Preface

This book is an introduction to the methods and results of *surgery theory*, which is our main source of detailed information about the topology of high-dimensional manifolds.

The term *surgery*, narrowly interpreted, refers to a certain geometric construction on manifolds — a smooth analog of the *cell attachment* that is so useful in algebraic topology. 'Surgery theory' in the broad sense, though, refers not just to this construction but also to the range of algebraic, geometric and homotopy-theoretic tools that have been developed to determine when surgery is possible and to analyze its effects. Thus, 'surgery' is to 'surgery theory' as 'cell attachment' is to 'homotopy theory (of CW complexes)'.

The apparatus of surgery theory is significantly more complicated, though. This is because Poincaré duality for manifolds introduces an essentially *quadratic* structure into the underlying algebra. Where an algebraic topologist might consider a module, or a chain complex, a surgery-theorist will likely consider a module equipped with a quadratic form, or a chain complex equipped with 'chain duality'. A by-product of this is the special attention given to 2-torsion issues in surgery theory. (The reader will no doubt recall that the theory of quadratic forms over a field of characteristic 2 is particularly tricky.)

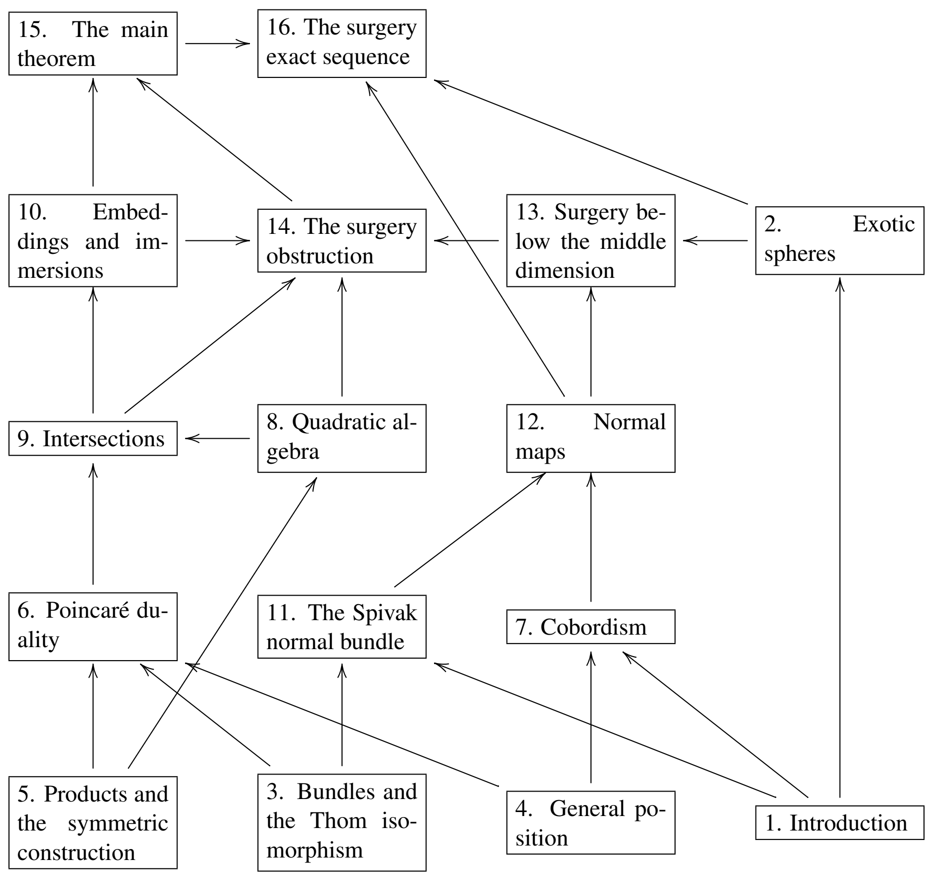
The output of surgery theory may be concisely expressed by the following mnemonic:

$$\boxed{\text{Manifolds} = \text{Bundles} + \text{Handles}}$$

This 'equation' tells us that there are two kinds of invariants associated to a manifold: homotopy-theoretic invariants associated to the tangent *bundle*, and ones connected to cobordism, which can be thought of in terms of attaching and detaching *handles*. Moreover, the two kinds of invariants together provide a complete classification of the different ways in which the same underlying space X can be given the structure of a manifold. An 'equation' of this sort, and the technique of surgery itself, first appeared in the Kervaire-Milnor classification of exotic spheres; it takes definitive form in the surgery exact sequence', which is the subject of Chapter 16.

