= \boldsymbol{a}_{\cdot 1} \circ v_{11} \oplus \ldots \oplus \boldsymbol{a}_{\cdot p} \circ v_{1p} \\ &= t(At^{-1}(\boldsymbol{v}_1)) \end{aligned}$$

$$\begin{aligned} c_1 \circ \boldsymbol{v}_1 \oplus \ldots \oplus c_q \circ \boldsymbol{v}_q &= t\Big(c_1 t^{-1}(\boldsymbol{v}_1)\Big) \oplus \ldots \oplus t\Big(c_q t^{-1}(\boldsymbol{v}_q)\Big) \\ &= t(c_1 \boldsymbol{x}_1) \oplus \ldots \oplus t(c_q \boldsymbol{x}_q) \\ &= t\Big(X\boldsymbol{c}\Big) \\ &= X \circ t(\boldsymbol{c}) \end{aligned}$$

Inner product definition

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle := \sum_{i=1}^{p} t^{-1}(v_{1i}) t^{-1}(v_{2i})$$

- $\|\boldsymbol{v}\| := \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$
- $\boldsymbol{v}_1 \perp \boldsymbol{v}_2 := \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$
- $\boldsymbol{x}_1 = t^{-1}(\boldsymbol{v}_1), \boldsymbol{x}_2 = t^{-1}(\boldsymbol{v}_2) \in \mathbb{R}^p \Rightarrow \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle$

  $\Rightarrow \|\boldsymbol{v}\| = \|\boldsymbol{x}\|_2$

  $\Rightarrow \boldsymbol{x}_1 \perp \boldsymbol{x}_2$ in $\mathbb{R}^p$ *iff* $\boldsymbol{v}_1 \perp \boldsymbol{v}_2$ in $\mathbb{V}^p$

Eigenvalues and eigenvectors

- $S \in \mathbb{R}^{p\times p}$, think operator $\mathbb{V}^p \mapsto \mathbb{V}^p$ defined by $S \circ \boldsymbol{v}$
- S^{-1} defined such that $S^{-1} \circ \left(S \circ \boldsymbol{v}\right) = S \circ \left(S^{-1} \circ \boldsymbol{v}\right) = \boldsymbol{v}$ (corresponds to usual matrix inverse)

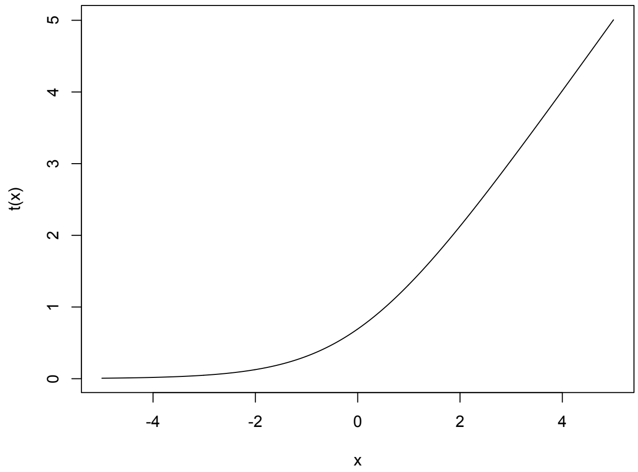
Define an eigenvalue/eigenvector pair $\lambda \in \mathbb{R}, \boldsymbol{e} \in \mathbb{V}^p$ of S to be such that $S \circ \boldsymbol{e} = \lambda \circ \boldsymbol{e}$.

$$
\begin{aligned}
S \circ \boldsymbol{e} &= t\Big(St^{-1}(\boldsymbol{e})\Big) \\
&= t(S\boldsymbol{u}) \\
&= t(\lambda\boldsymbol{u}) \\
&= t\Big(\lambda t^{-1}(\boldsymbol{e})\Big) \\
&= \lambda \circ \boldsymbol{e}.
\end{aligned}
$$

$\Rightarrow$ if $\lambda, \boldsymbol{u} \in \mathbb{R}^p$ eigenvalue/vector pair, then $\lambda, \boldsymbol{v} \in \mathbb{V}^p$ e-value/vector pair.

