Appendix A

Fourier transforms

A.1 Discrete FT

Consider a discrete periodic time series \( f_j \), with period \( n \). It is often useful to express such a data set as an “expansion in waves”, i.e., as a sum over a set of \( n \) periodic functions \( e^{2\pi ikj/n} \), with \( k \) ranging from zero to \( n - 1 \). We write

\[
    f_j = \sum_{k=0}^{n-1} \tilde{f}_k e^{2\pi ikj/n} \quad (A.1)
\]

We would like to extract the amplitude \( \tilde{f}_k \), by an appropriate sum over the series \( f_j \), making use of the orthogonality of the complex exponentials:

\[
    \tilde{f}_k = \frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi ikj/n} f_j \quad (A.2)
\]

To prove this relation, substitute the expansion for \( f_j \) into the above equation and reverse the order of summation:

\[
    \frac{1}{n} \sum_{j} e^{-2\pi ikj/n} f_j = \sum_{m} \tilde{f}_m \frac{1}{n} \sum_{j} e^{2\pi i(m-k)j/n} \quad (A.3)
\]

(Note that multiple normalization conventions exist for this transform as it appears in various sources.)

Consider the inner sum, which can be written in terms of the function

\[
    S_q = \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i q j/n} \quad (A.4)
\]

with argument \( q = m - k \). The function \( S_q \) is evidently periodic with period \( n \), since the replacement \( q \to q + n \) generates a factor \( e^{2\pi ij} = 1 \) multiplying the original summand.

We shall show that \( S_q \) vanishes unless \( q = 0 \) (for \( q \) in the range from 0 to \( n - 1 \)). To see this, note that the summands are the \( q \)th powers of the \( n \) complex \( n \)th roots of unity, which are uniformly spaced about the unit circle. Therefore, the summation index is periodic, and can be shifted by a constant without changing the sum. If we shift \( j \) by 1, we can show \( S_q = \exp(2\pi i q/n)S_q \). Thus \( S_q \) equals zero unless \( q = 0 \), in which case \( S_0 = 1 \).
This result demonstrates the orthogonality of the complex exponential basis functions,

\[
\frac{1}{n} \sum_{j} \exp(2\pi imj/n) \exp(-2\pi ikj/n) = \delta_{mk}
\]  

(A.5)

in which we restrict \( m \) and \( k \) to the integer interval \([0, n - 1]\).

The above expressions also serves as a “completeness relation”, because if the Kronecker delta can be expanded in terms of the basis set of complex exponential waves, then any function can be so expanded.

That is, any function \( f_m \) on the integers \([0, n - 1]\) can be written as

\[
f_m = \sum_{k} f_k \delta_{mk}
\]

(A.6)

into which we can substitute the completeness relation, to obtain after some rearrangement

\[
f_m = \sum_{j} \tilde{f}_j e^{2\pi jm/n}
\]

(A.7)

with \( \tilde{f}_j \) defined as in Eq. (A.2).

Note that if \( f_j \) is real-valued (so that \( f_j^* = f_j \)), we have

\[
\tilde{f}_j^* = \frac{1}{n} \sum_{j} \exp(2\pi ikj/n) f_j = \tilde{f}_{-k}
\]

(A.8)

### A.2 Fourier series

Consider a periodic continuous time series \( f(x) \) with period \( L \). We can think of such a periodic continuous time series as the limit of a large number of points in a periodic discrete series such as we discussed in the previous section.

Let \( L = ndx \), where \( n \) is the number of data points and \( dx \) is their spacing on the continuous axis (space or time, depending on the application). The location \( x_j \) is simply \( jdx \).

Now take the limit \( n \to \infty \) and \( dx \to 0 \) such that \( L \) is fixed, whereupon the sums of the previous section become integrals:

\[
\frac{1}{n} \sum_{j=0}^{n-1} \to \frac{1}{L} \int_0^L dx
\]

(A.9)

We can use this correspondence to translate the results of the previous section to the present case of a continuous time series with period \( L \). Or, we can proceed directly to derive the analogous results for Fourier series, following the analogous sequence of manipulations in the new notation.