As its title indicates, this book is neither a fully detailed textbook on surgery theory for the novice nor a comprehensive reference for the expert. It is intended for the mathematician who would like to gain some overall perspective on what the subject is about. This might be the graduate student beginning research in the topology of manifolds; equally, it might be the established mathematician who has met some surgical ideas in other contexts (as operator algebraists, for example, have often encountered the Novikov conjecture) and wants to learn more about where they arose. To get the most out of the book, you need some background in algebraic topology with an emphasis on smooth manifold theory ([**8**] and [**13**] are good sources) and characteristic classes ([**26**] is the classic reference here).

A word about the structure of the book. The first two chapters introduce some key examples and questions about manifold topology (Chapter 1) and sketch part of the classification of exotic spheres (Chapter 2). Chapter 2 introduces the technique of surgery in its historical context, and the final section of that chapter provides an overview of the whole surgery project and the rest of the book. Surgery theory demands a rather elaborate technique, and it is not always easy to keep the overall objective in view, so the reader may wish to refer back to this section several times. Less central material is set in *small print*; the first-time reader may wish to omit all the small-print sections. The diagram below indicates the main dependencies between the chapters (up to Chapter 16).



CHAPTER 1

Questions about the Topology of Manifolds

This is a book about the topology of manifolds. One of the most important discoveries in topology — one that was the work of many mathematicians in the third quarter of the twentieth century — is that there is a systematic procedure for answering many natural questions about manifold topology, provided that the manifolds in question are sufficiently *high-dimensional*. Alexandroff wrote in 1932

> Let it be remarked here that, at present, in contrast to the two-dimensional case, the problem of enumerating the topological types of manifolds of three or more dimensions is in an apparently hopeless state. We are not only far removed from the solution, but even from the first step toward a solution, a plausible conjecture.

The natural expectation, which seems to be expressed by Alexandroff here, is that the topology of manifolds will become more and more complicated as the dimension of the manifold increases. Forty years after Alexandroff wrote it had become clear that this is true only up to a point. The topology of two, three, and four dimensions does indeed seem to require special geometrical techniques. However in dimensions five and up there is finally sufficient room for the flabbier techniques of differential topology to get to work and to provide, in a sense, a complete classification. A key geometric construction involved in this procedure is known as *surgery*, and the entire subject has taken on this name and is therefore often called 'surgery theory'.

Let's begin by reminding ourselves of the definitions of the objects that we want to study.

1.1. Definition. A *topological n-manifold M* is a metrizable topological space that is locally homeomorphic to Euclidean space $\mathbb{R}^n$ — there is a cover of M by open sets U_α and there are homeomorphisms $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$. (Such a cover $\{(U_\alpha, \varphi_\alpha)\}$ is called an *atlas*.)

The *transition functions* of an atlas are the functions $\varphi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1}$, which are homeomorphisms between open subsets of $\mathbb{R}^n$. An atlas is *smooth* if its transition functions are smooth (infinitely differentiable).

1.2. Definition. A *smooth structure* on a topological manifold is a maximal smooth atlas. A *smooth manifold* is a manifold with a smooth structure.

Already some natural questions arise: Does every topological manifold admit a smooth structure? Is such a structure unique? As we shall see, the answers to both these questions are in general negative.

A natural way to focus attention is to think about the *classification problem* — give a complete set of invariants which allows one to determine whether two manifolds are diffeomorphic, and give a list of representatives for the diffeomorphism classes. Of course, there is much more to differential topology than this, just as there is much more to group theory than trying to give a list of finite groups up to isomorphism; one wants to use the theory to say interesting things about non-trivial and natural examples. But classification is a good point at which to start our thinking. To solve a classification problem one needs to produce a list of *invariants* of the structure under consideration. What kind of invariants, then, are given to us by the statement that M is a smooth manifold?