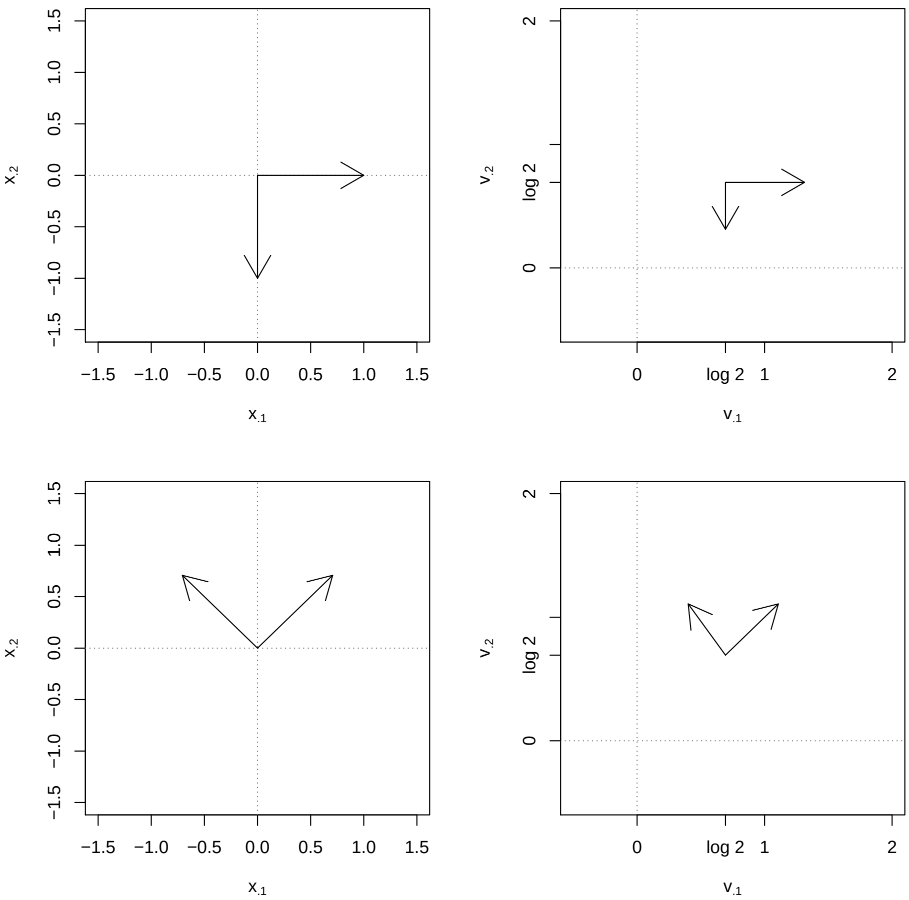
Our particular transformation

$$t(x) = \log(1 + \exp(x))$$



- anti-derivative of logistic fn $t(x) = \int \exp(x)/(1{+}\exp(x))dx$.
- $t^{-1}(v) = \log(\exp(v) - 1)$.
- $\mathbb{V}^p = (0, \infty)^p$.
- leaves upper tail alone: $\lim_{x\to\infty} \frac{t(x)}{x} = \lim_{x\to\infty} \frac{t^{-1}(x)}{x} = 1$.
- $t(0) = \log 2$.

Geometry of $\mathbb{V}^p$



- Vector pairs orthogonal.
- All vectors unit length.

Outline

1. Inner product space via transformation
2. Regular variation and transformed linear operations
   - Regular variation background
   - Transformed linear ops on reg var random vectors
   - A class of reg. var. random vectors
3. Tail pairwise dependence matrix
4. Decomposition 1: Eigen decomposition
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6. Application: Extreme precipitation in Switzerland

Regular variation: definition

$\boldsymbol{X}$ is a p-dimensional non-negative random vector.
$\boldsymbol{X}$ is *regularly varying* if there exists $\{b_n\}$ such that

$$nP\left(\frac{\boldsymbol{X}}{b_n} \in \cdot\right) \xrightarrow{v} \nu(\cdot)$$

where ν is a Radon measure on $[0,\infty]^p \setminus \{\mathbf{0}\}$.

Polar representation:
For any norm, let unit ball $\mathbb{S}_{p-1} = \{\boldsymbol{x} \in \mathbb{R}^p : \|\boldsymbol{x}\| = 1\}$.
Let $D(r,B) := \{\boldsymbol{x} \in \mathbb{R}_+^p : \|\boldsymbol{x}\| > r, \|\boldsymbol{x}\|^{-1}\boldsymbol{x} \in B\}$ for some $r > 0$, and some Borel set $B \subset \mathbb{S}_{p-1}$.

$$\nu(D(r,B)) = r^{-\alpha} H(B),$$

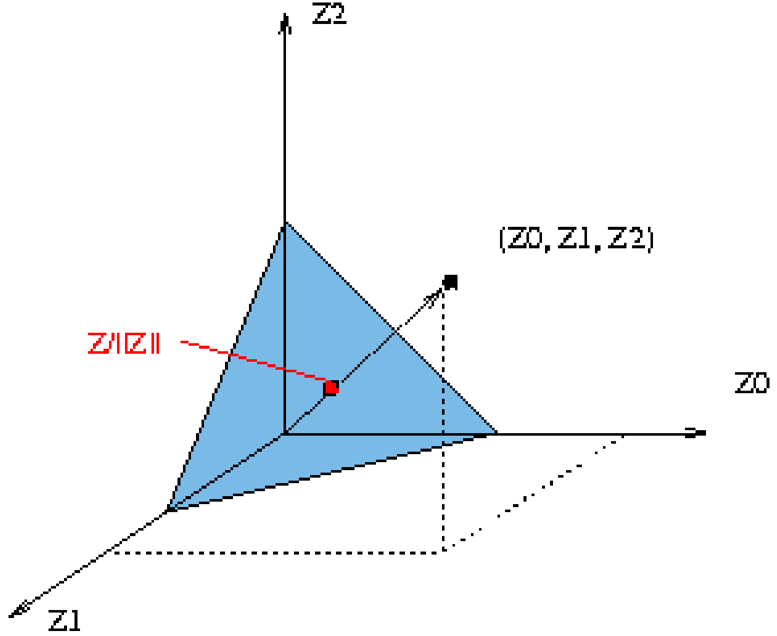
where H is 'angular' measure on $\mathbb{S}_{p-1}$.

$$\Rightarrow \nu(dr \times d\boldsymbol{w}) = \alpha r^{-\alpha-1} dH(\boldsymbol{w})$$

- α is index of reg var.