Either way, we would like to represent this time series as a sum of waves, i.e., as a sum over the infinite set of complex exponentials \( e^{iqx} \), in which \( q = 2\pi k/L \), and \( k \) ranges over the integers. We write

\[
f(x) = \sum_{q} \tilde{f}_q e^{iqx}
\]

(A.10)

We wish to extract the Fourier amplitude \( f_q \) by again taking the inner product of \( e^{iqx} \) with \( f(x) \):

\[
\tilde{f}_q = \frac{1}{L} \int_0^L dx e^{-iqx} f(x)
\]

(A.11)
As before, to prove that this relation holds, substitute the expansion for \( f(x) \) on the right hand side, and reverse the order of summation and integration; we have

\[
\frac{1}{L} \int_0^L dx e^{-iqx} f(x) = \sum_{q'} f_{q'} \frac{1}{L} \int dx e^{i(q'-q)x} \tag{A.12}
\]

This integral is elementary,

\[
\frac{1}{L} \int dx e^{i(q'-q)x} = \delta_{qq'} \tag{A.13}
\]

which expresses the orthogonality of the complex basis functions, and proves the formula Eq. (A.11) for extracting the Fourier amplitude \( \tilde{f}_q \) from the function \( f(x) \).

The above result can also serve as a completeness relation, in the sense that any arbitrary set of wave amplitudes \( f_q \) can be expressed as an integral over waves Eq. (A.11) of some periodic function \( f(x) \), in fact given by Eq. (A.10).

The reverse is also true; that is, any (sufficiently smooth) periodic function \( f(x) \) can be expressed as the sum of waves Eq. (A.10) for some set of mode amplitudes \( f_q \). To see this, we consider the limit of the completeness relation for the discrete Fourier transform Eq. (A.5) as the number of modes \( n \) becomes large.

The exponential factors \( e^{2\pi jm/n} \) in Eq. (A.5) can be written as \( e^{iqx} \) with \( q = 2\pi j/L \) and \( x = m dx \), whereupon we have

\[
\sum_j e^{iq(x-x')} = n\delta m, k = (L/dx)\delta_{x,x'} \rightarrow L\delta(x-x') \tag{A.14}
\]

in which \( x \) and \( x' \) are assumed to reside on the interval \([0, L]\).

We identify \( \lim_{n \to \infty} n\delta_{mk} \) as \( \delta(x-x') \), because as \( n \) becomes large with fixed \( L \), \( dx \) becomes small, and a sum \( \sum_m dx \) over the set of finely spaced \( x \) values becomes the integral \( \int_0^L dx \). So we have

\[
dx \sum_m n\delta_{m,k} = ndx = L
\]

\[
\int_0^L dx L\delta(x-x') = L \tag{A.15}
\]

With the identification of \( \sum_j e^{iq(x-x')} \) with \( L\delta(x-x') \), we can use the same pattern of argument to show that any function \( f(x) \) can be expanded in waves:

\[
f(x) = \int_0^L dx' \delta(x-x')f(x') = \sum_j e^{iqx} \frac{1}{L} \int_0^L dx' e^{-iqx'} f(x') = \sum_j e^{iqx} \tilde{f}_q \tag{A.16}
\]

To reach the second line, we substitute Eq. (A.14) and rearrange; to reach the final line, we recognize the Fourier amplitude from Eq. (A.11).

As for discrete series, if \( f(x) \) is real-valued (so that \( f^*(x) = f(x) \)), we have

\[
\tilde{f}_k = \tilde{f}_{-k} \tag{A.17}
\]
A.3 Fourier transform

As the width of the domain $x = (0, L)$ (or equivalently $-L/2$ to $L/2$) increases, the discrete infinite set of wavenumbers grows more closely spaced. To take the limit of large $L$, we need to redefine the Fourier amplitudes in such a way as to eliminate explicit appearance of $L$, and create the necessary factors to turn $\sum_q$ into an integral.