Given any finite group presentation, one can effectively construct a compact n-manifold, $n \geqslant 4$, whose fundamental group is given by the presentation. An effective classification of manifolds up to diffeomorphism (or even up to homotopy equivalence) would thus in particular include a classification of the groups given by finite presentations. It is known that there is no algorithm to accomplish such a classification. To avoid these logical issues, and for other reasons, one traditionally[1] formulates the classification problem in terms of classification of manifolds *within a given a homotopy type*: for some specified space X, how many 'essentially different' smooth manifolds are there homotopy equivalent to X?

In this chapter we want to review some of the invariants that can be used to approach this problem. We will also describe some key examples from the fifties and early sixties. These examples illustrate a number of mechanisms whereby the homotopy, homeomorphism and diffeomorphism of manifolds can be distinguished. Surgery theory proper tells us, in essence, that these mechanisms account for all the differences that there are between these various classifications.

[1]Thus surgery theory (as presented in this book) addresses a relative classification problem, diffeomorphism type relative to homotopy type. This assumes of course that information about the homotopy types of manifolds is supplied initially. It is however also possible to apply surgical methods to investigate these homotopy types; if one wishes to do this, the 'modified surgery theory' of Kreck [**?**] organizes matters more conveniently.

1.1. Algebraic topology

To begin with, we of course have the usual invariants of algebraic topology: homology, cohomology and homotopy groups. As a reference for these objects we suggest the texts by Bredon [**8**] or Hatcher [**13**].

When the homology groups of a space X (or rather the associated numerical invariants — Betti numbers and torsion coefficients) were first defined by Poincaré and others, the definitions made use of a *triangulation* of X (that is, a representation of X as a simplicial complex). This led to the question whether homeomorphic polyhedra (or manifolds) are *combinatorially* equivalent (piecewise-linearly homeomorphic). The hypothesis that this is the case was known as the 'Main Conjecture' or *Hauptvermutung*. In fact the Hauptvermutung turned out to be false, even for manifolds — that is part of the story we have to tell in this book. However, long before these examples topologically invariant definitions of homology and cohomology had appeared (singular and Čech theories, for example). Thus the Hauptvermutung was no longer needed to prove the topological invariance of (co)homology.

When we deal with a *smooth* manifold M, it is also relevant to consider the *de Rham cohomology* groups. These are the cohomology groups of the complex

$$\Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \cdots$$

of differential forms on M. The *de Rham theorem* says that the de Rham cohomology of M is isomorphic to the usual cohomology with real coefficients. The usual proof of this establishes an isomorphism between de Rham and Čech cohomology; for this, and other matters relating to de Rham theory, our reference will be the book of Bott and Tu [**7**]. Cohomology has a ring structure (the cup product, given in de Rham theory by exterior product of forms) and this feature of cohomology will be crucially important in the discussion that follows.

One of the most notable features of the homology and cohomology of manifolds is *Poincaré duality.* Already in his 1895 memoir *Analysis Situs* [**?**], which founded the subject of topology, Poincaré had drawn attention to the fact that the Betti numbers of a compact oriented manifold exhibit a certain symmetry: $b_p = b_{n-p}$, if n is the dimension. Poincaré's 'proof' of this fact was severely criticized by Heegard, and in response he offered a second proof in [**?**]. This proof made use of dual cell decompositions in a manner that is still recognizable today. Poincaré also drew attention to the special rôle of the *middle dimension* in terms of duality. If $n = 2k$, then the k-dimensional homology of M carries a nondegenerate bilinear form, the *intersection form*, which is symmetric if k is even but skew-symmetric if k is odd. In particular, Poincaré pointed out, the middle Betti number of a (compact oriented) $4l + 2$-dimensional manifold must be *even*. This is because the intersection form is nondegenerate and skew-symmetric, and such a form on a real vector space is a direct sum of copies of the form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; in particular, such a form can exist only on an even-dimensional space.