Making sense of regular variation

Idea: multivariate heavy-tailed distribution



Definition says: distribution of large points

- decomposes into *independent* radial/angular components.
- radial component decays like power function (α).
- angular component's dist'n described by H.

Why use regular variation for modeling?

- theoretical justification–tied to MVEVD's.
- defined in terms of tail, says nothing about distn's 'bulk'.
- framework for modeling (norm) threshold exceedances.
- allows for extrapolating further into the tail.
- *a multivariate model for asymptotic dependence.*

Modeling approach:

- Model assumes heavy-tailed marginals w/ common index.
- Often, transform to a common marginal:
  - often chosen s.t. $\alpha = 1$.
  - induces a balance condition on H.
- After transformation:
  - radial component behavior known.
  - need to model angular measure.
- In high dimensions, modeling H is hard!

Transformed regular varying random vectors

$\boldsymbol{X}$ is p-dimensional reg var with measure ν.
Let $\bar{\mathbb{V}}^p := [0, \infty]^p \setminus \{\mathbf{0}\}$.
Extend definition of t such that $t(-\infty) = 0$, and $t^{-1}(0) = -\infty$.

For $\boldsymbol{x} > \mathbf{0}$,

$$\begin{aligned} nP\left(\frac{t^{-1}(\boldsymbol{X})}{b_n} \in [-\boldsymbol{\infty}, \boldsymbol{x}]^c\right) &= nP\left(t^{-1}(\boldsymbol{X}) \in [-\boldsymbol{\infty}, b_n\boldsymbol{x}]^c\right) \\ &= nP\left(\boldsymbol{X} \in [\mathbf{0}, t(b_n\boldsymbol{x})]^c\right) \\ &\sim nP\left(\boldsymbol{X} \in [\mathbf{0}, b_n\boldsymbol{x}]^c\right) \\ &\to \nu([\mathbf{0}, \boldsymbol{x}]^c]). \end{aligned}$$

Transformed-linear operations on regularly varying random vectors

Proposition 1 *: Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be indep p-dimensional reg var random vectors, with normalizing sequence $\{b_n\}$ st*

$$nP(b_n^{-1}\boldsymbol{X}_1 \in \cdot) \xrightarrow{v} \nu_1(\cdot) \text{ and } nP(b_n^{-1}\boldsymbol{X}_2 \in \cdot) \xrightarrow{v} \nu_2(\cdot).$$

Define $\boldsymbol{X}_1 \oplus \boldsymbol{X}_2 = t\Big(t^{-1}(\boldsymbol{X}_1) + t^{-1}(\boldsymbol{X}_2)\Big)$. *Then*

$$nP\left(\frac{\boldsymbol{X}_1 \oplus \boldsymbol{X}_2}{b_n} \in \cdot\right) \xrightarrow{v} \nu_1(\cdot) + \nu_2(\cdot).$$

Proposition 2 *: Let $\boldsymbol{X}$ be st $nP(b_n^{-1}\boldsymbol{X} \in \cdot) \xrightarrow{v} \nu(\cdot)$. Assume $nP(X_i \leq \exp(-kn^{1/\alpha})) \to 0$ for any $k > 0$. Then for $a \in \mathbb{R}$,*

$$nP\left(\frac{a \circ \boldsymbol{X}}{b_n} \in \cdot\right) \xrightarrow{v} a^\alpha \nu(\cdot) \text{ if } a > 0, \text{ and}$$

$$nP\left(\frac{a \circ \boldsymbol{X}}{b_n} \in \cdot\right) \xrightarrow{v} 0 \text{ if } a \leq 0.$$

A matrix-defined class of reg var random vectors

Corollary 1 *: Let* $A = (\boldsymbol{a}_{\cdot 1}, \ldots, \boldsymbol{a}_{\cdot p})$ *be a* $p \times q$ *matrix where* $\max_{i=1,\ldots,p} a_{i,j} > 0$ *for all* $j = 1, \ldots q$*.*
Let $\boldsymbol{Z} = (Z_1, \ldots, Z_q)^T$ *be vector of iid reg var* α *random variables with* b_n *s.t.* $nP(Z_j > b_n z) \to z^{-\alpha}$ *for* $j = 1, \ldots, q$*.*
Then $A \circ \boldsymbol{Z}$ *is reg var* α *and angular measure*

$$H_{A \circ \boldsymbol{Z}}(\cdot) = \sum_{j=1}^{q} \|\boldsymbol{a}_{\cdot j}^{(0)}\|^{\alpha} \delta_{\boldsymbol{a}_{\cdot j}^{(0)}/\|\boldsymbol{a}_{\cdot j}^{(0)}\|}(\cdot),$$

where $\boldsymbol{a}^{(0)} = \max(\boldsymbol{a}, \boldsymbol{0})$. *(Geometry not quite right.)*

- angular measure discrete, corresponds to columns of A.
- similar to max-linear constructions (e.g., Strokorb and Schlat 2015).
- can show construction with *nonnegative* A forms dense class of reg var rand vecs (e.g., Fougères et al., 2013).
- realizations would differ from max-linear.