Hence we rewrite $\tilde{f}_q$ as $(1/L)\tilde{f}(q)$, define $dq = 2\pi/L$, and therefore write

$$f(x) = \int \frac{dq}{2\pi} \tilde{f}(q) \exp(iqx)$$
$$\tilde{f}(q) = \int dx \exp(-iqx)f(x)$$
$$\int \frac{dq}{2\pi} \exp(iqx) = \delta(q)$$ (A.18)

In the last expression, we have defined the delta-function $\delta(q)$ as the limit of $L/(2\pi)\delta_{q0}$, i.e., such that

$$\delta(q)dq = \delta_{q0}$$ (A.19)

This is a narrow function, normalized in that

$$\int dq \delta(q) = \sum q \delta_{q0} = 1$$ (A.20)

The last of the above array of equations expresses the orthogonality of the complex exponential on the infinite interval, as well as the completeness relation (expansion of the delta function in basis functions).

Once again, if $f(x)$ is real-valued, we have $\tilde{f}^*(k) = \tilde{f}(-k)$.

A.4 Convolution theorem

One useful property of Fourier transforms is that the Fourier transform of a convolution of two functions, is the product of the Fourier transforms of the two functions. Such a theorem holds equally for discrete Fourier transform, Fourier series, and continuous Fourier transform.

Consider a convolution of two periodic series $f$ and $g$, defined as

$$C_j = (1/n) \sum_m f_m g_{m+j}$$ (A.21)

Substitute both $f$ and $g$ with their discrete Fourier transforms, to obtain

$$C_j = (1/n) \sum_m \sum_{k_1} \sum_{k_2} e^{2\pi ik_1j/n} e^{2\pi ik_2(j+m)/n} \tilde{f}_{k_1} \tilde{g}_{k_2}$$ (A.22)

Perform the sum over $m$, making use of the completeness relation, to obtain after a bit of algebra

$$C_j = \sum_k \tilde{f}_k \tilde{g}_{-k} e^{2\pi i kj/n}$$ (A.23)

That is, $C_j$ is the discrete transform of $\tilde{f}_k \tilde{g}_{-k}$. If $f$ and $g$ are real-valued, we have

$$\tilde{C}_k = \tilde{f}_k \tilde{g}_k^*$$ (A.24)
The same result holds for Fourier series; we define

\[ C(y) = \frac{1}{L} \int_0^L dx \, f(x)g(x + y) \]  
(A.25)

and obtain

\[ \tilde{C}_k = \tilde{f}_k \tilde{g}_k - \tilde{f}_k \tilde{g}_k^\ast \]  
(A.26)

in which the second equality holds if \( g \) is real-valued. This result is proved by essentially the same sequence of manipulations as for discrete transforms; namely, substitute \( f \) and \( g \) with their Fourier series, use the completeness relation, and use the resulting Kronecker delta to perform one of the sums over wavenumber.

Finally, essentially the same result holds for continuous Fourier transforms; we define

\[ C(y) = \int dx \, f(x)g(x + y) \]  
(A.27)

which leads in analogous fashion to

\[ \tilde{C}(k) = \tilde{f}(k)\tilde{g}(-k) = \tilde{f}(k)\tilde{g}^\ast(k) \]  
(A.28)

in which the second equality holds if \( g \) is real-valued.

### A.5 Hamiltonians in Fourier space

The simplest Landau-Ginzburg effective Hamiltonian for a symmetric scalar field \( \psi(r) \) takes the form

\[ \beta H = \int dr \left[ \frac{1}{2} \tau \psi^2(r) + \frac{1}{2} a^2 (\nabla \psi(r))^2 + \frac{\lambda}{4} \psi^4(r) \right] \]  
(A.29)

We can transform this to Fourier space by substituting \( \psi(r) \) by its Fourier transform, and using the delta function relationship to simplify the resulting multiple integrals. For example, the first term is

\[ \int dr \psi^2(r) = \int dr \int_{q_1} \int_{q_2} \psi(q_1)\psi(q_2)e^{i(q_1+q_2)\cdot r} \]

\[ = \int dq |\psi(q)|^2 \]  
(A.30)

in which we used the fact that \( \psi(r) \) is real to replace \( \psi(-q) \) with \( \psi^*(q) \), and the shorthand notation \( \int_q \) denotes the \( d \)-dimensional Fourier integral \( \int d^d q / (s\pi)^d \).