1.3. Remark. Here is an outline of a proof of Poincaré duality using de Rham theory. The *de Rham homology groups* of a manifold M are the homology groups of the complex of (compactly supported) *currents* on M — a k-*current* is, by definition, a continuous linear functional on the space of k-forms on M (equipped with its natural locally convex topology). If M^n is oriented we can define integration for n-forms on M, and we can interpret this as a map $D\colon \Omega^*(M) \to \Omega_{n-*}(M)$ from the complex of forms to the complex of currents:

$$D(\alpha)(\beta) = \int_M \beta \wedge \alpha.$$

Stokes' theorem shows that this is in fact a chain map. The *Poincaré duality theorem* now states that for a closed manifold D induces an isomorphism $H^*(M;\mathbb{R}) \to H_{n-*}(M;\mathbb{R})$ from de Rham cohomology to homology. To prove it, observe that D can be defined whether or not M is compact, so long as we use *compactly supported* cohomology. Moreover, direct calculation with the Poincaré lemma (see 3.3) shows that this map is an isomorphism when M is Euclidean space. Now cover a closed manifold M by finitely many open sets each of which, together with all their possible intersections, is either empty or diffeomorphic to Euclidean space. A Mayer-Vietoris 'assembly' argument completes the proof.

If $n = 2k$ is even the *intersection form* is the bilinear form

$$(x, y) \mapsto (D^{-1}(x))(y)$$

on $H_k(M; \mathbb{R})$. Since D is an isomorphism, the form is nondegenerate, as we asserted above.

1.4. Remark. We shall ultimately need a sharper form of Poincaré duality than this — in particular we shall need to know that it gives an isomorphism from cohomology to homology with *integer* (not just *real*) coefficients. We return to the topic in Chapters 6 and 9.

1.5. Remark. The intersection form has an appealing geometric interpretation in the case of homology classes represented by closed oriented submanifolds N_1 and N_2 having $\dim N_1 + \dim N_2 = \dim M$: it simply counts (with sign) the number of points of intersection of N_1 and N_2 — possibly after a small perturbation to put them in 'general position' with respect to one another. This geometry will be developed in detail in Chapters 6 and 9.

In the case $n = 4l$ the intersection form is nondegenerate and symmetric. It is an elementary fact of linear algebra ("Sylvester's Law of Inertia") that any symmetric bilinear form over a finite-dimensional real vector space can be reduced, by a change of basis, to the form

$$B(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q},$$

and the number p of positive signs and q of negative signs appearing here are *invariants* of the form (in fact, they are the maximal dimensions of subspaces restricted to which the form is positive or negative definite).

1.6. Definition. The difference $p - q$ is called the *signature* of the form, or of the manifold from which it arises.

1.7. Exercise. What is the signature of the complex projective space $\mathbb{CP}^{2k}$? Show that this space does not possess any orientation-reversing diffeomorphism.

1.8. Remark. Notice that in defining the signature we have neglected any finer arithmetic structure which arises from the fact that the intersection form is defined over $\mathbb{Z}$, not simply over $\mathbb{R}$. The classification of symmetric bilinear forms over $\mathbb{Z}$ is a much more subtle matter. For instance, a symmetric bilinear form over $\mathbb{Z}$ is called *even* if the diagonal entries in a matrix representation are even integers; equivalently, $B(\mathbf{x}, \mathbf{x})$ is even for every integer vector $\mathbf{x}$. This notion is invariant under change of (integer) basis.

Let X be a space with basepoint. The *homotopy groups* of X are the groups $\pi_n(X) := [S^n, X]$ of homotopy classes of maps from the n-sphere to X (all maps and homotopies are required to be basepoint-preserving). These groups are abelian when $n > 1$.

The notation S^n of course denotes the n-sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$; it is a smooth manifold, the boundary of the $(n+1)$-disk $D^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leqslant 1\}$. We shall also require the *relative homotopy groups* of a pair (X, A), or more generally of a map $i \colon A \to X$. An element of $\pi_n(X, A)$ is a homotopy class of commuting diagrams

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & A \arrow[d, "i"] \\ D^n \arrow[r] & X \end{tikzcd}$$

The definition is so arranged that there is an exact sequence

$$\dots \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \dots$$

In general, homotopy groups are much more mysterious that homology groups. The following example was known in the 1930s.

1.9. Exercise (The Hopf fibration)**.** Regard S^3 as the group of unit quaternions and obtain a group homomorphism $S^3 \to SO(3)$ by sending a quaternion q to the transformation $x \mapsto qx\bar{q}$ of the purely imaginary quaternions. Since $SO(3)$ acts on S^2 by rotations, we obtain a map $S^3 \to S^2$. This map is called the *Hopf fibration*. Show that it represents a nonzero element in $\pi_3(S^2)$. (In fact, $\pi_3(S^2) = \mathbb{Z}$ and the Hopf map is the generator.)