Outline

1. Inner product space via transformation
2. Regular variation and transformed linear operations
3. Tail pairwise dependence matrix
   - Special reg var case: $\alpha = 2$, L_2 norm.
   - Properties
     - scale.
     - pairwise asymptotic independence.
     - positive-definite.
     - relation to construction by A.
     - completely positive.
4. Decomposition 1: Eigen decomposition
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Tail pairwise dependence matrix

Assume $\boldsymbol{X}$ is such that

$$nP\left(\frac{\boldsymbol{X}}{\sqrt{n}} \in \cdot\right) \xrightarrow{v} \nu(\cdot), \text{ where } \nu(dr \times d\boldsymbol{w}) = 2r^{-3} dr dH_{\boldsymbol{X}}(\boldsymbol{w}),$$

and $H_{\boldsymbol{X}}$ is Radon measure on $\Theta_{p-1} = \{\boldsymbol{w} \in \mathbb{R}^p_+ : \|\boldsymbol{w}\|_2 = 1\}$.

- $\alpha = 2$, L_2 norm

Define TPDM

$$\sigma_{ik} := \int_{\Theta_{d-1}} w_i w_k dH_{\boldsymbol{X}}(\boldsymbol{w}), \text{ and } \Sigma_{\boldsymbol{X}} := [\sigma_{ik}]_{i,k=1,\ldots,p}.$$

- each σ_{ik} an extremal dependence measure (Larsson and Resnick, 2012). (χ, ext coef, madogram).
- analogous to a covariance matrix in non-extreme setting.
- pairwise!
- gives useful but incomplete dependence information.
  - much of standard MV analysis based on cov matrix.

Properties of TPDM

- *Diagonals describe scale:*

$$\lim_{n\to\infty} nP\left(\frac{X_i}{\sqrt{n}} > x\right) = \int_{\Theta_{p-1}} \int_{x/w_i}^{\infty} 2r^{-3} dr dH(\boldsymbol{w})$$
$$= x^{-2} \int_{\Theta_{p-1}} w_i^2 dH(\boldsymbol{w})$$
$$= x^{-2}\sigma_{ii}$$

- *Asymptotic independence:*

$$\lim_{n\to\infty} P\left(\frac{X_i}{\sqrt{\sigma_{ii}}} > \sqrt{n}z \middle| \frac{X_k}{\sqrt{\sigma_{kk}}} > \sqrt{n}z\right) = 0 \text{ iff } \sigma_{ik} = 0.$$

- Σ_X is *non-negative definite*.

TPDM and construction by matrix A

Let $\boldsymbol{Z} = (Z_1, \ldots, Z_q)^T$ indep rand vars st $nP(Z_j > \sqrt{n}z) \to z^{-2}$. A a $p \times q$ matrix with $\max_{i=1,\ldots,p} a_{i,j} \geq 0$. From before:

$$H_{A \circ \boldsymbol{Z}}(\cdot) = \sum_{j=1}^{q} \|\boldsymbol{a}_{\cdot j}^{(0)}\|_2^2 \delta_{\boldsymbol{a}_{\cdot j}^{(0)} / \|\boldsymbol{a}_{\cdot j}^{(0)}\|_2}(\cdot).$$

The (i,k)th element of $\Sigma_{A \circ \boldsymbol{Z}}$ is

$$\begin{aligned}
\sigma_{ik} &= \int_{\Theta_{p-1}} w_i w_k dH_{A \circ \boldsymbol{Z}}(\boldsymbol{w}) \\
&= \sum_{j=1}^{q} \left(\frac{a_{i,j}^{(0)}}{\|\boldsymbol{a}_{\cdot,j}^{(0)}\|_2} \right) \left(\frac{a_{k,j}^{(0)}}{\|\boldsymbol{a}_{\cdot,j}^{(0)}\|_2} \right) \|\boldsymbol{a}_{\cdot,j}^{(0)}\|_2^2 \\
&= \sum_{j=1}^{q} a_{i,j}^{(0)} a_{k,j}^{(0)},
\end{aligned}$$

thus $\Sigma_{A \circ \boldsymbol{Z}} = A^{(0)}(A^{(0)})^T$.

(Again geometry gets slightly in the way.)

Completely positive (New!)

Defn: Σ is completely positive if $\exists$ a *finite* $p \times q$ matrix A with nonnegative entries st $\Sigma = AA^T$. (Usually, $q > p$).

Dense result says $\exists$ *nonnegative* $\{A_q\}$ st $H_{A_q \circ \boldsymbol{Z}_q} \xrightarrow{w} H_{\boldsymbol{X}}$.
Define $\Sigma_q = A_q A_q^T$. $\{\Sigma_q\}$ a sequence of comp pos matrices.
$\Rightarrow$ $\Sigma = \lim_{q \to \infty} \Sigma_q$ is comp pos. (exist on closed cone) (Berman and Shaked-Monderer, 2003, Theorem 2.2).

$$\Rightarrow \exists\, q^* < \infty \text{ and nonnegative } A_{q^*} \text{ st } \Sigma = A_{q^*} A_{q^*}^T.$$

Take-away message:

- To match any $H_{\boldsymbol{X}}$, A needs infinite number of columns.
- To match $\Sigma_{\boldsymbol{X}}$, A can have finite number of columns.