With similar manipulations, we can transform the Landau-Ginzburg Hamiltonian as

\[ \beta H = \frac{1}{2} \int_q (\tau + a^2 q^2)|\psi(q)|^2 + \frac{\lambda}{4} \int_{q_1} \int_{q_2} \int_{q_3} \int_{q_4} (2\pi)^d \delta(q_1 + q_2 + q_3 + q_4) \psi(q_1)\psi(q_2)\psi(q_3)\psi(q_4) \]  
(A.31)

(In this course we will not make much use of the quartic term.)

Sometimes it is more convenient to have a discretely infinite set of Fourier modes, as we do for a periodic system of finite size. Employing the Fourier series in a way analogous to the above, we obtain

\[ \int dr \psi^2(r) = \int dr \sum_{q_1} \sum_{q_2} \psi_{q_1}\psi_{q_2}e^{i(q_1+q_2)\cdot r} \]

\[ = V \sum dq |\psi_q|^2 \]  
(A.32)
in which the sum ranges over values \( q = 2\pi n/L \) where \( n \) is any integer.

The Hamiltonian is similarly transformed to

\[
\beta H = \frac{V}{2} \sum_{q} (\tau + a^2 q^2) |\psi_q|^2 + \frac{V\lambda}{4} \sum_{q_1,q_2,q_3,q_4} \delta_{q_1+q_2+q_3+q_4,0} \psi_{q_1} \psi_{q_2} \psi_{q_3} \psi_{q_4} \tag{A.33}
\]

### A.6 Gaussian wave packets

Fourier transforms express a function \( f(x) \) that varies in “real space”, as a weighted sum of plane waves, with weights \( \tilde{f}(q) \). To gain some insight into what Fourier transforms do, it is instructive to consider a particular convenient form of a real-space function, namely a one-dimensional normalized “Gaussian wave packet”, of the form

\[
f(x) = (2\pi \sigma^2)^{-1/2} e^{-x^2/(2\sigma^2)} \cos(\kappa x) \tag{A.34}
\]

(See Fig. A.1).

The quantity \( \kappa \) is the wavenumber of the packet, given by \( 2\pi/\lambda \) where \( \lambda \) is the wavelength of the oscillation. The width of the packet is set by \( \sigma \). We can transform \( f(x) \) using our method for Gaussian integrals of completing the square in the exponent, as

\[
\tilde{f}(q) = (2\pi \sigma^2)^{-1/2} \int dx e^{-iqx} e^{-x^2/(2\sigma^2)} \cos(\kappa x)
\]

\[
= (2\pi \sigma^2)^{-1/2} \int dx e^{-iqx} e^{-x^2/(2\sigma^2)} (1/2) \left( e^{i\kappa x} + e^{-i\kappa x} \right)
\]

\[
= (1/2) \left( e^{-(q-\kappa)^2 \sigma^2/2} + e^{-(q+\kappa)^2 \sigma^2/2} \right) \tag{A.35}
\]

(See Fig. A.1).

We see that the Gaussian wavepacket is actually composed of complex exponential waves \( e^{iqx} \) with wavenumbers centered on \( \kappa \) and \(-\kappa\), as one might expect from observing that \( \cos(\kappa x) \) equals \( (1/2) (e^{i\kappa x} + e^{-i\kappa x}) \). However, we see that there is a spread in the wavenumbers for which the amplitude is significant, set by \( 1/\sigma \). When the packet is broad in real space (larger \( \sigma \)), the wavenumber becomes more well defined (smaller \( 1/\sigma \)) and vice versa.
A.7 A useful transform pair

A Fourier transform that arises in many contexts, for example our calculation of the correlation function for composition fluctuations in the vicinity of the critical point \( \langle \psi(r) \psi(r') \rangle \), is the Fourier integral

\[
I(r) = \int \frac{dq}{(2\pi)^3} \frac{e^{iq \cdot r}}{\kappa^2 + q^2} = \frac{e^{-\kappa|r|}}{4\pi |r|} \tag{A.36}
\]

In various contexts, this functional form is called a “screened Coulomb potential” (the potential of a point charge in the presence of mobile screening ions), or the “Yukawa potential” (from nuclear physics).