1.10. Remark. Let $f\colon S^{2k-1} \to S^k$ be a map. Then we may form a CW-complex $X = D^{2k} \cup_f S^k$ by using the map f to attach the $2k$-disk to a k-sphere. Assuming that $k > 1$, the integral cohomology of X has one generator, say x, in dimesion k and another generator, say y, in dimension $2k$. There is then an integer m such that $x \smile x = ky$. The absolute value of this integer depends only on the homotopy class of the map f; it is called the *Hopf invariant* of f. In the case of the classical Hopf map $S^3 \to S^2$, the space X is $\mathbb{CP}^2$ and the Hopf invariant is 1. This is essentially[2] how Hopf showed that the Hopf map is not homotopically trivial.

1.11. Exercise. Following on from the above exercise, show that the Hopf fibration is a principal S^1-bundle over S^2. Give a complete classification of such bundles. (Any such bundle is trivial over the upper and lower hemispheres, so that it is determined by its *clutching function*, which is the map $S^1 \to S^1$ which shows how these two trivial bundles are joined together over the equator. Thus these bundles are classified by an integer $k \in \pi_1(S^1) = \mathbb{Z}$. This is an example of a *characteristic class*, in fact an Euler class; see Chapter 3. The Hopf fibration corresponds to $k = 1$.)

1.12. Exercise. From a principal S^1-bundle over S^2 one can build an S^2-bundle over S^2 by fiberwise suspension. Show that the resulting S^2-bundles are classified by the residue class mod 2 of the integer k introduced in the previous exercise. (This is a matter of the homomorphism $\pi_1(SO(2)) = \mathbb{Z} \to \pi_1(SO(3)) = \mathbb{Z}/2$.)

1.13. Exercise. Show that the total space of the S^2-bundle over S^2 obtained in the previous section with $k = 1$ is diffeomorphic to the connected sum $\mathbb{CP}^2 \# (-\mathbb{CP})^2$, where $-\mathbb{CP}^2$ is the complex projective plane with the opposite of the standard orientation. (First show that the complement of a small 4-disk in $\mathbb{CP}^2$ is diffeomorphic to the total space of the complex line bundle associated to the Hopf bundle.)

We will need a number of key facts about the relationship between homotopy and homology. First notice the obvious map (the *Hurewicz map*) $h_n\colon \pi_n(X) \to H_n(X;\mathbb{Z})$, given by sending a map $f\colon S^n \to X$ to $f_*(x)$, where $x \in H_n(S^n;\mathbb{Z}) = \mathbb{Z}$ is a canonical generator.

1.14. Theorem (Hurewicz Theorem)**.** *Suppose that X is (path) connected and that $\pi_n(X) = 0$ for $n < N$, where $N > 1$. Then $H_n(X;\mathbb{Z}) = 0$ for $n < N$ also, and moreover the Hurewicz map in dimension N, $\pi_N(X) \to H_N(X;\mathbb{Z})$, is an isomorphism.*

1.15. Remark. There is also a *relative form* of the Hurewicz theorem, but it is slightly more complicated: if $\pi_n(X, A) = 0$ for $n < N$ then $H_n(X, A) = 0$ for $n < N$ also and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is an epimorphism with kernel generated by the action of $\pi_1(A)$ on $\pi_n(X, A)$; in particular if A is simply connected the Hurewicz map is an isomorphism.

[2] Cohomology and cup products were not invented when Hopf wrote; he expressed the Hopf invariant as the *linking number* of the inverse images of two generic points of S^2.

1.16. Theorem (Whitehead Theorem)**.** *Let $f\colon X \to Y$ be a map of connected CW-complexes inducing an isomorphism on all homotopy groups, or equivalently*[3] *inducing an isomorphism on π_1 and on all homology groups. Then f is a homotopy equivalence.*

The reader will find the proofs of these results in [**13**, Chapter 4].

1.17. Exercise. Let M be a manifold of dimension $2k$ or $2k+1$. Show that if M is k-connected, then it is a *homotopy sphere* (i.e., homotopy equivalent to a sphere). (Use Poincaré duality and the Hurewicz and Whitehead theorems.)