Open questions about completely positive matrices:

- *cp*-rank. Can be pretty big (Berman et al., 2015).
- Factorization algorithms (Dür and Groetzner, 2016).

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Decomposition 1: Eigen decompostion

Why?

A: PCA/EOF. In standard PCA, a random vector can be created from a linear combination of an *orthonormal* basis with random coefficients of *decreasing* variance.

Σ is positive definite, can perform the usual eigendecomp.

$$\Sigma = UDU^T,$$

where $U = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p)$ is unitary.
D is diagonal w/ $\lambda_1 \geq \ldots \geq \lambda_p > 0$.

As $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p$ are basis for $\mathbb{R}^p$, $\boldsymbol{e}_i = t(\boldsymbol{u}_i)$ are basis for $\mathbb{V}^p$
$\Rightarrow$ for any (nonrandom) realization $\boldsymbol{x} \in \mathbb{V}^p$ $\exists$ representation

$$\begin{aligned} \boldsymbol{x} &= y_1 \circ \boldsymbol{e}_1 \oplus \ldots \oplus y_p \boldsymbol{e}_p \\ &= U \circ t(\boldsymbol{y}) \end{aligned}$$

Defining principal components

Following PCA, define

$$\boldsymbol{Y} = U^T \circ \boldsymbol{X}.$$

In standard PCA, cov mtx of $\boldsymbol{Y}$ is D.
Here, not quite. Because U has negative entries.

Consider $\boldsymbol{X} = A_q \circ \boldsymbol{Z}$. $\Sigma_{\boldsymbol{X}} = A_1 A_q^T$. TPDM of $\boldsymbol{Y}$:

$$\begin{aligned}
\Sigma_Y &= (U^T A_q)^{(0)} \Big((U^T A_q)^{(0)} \Big)^T \\
&= (U^T A_q)^{(0)} \Big((U^T A_q)^T \Big)^{(0)} \\
&= (U^T A_q)^{(0)} (A_q^T U)^{(0)} \neq D.
\end{aligned}$$

Scales of principal components

$$\boldsymbol{Y} := U^T \circ \boldsymbol{X}; \;\; \Sigma_{\boldsymbol{Y}} \text{ is TPDM of } \boldsymbol{Y}.$$

Result 1: $\sum_{i=1}^{p} \sigma_{ii} = \sum_{i=1}^{p} \lambda_i$. (from properties of trace)

Result 2: $\sigma_{11\boldsymbol{Y}} = \lambda_1$; $\sigma_{11\boldsymbol{Y}} \geq \sigma_{ii\boldsymbol{Y}}$ for $i = 2, \ldots, p$.

Result 3: $\sigma_{ii} \leq \lambda_i$ for $i = 2, \ldots, p$.

Although we cannot show the scales of $\boldsymbol{Y}$ are ordered, we can show there is an ordered upper bound.

My conclusion: Constructing PC's is useful for exploring the modes of dependence in extremes.

- Represent as linear combination of orthogonal basis (new to extremes).
- Some idea of ordering of importance: can still do dimension reduction.

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Completely positive decomposition

$\boldsymbol{X}$ has TPDM $\Sigma_{\boldsymbol{X}}$; $\exists$ A_{q^*} st $\Sigma_{\boldsymbol{X}} = A_{q^*} A_{q^*}^T$.

Q: Why find A_{q^*}?
A: Simulation or Estimation of probabilities.
$\boldsymbol{X}^* := A_{q^*} \circ \boldsymbol{Z}$, $\boldsymbol{Z}$ iid w/ scale $= 1$ has TPDM $\Sigma_{\boldsymbol{X}}$.

Q: Can you find an A_{q^*}?
A: Active area of research.

- Algorithms can do moderate size ($\sim p = 40$).
  (Big for extremes!)

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Swiss Data

- 44 stations.
- 4692 days.
- Rank-transformed to be reg var $\alpha = 2$ with scale 1.
- $\hat{\Sigma}_X$ estimated by taking largest 5%.