The above integral can be computed analytically, using the method of contour integration from complex analysis, which is beyond the scope of these lectures. However, we can at least verify the answer by computing the inverse of this Fourier transform,

\[
I(q) = \int dr e^{-iq \cdot r} \frac{e^{-\kappa|r|}}{4\pi |r|} = \frac{1}{\kappa^2 + q^2} \tag{A.37}
\]

which be evaluated by more elementary methods of integration.

We perform the above integral in spherical coordinates, writing \( q \cdot r \) as \( q r \mu \), with \( \mu = \cos \theta \), and the integration as \( \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \). The integral over the azimuthal angle \( \phi \) gives a factor of \( 2\pi \), since the integrand does not depend on \( \phi \).

Thus we have, by straightforward manipulations,

\[
I(q) = \int_0^\infty dr r e^{-\kappa r} (1/2) \int_{-1}^1 d\mu e^{iq r \mu} \\
= \int_0^\infty dr r e^{-\kappa r} \frac{e^{iq r} - e^{-iq r}}{2iq r} \\
= \int_0^\infty dr r e^{-\kappa r} \sin qr \frac{qr}{qr} \\
= (1/q) \text{Im} \int_0^\infty dr r e^{-\kappa r} \tag{A.38}
\]

This functional form is called a “Lorentzian”, and emerges in many contexts in physics.

So the Yukawa potential and the Lorentzian are a “Fourier transform pair”; having shown that the Lorentzian is the transform of the Yukawa potential, we know from the general properties of Fourier transforms that the Yukawa potential is the inverse transform of the Lorentzian.
Appendix B

Gaussians

B.1 Moment integrals

We begin with the integral of a Gaussian,

\[ I_0(\alpha) = \int_{-\infty}^{\infty} dx \, e^{-\alpha x^2} = (\pi/\alpha)^{1/2} \]

This result can be obtained by squaring \( I_0 \), interpreting the result as an integral over the \( x-y \) plane, and converting to polar coordinates:

\[ I_0^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, e^{-\alpha(x^2+y^2)} \]

\[ = \int_0^{2\pi} d\theta \int_0^{\infty} r dr \, e^{-\alpha r^2} \]  

(B.2)

Now a change of variable \( z = \alpha r^2 \) gives \( dz = 2\alpha \, r \, dr \), with the entire factor \( r \, dr \) now downstairs (which was the point of squaring \( I_0 \)), so that we can perform the integral over \( r \) (giving a factor \( 1/(2\alpha) \) and \( \theta \) (giving a factor \( 2\pi \)), leading to the result above.

To compute averages of powers of \( x \) in a Gaussian distribution, we require integrals of \( x^n \) times a Gaussian. If \( n \) is odd, the integral \( \int_{-\infty}^{\infty} x^n e^{-\alpha x^2} \) vanishes by symmetry (integrand is odd). If \( n \) is even, we can generate the factor of \( x^n \) by taking successive derivatives of \( I_0 \) with respect to \( \alpha \). Each derivative brings down a factor of \( -x^2 \). Hence we have

\[ \int_{-\infty}^{\infty} dx \, x^{2k} e^{-\alpha x^2} = (-\partial/\partial \alpha)^k I_0(\alpha) \]

For example,

\[ \int_{-\infty}^{\infty} dx \, x^2 e^{-\alpha x^2} = (-\partial/\partial \alpha) I_0(\alpha) = \frac{\pi^{1/2}}{2\alpha^{3/2}} = \frac{I_0(\alpha)}{2\alpha} \]

We often require integrals in which a Gaussian integrand is multiplied by a simple exponential, as in

\[ J = \int_{-\infty}^{\infty} dx \, e^{-\beta x} e^{-\alpha x^2} \]  

(B.3)

One example of this form is the Fourier transform of a Gaussian of width \( \sigma \),

\[ I = \int_{-\infty}^{\infty} dx \, e^{-iqx} e^{-x^2/(2\sigma^2)} \]  

(B.4)
Integrals like $J$ can be evaluated by “completing the square” in the exponent, i.e., by writing the exponent as

$$\beta x + \alpha x^2 = \alpha(x - \beta/(2\alpha))^2 - \beta^2/(4\alpha)$$  \hspace{1cm} (B.5)