1.18. Exercise. Show that the smooth 4-manifolds $S^2 \times S^2$ and $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ have isomorphic homotopy groups (in all dimensions), but are *not* homotopy equivalent. This shows that the condition of Whitehead's theorem cannot be weakened to *abstract* isomorphism of homotopy groups; it is necessary that the isomorphisms be induced by a map of spaces.

One way to show that these manifolds are not homotopy equivalent is to show that $S^2 \times S^2$ has even intersection form but $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ does not. On the other hand, one can represent $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ as the total space of an S^2-bundle over S^2 which admits a cross section (see Exercise 1.13). Then its homotopy groups can be computed using the long exact homotopy sequence of a fibration.

Let X and Y be spaces with basepoint. Their *wedge* $X \vee Y$ is obtained from the disjoint union by identifying the basepoints. One can regard it as the subspace $X \times \bullet \cup \bullet \times Y$ of $X \times Y$. The *smash product* of X and Y is the identification space $X \wedge Y = X \times Y / X \vee Y$. The (reduced) *suspension* of Y is the space $\Sigma Y = S^1 \wedge Y$.

Standard homological machinery produces an identification $\tilde{H}_r(X) = \tilde{H}_{r+1}(\Sigma X)$ (using reduced homology here). The effect of suspension on *homotopy* is less straightforward. There is a natural *suspension homomorphism*

$$E : \pi_r(X) \to \pi_{r+1}(\Sigma X),$$

but it is not an isomorphism in general. It follows from a theorem of Freudenthal, however, that E is an isomorphism provided that X is sufficiently highly connected (roughly $\frac{1}{2}r$-connected). In particular the sequence of groups

$$\pi_r(X) \to \pi_{r+1}(\Sigma X) \to \pi_{r+2}(\Sigma^2 X) \to \dots$$

eventually stabilizes; the common limit is the *stable homotopy group* $\pi_r^s(X)$.

In the 1950s, Serre proved the following basic result using the then-new method of spectral sequences.

1.19. Proposition. *The stable homotopy groups of spheres, $\pi_r^s = \pi_r^s(S^0)$, are finite for $r > 0$.*

Here is a table of the stable homotopy groups for small values of r.

r	0	1	2	3	4	5	6	7	8
π_r^s	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Much more extensive tables can be found in [].

[3]The equivalence follows from the Hurewicz theorem.

1.2. Pontrjagin classes

If M is a smooth manifold then its smooth structure provides a canonical (real) vector bundle, the *tangent bundle* TM over M. One can think of this as follows: let $\{U_\alpha\}$ be a coordinate cover of M; then the differentials of the transition functions of this cover provide maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ that satisfy the *cocycle condition*

$$\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = 1$$

where defined. Such a cocycle with values in $GL(n, \mathbb{R})$ can be used to construct a vector bundle by using the isomorphisms $\varphi_{\alpha\beta}$ to patch together trivial $\mathbb{R}^n$-bundles over U_α and U_β. Any cocycle with values in $GL(n, \mathbb{R})$ is cohomologous to one with values in the maximal compact subgroup $O(n)$ (one then speaks of a *reduction of structure group* to $O(n)$); this corresponds to the fact that every manifold can be given a Riemannian metric.

Diffeomorphic smooth manifolds have isomorphic tangent bundles. Therefore, invariants of smooth structure will be found from the *characteristic classes* of the tangent bundle.

Recall that a *characteristic class* for a certain category of bundles (the categories of real vector bundles and of complex vector bundles are the immediate examples) is just a *natural* map which associates, to each such bundle E over a base space B, a cohomology class $c(E) \in H^*(B)$, in such a way that isomorphic bundles E and E' have equal characteristic classes $c(E) = c(E')$. The classic reference for the theory is the book of Milnor and Stasheff [**26**]; for a more recent account see Chapter 3 of the book by Hatcher [] (unfinished, but available online). Notice that the term 'natural' which appears above is a technical one: it means that if $f : X \to Y$ is a map and E is a vector bundle over Y, then $c(f^*(Y)) = f^*(c(Y))$.

The most important characteristic classes for real vector bundles are the *Pontrjagin classes*. For a real vector bundle E over base B, these classes $p_k(E) \in H^{4k}(B; \mathbb{Z})$, $k = 1, 2, \ldots$ vanish for $k > \frac{1}{2}\dim E$, all vanish for a trivial bundle, and satisfy the *Whitney sum formula*: if we denote by $p(E)$ the 'total Pontrjagin class'

$$p(E) = 1 + p_1(E) + p_2(E) + \in H^*(B; \mathbb{Z})$$

then

$$p(E_1 \oplus E_2) = p(E_1) \cdot p(E_2) \quad \text{modulo 2-torsion.}$$

(The dot of course denotes the cup-product in the cohomology ring.)