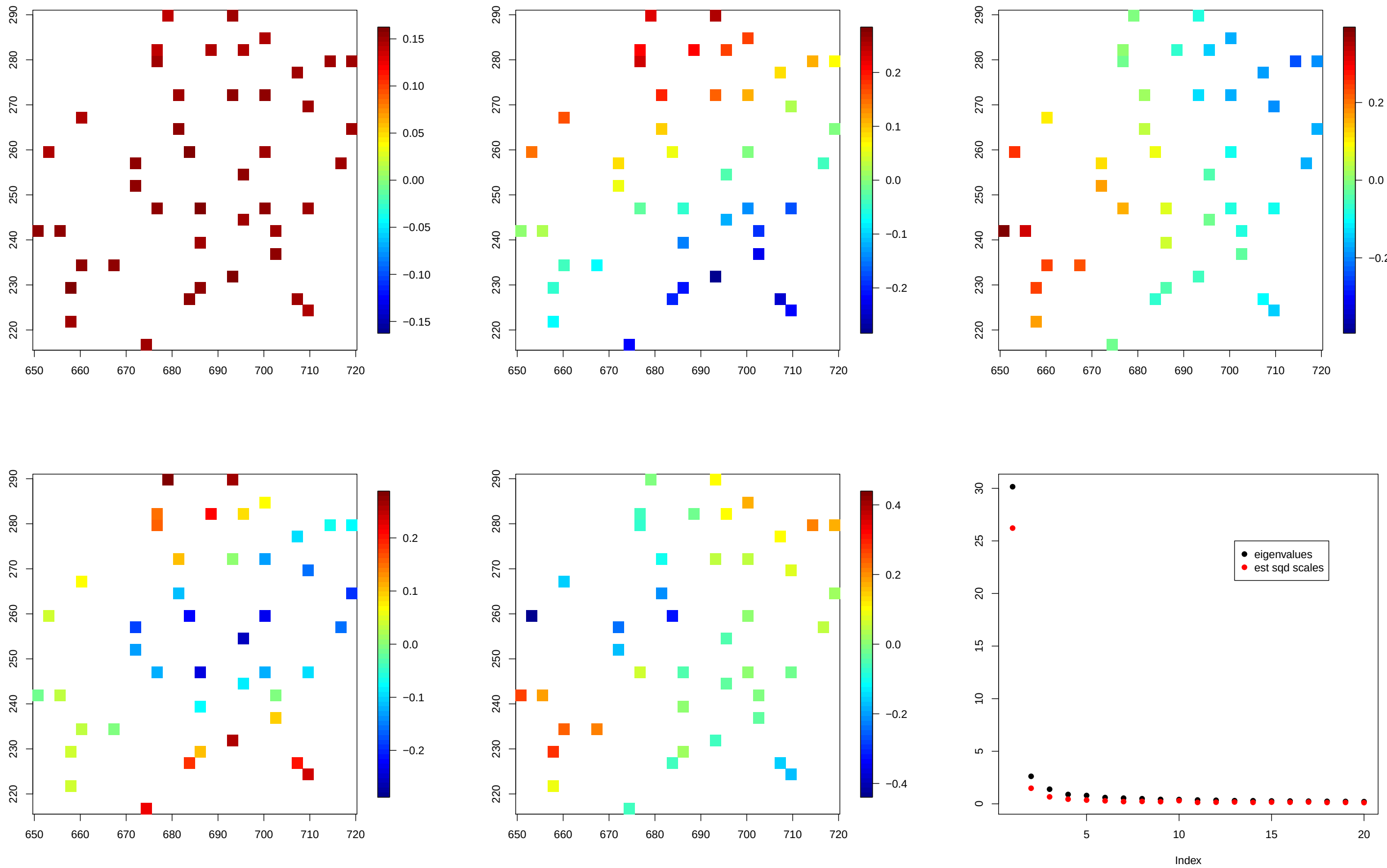
Note: Although data are spatial, we are doing multivariate, not process, modeling.

Plan:

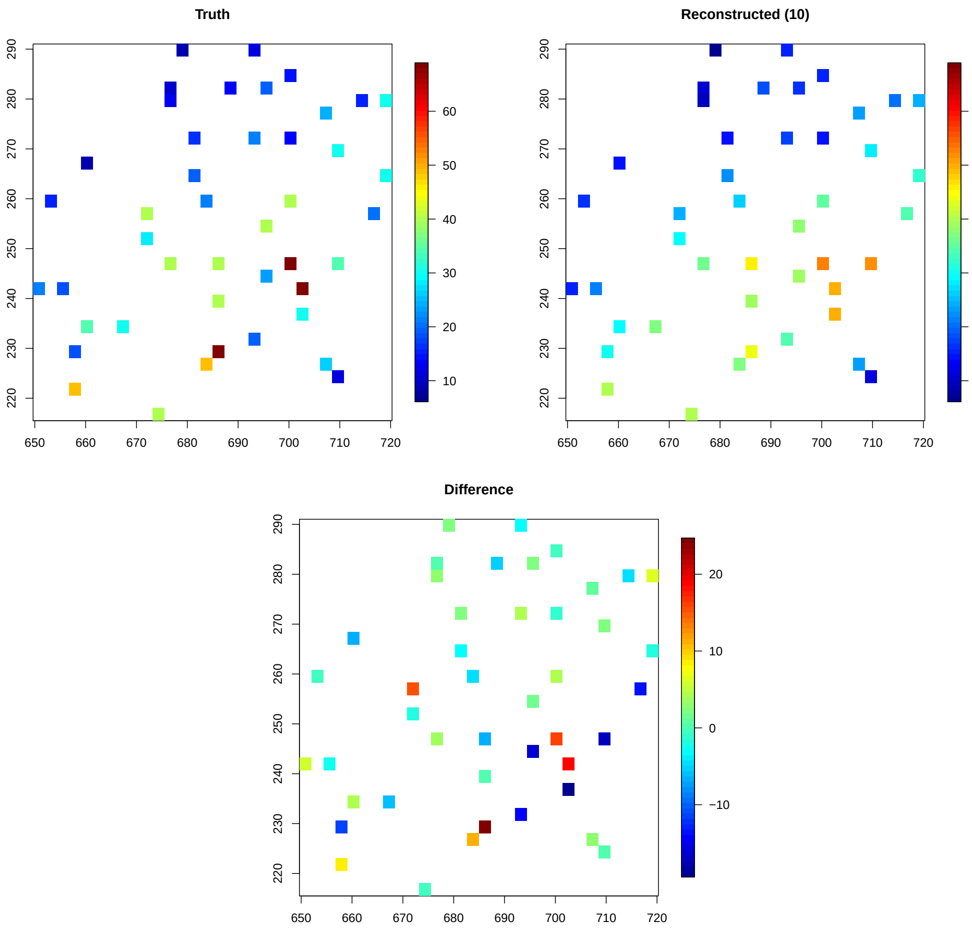
1. Eigendecomposition (procedure: "eig" in R).
2. Completely positive factorization.