Then we can shift the integration variable, defining $y = x - \beta/(2\alpha)$, so that

$$J = \int_{-\infty}^{\infty} dy \, e^{-\alpha y^2 - \beta^2/(4\alpha)} = I_0(\alpha)e^{-\beta^2/(4\alpha)}$$  \hspace{1cm} (B.6)

Using this result we have for the Fourier transform of a Gaussian of width $\sigma$, (with $\beta = iq$ and $\alpha = 1/(2\sigma^2)$),

$$I = \int_{-\infty}^{\infty} dx \, e^{-iqx} e^{-x^2/(2\sigma^2)} = I_0(1/(2\sigma^2))e^{-q^2\sigma^2/2}$$  \hspace{1cm} (B.7)

Note that the width of the Fourier transformed Gaussian is $1/\sigma$; a narrow Gaussian in “real space” transforms to a wide Gaussian in “reciprocal space”, and vice versa.

### B.2 Surface area in $n$ dimensions.

From the previous section we have the integral of a Gaussian in $n$ dimensions as

$$I(n) = \int dR^n \, e^{-R^2} = \pi^{n/2}$$  \hspace{1cm} (B.8)

We can write this integral in “hyperspherical coordinates”, as

$$I(n) = \int dR^n \, e^{-R^2} = S_n \int_0^{\infty} dR \, R^{n-1} e^{-R^2}$$  \hspace{1cm} (B.9)

in which $S_n$ is the area of the unit hypersphere.

Changing variables to $y = R^2$, we have

$$I(n) = \frac{S_n}{2} \int_0^{\infty} dy \, y^{n/2-1} e^{-y} = \frac{S_n}{2} \Gamma(n/2)$$  \hspace{1cm} (B.10)

in which we have defined the gamma function as

$$\Gamma(\beta) = \int_0^{\infty} dy \, y^{\beta-1} e^{-y}$$  \hspace{1cm} (B.11)

Thus we have the area of the unit hypersphere in $n$ dimensions as

$$S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$  \hspace{1cm} (B.12)

To get the volume $V$ of the unit hypersphere in $n$ dimensions, note that the area of a hypersphere of radius $R$ would be $A = S_n R^{n-1}$ (area scales as $R^2$ in $n = 3$ dimensions, and so forth); the volume $V$ satisfies $V = \int_0^R A(R')dR'$ (which builds up the volume from spherical shells of thickness $dR'$). This gives (setting $R = 1$)

$$V_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$$  \hspace{1cm} (B.13)
The gamma function satisfies a recurrence relation, as can be shown by integrating by parts:

\[
\Gamma(\beta) = -y^{\beta}e^{-y}\bigg|_0^\infty + (\beta - 1) \int_0^\infty dy y^{\beta-2}e^{-y} = (\beta - 1)\Gamma(\beta - 1)
\]  

(B.14)

which holds as long as \(\beta\) is positive (so that the integrals converge).

For an “end condition” on \(\Gamma(\beta)\), we have

\[
\Gamma(1) = \int_0^\infty dy e^{-y} = 1
\]  

(B.15)

Thus for integer arguments, \(\Gamma(n)\) equals \((n-1)!\). (Note however that \(\Gamma(\beta)\) is equally well defined for real and even complex arguments.)

The recurrence relation is valid for arbitrary arguments as well. In particular, we have for example \(\Gamma(3/2) = (1/2)\Gamma(1/2)\); and since \(S_3 = 4\pi\), we must have \(\Gamma(1/2) = \pi^{1/2}\).

### B.3 Delta function

The simplest way to define a delta function is in terms of a limit of a narrow, conveniently integrated, normalized function:

\[
\delta(t) = \lim_{\sigma \to 0} \frac{e^{-t^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}
\]  

(B.16)

Taking the Fourier transform of this definition,

\[
\int dt e^{i\omega t} \frac{e^{-t^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} = e^{-\omega^2\sigma^2/2}
\]  

(B.17)

which in the limit \(\sigma \to 0\) gives simply unity.