Leading eigenvectors

Plots of $\boldsymbol{u}_1, \ldots \boldsymbol{u}_p$ (in $\mathbb{R}^p$)



Partial basis reconstruction



Partial reconstruction of 3rd largest event in record.

Completely positive factorization

Procedure: Send to Dur and Groetzner (University of Trier).
Returned $\widehat{A}$ dimension 44×51 (dim surprisingly small).
Error: $\|\hat{\Sigma} - \widehat{A}\widehat{A}^T\| = 2.5 \times 10^{-14}$.

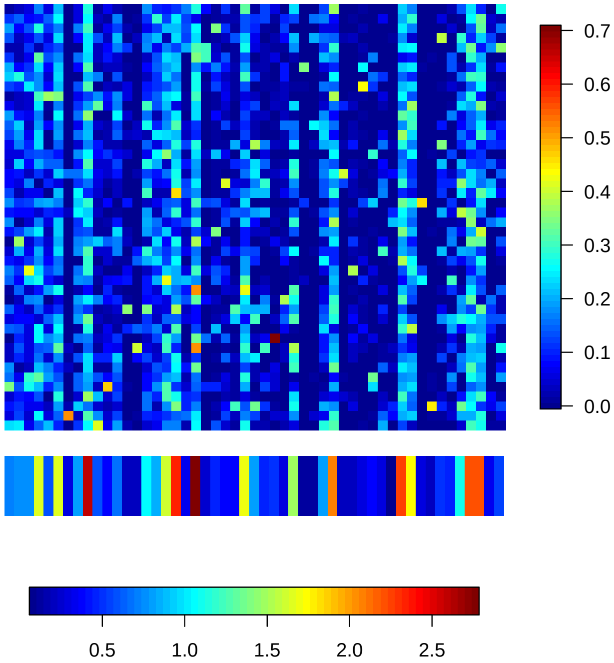


Image of $\widehat{A}_{q^*}$ and column norms.

Probability estimation and simulation

Define $\boldsymbol{X}^* = \widehat{A}_{q^*}\boldsymbol{Z}_{q^*}$. Important: $\boldsymbol{X}^* \overset{d}{\neq} \boldsymbol{X}$, they don't have same angular measure. But $\Sigma_{\boldsymbol{X}^*} = \widehat{\Sigma}_{\boldsymbol{X}}$.

Probability of event in risk region:

$D^{(orig)} = \{\boldsymbol{x} \in \mathbb{R}^{44} \mid x_i > 30\}$, define $D = tr(D^{(orig)})$.

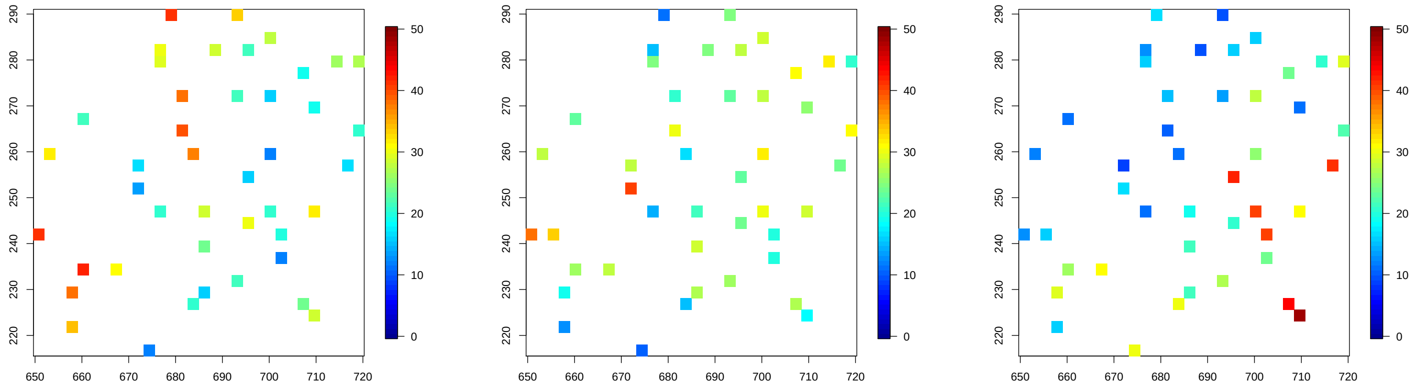
$\widehat{P}(\boldsymbol{X}_t^* \in D) = 4.8 \times 10^{-4}$.

Empirical est of $P(\boldsymbol{X}_t^{(orig)} \in D^{(orig)})$ is $2/4691 = 4.3 \times 10^{-4}$.

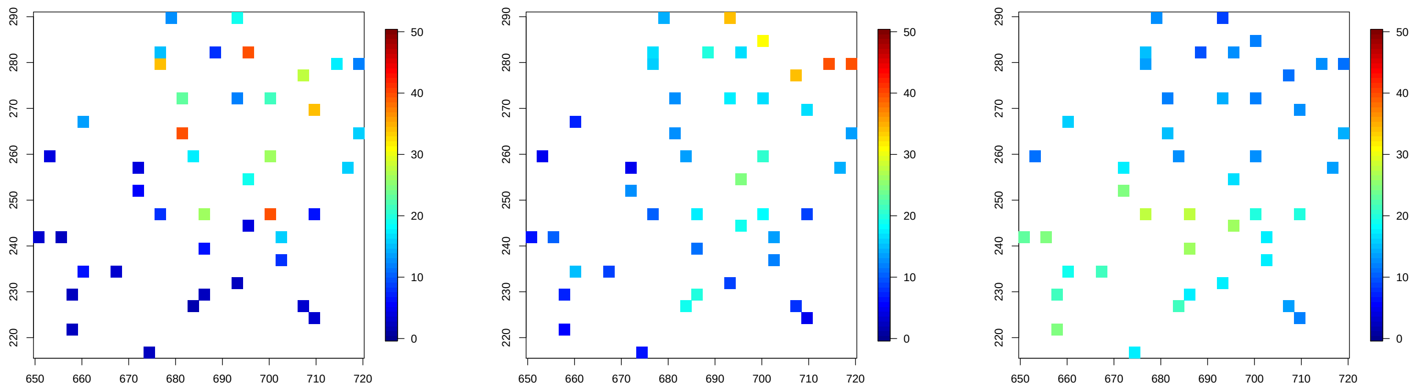
Simulated and observed

Generate $\boldsymbol{X}^* = \hat{A} \circ \boldsymbol{Z}$.

Simulated

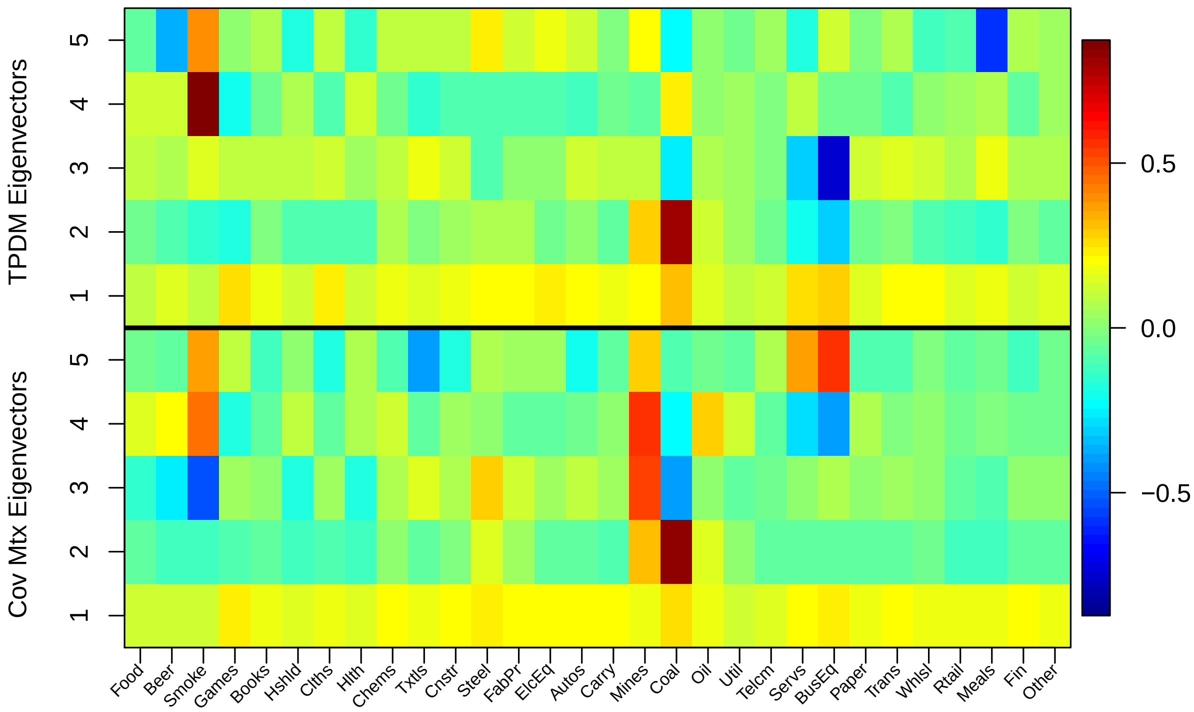


Observed



Different Application: Financial

Data: Daily returns from 30 industrial categories.



Summary

- Can create a vector space for positive orthant by applying a transformation to $\mathbb{R}^p$.
- With right transformation, transformed linear operations on reg var random vectors remain regular varying.
- $\Rightarrow$ can do linear-algebra-like things for extremes.
- Can *summarize* tail dependence in TPDM ($\alpha = 2, L_2$).
- Two ways to factorize TPDM:
  - Eigendecomposition $\rightarrow$ exploring modes of variability, dimension reduction.
  - Completely positive decomposition $\rightarrow$ simulation, estimation of probabilities.

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