Inverting this Fourier transform, we have

\[
\int d\omega 2\pi e^{-i\omega t} = \delta(t)
\]  

(B.18)

which is a useful integral representation of the delta function.
Appendix C

Delta functions

A delta function, denoted $\delta(x)$, may be regarded as a function with an extremely sharp peak at the origin ($x = 0$), normalized to have unit integral ($\int dx \delta(x) = 1$). More formally, the delta function should be thought of as the limit of some peaked function (such as a Gaussian) as the width of the function ($\sigma$, for the Gaussian) approaches zero.

A delta function is useful for imposing a constraint on the value of its argument; $\delta(x - x_0)$ “requires” $x$ to equal $x_0$, since the delta function is zero otherwise.

We do not ordinarily evaluate delta functions, but rather evaluate integrals with a delta function as a factor in the integrand. To do this we use the integral formula

$$\int dx \delta(x - x_0)f(x) = f(x_0) \quad \text{(C.1)}$$

in which the integration range is presumed to include a neighborhood around the point $x_0$.

The above formula holds for any ordinary (smooth, finite) function $f(x)$, because in comparison to the extremely sharply peaked delta function, the function $f(x)$ is practically constant at $x_0$, which is the only place the delta function does not vanish. Thus we may evaluate the integral above as

$$\int dx \delta(x - x_0)f(x) = f(x_0) \int dx \delta(x - x_0) = f(x_0) \quad \text{(C.2)}$$

Sometimes we are faced with a delta function that constrains the value of a function, as in $\delta(g(x))$. To evaluate integrals against such a delta function, we use a change of variables in the integral to make $g$ rather than $x$ the integration variable:

$$\int dx f(x)\delta(g(x)) = \int dg \frac{dx}{dg} f(x(g))\delta(g) = \frac{f(x_0)}{g'(x_0)} \quad \text{(C.3)}$$

Here $x_0$ is the solution of $g(x) = 0$, i.e., the value of $x$ ultimately imposed by the delta function constraint that $g(x)$ must vanish.

Since the above relation must hold for arbitrary $f(x)$, we may write a shorthand relation for the delta function change of variable, as

$$\delta(g(x)) = \frac{\delta(x - x_0)}{g'(x_0)} \quad \text{(C.4)}$$

which encodes the results of the change of variable in the integral above.

Here we note the separation of a delta function in spherical coordinates, as a product of a radial and an angular delta functions.

$$\delta(r - r_0) = (1/r^2)\delta(|r| - |r_0|)\delta(\hat{n} - \hat{n}_0) \quad \text{(C.5)}$$
in which \( \hat{n} \) is the unit vector \( r/|r| \), and \( \hat{n}_0 \) likewise for \( r_0 \).

The three delta functions in Eqn. (C.5) are all different “types” of delta function. The first, \( \delta(r - r_0) \), constrains a three-dimensional vector argument, which we may think of as a product of three one-dimensional delta functions constraining each component of the vector. The second, \( \delta(|r| - |r_0|) \), is the usual one-dimensional delta function, which constrains the length \( |r| \). The third, \( \delta(\hat{n} - \hat{n}_0) \), constrains the unit vector \( \hat{n} \) to a particular direction on the unit sphere.

Here we maintain our customary notational economy, and forgo any notation to distinguish these different types of delta function, relying instead on the form of the argument (vector, scalar, or unit vector) to carry the information as to which kind of delta function we have.

Why the factor of \( 1/r^2 \) in Eqn. (C.5)? We want our delta functions normalized so that the following are true:

\[
\int d^3r \, \delta^{(3)}(r - r_0) = 1
\]
\[
\int dr \, \delta(|r| - |r_0|) = 1
\]
\[
\int d\Omega \, \delta^{(2)}(\hat{n} - \hat{n}_0) = 1
\] (C.6)

Because the volume element \( d^3r \) in spherical coordinates is written \( r^2 dr d\Omega \), the factor of \( 1/r^2 \) in Eqn. (C.5) is necessary.

Note that even the dimensions of the three delta functions are different: we see from Eqn. (C.6) that \( \delta(r - r_0) \) has dimensions of inverse volume, \( \delta(|r| - |r_0|) \) has dimensions of inverse length, and \( \delta(\hat{n} - \hat{n}_0) \) is dimensionless. Eqn. (C.5) is consistent with these dimensions.