Here is a very abbreviated account of the construction of the Pontrjagin classes. In classical differential geometry one encounters the *Gauss map* of an embedded k-submanifold $M \subseteq \mathbb{R}^n$. This is the map which to each point $m \in M$ associates the tangent plane to M at m, translated so as to pass through the origin in $\mathbb{R}^n$. It is a map from M to the *Grassmannian* $G_{k,n}(\mathbb{R})$ of k-dimensional subspaces of $\mathbb{R}^n$. The Grassmannian carries a 'universal' k-dimensional vector bundle, whose fiber over a point p representing a k-dimensional subspace of $\mathbb{R}^n$ just *is* that k-dimensional subspace; by construction, the tangent bundle of M is the pull-back of this tautological bundle via the Gauss map. More generally, it is possible to show that *any* real vector bundle (at least over a compact base) is pulled back by some map from the universal bundle over some Grassmannian, and moreover the map is uniquely determined up to homotopy by the isomorphism class of the original bundle. This argument (sometimes called the *Yoneda lemma*) reduces the problem of finding characteristic classes to that of computing the cohomology of Grassmannians.

We denote the limit $\lim_{n\to\infty} G_{k,n}(\mathbb{R})$ by $BO(k)$ and call it the *classifying space* for bundles with structure group $O(k)$, that is k-dimensional real bundles. This construction is in fact a homotopy-theoretic one: for any topological group G, a space BG is defined uniquely up to homotopy equivalence by the requirement that it carry a *universal* principal G-bundle, one from which any other G-bundle is pulled back. (It turns out to be equivalent to require that the total space, denoted EG, of the universal bundle is contractible.) For similar reasons we denote by $BU(k)$ the limit $\lim_{n\to\infty} G_{k,n}(\mathbb{C})$, using the Grassmannian of k-dimensional *complex* subspaces of $\mathbb{C}^n$.

1.20. Example. The spaces $BO(1)$ and $BU(1)$ are the infinite-dimensional real and complex projective spaces $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$.

Although our interest is ultimately in real vector bundles, it turns out to be important to focus first on the classifying space $BU(1) = \mathbb{CP}^\infty$ for *complex* line bundles. This has a cell structure with cells only in even dimensions, and so its cohomology is $\mathbb{Z}$ in even dimensions and 0 in odd dimensions. Moreover, the cup-product of the generators in dimensions $2m$ and $2n$ is the generator in dimension $2(m+n)$ (geometric interpretation: in projective geometry the intersection of a codimension-m linear subspace and a codimension-n linear subspace is always a codimension-$(m+n)$ linear subspace). Thus

1.21. Proposition. *The integral cohomology ring $H^*(BU(1);\mathbb{Z})$ is a polynomial ring $\mathbb{Z}[c]$ on one 2-dimensional generator.*

What this means for characteristic classes is that every complex line bundle L over a space X has a *first Chern class* $c_1(L) \in H^2(X;\mathbb{Z})$, and every other characteristic class for complex line bundles is just a polynomial in that first Chern class.

There are many other ways to define $c_1(L)$. For instance, the exponential map gives a short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}(\mathbb{R}) \to \mathcal{O}(S^1) \to 0;$$

and the associated Bockstein homomorphism $H^1(X;\mathcal{O}(S^1)) \to H^2(X;\mathbb{Z})$ maps a line bundle to its first Chern class.

What can be said about k-dimensional complex vector bundles? A simple example of such a bundle is a direct sum of k line bundles. It is a surprising fact that, for the purpose of characteristic class theory, one often need only consider bundles that *split* in this way. Here is the reason: Consider the product $BU(1) \times \cdots \times BU(1)$ (k copies). The cohomology of this space is a polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$, where $x_1, \ldots, x_k$ are the first Chern classes of the canonical line bundles over the various factors. The direct sum of all these line bundles is a k-dimensional vector bundle and this gives us a map

$$BU(1) \times \cdots \times BU(1) \to BU(k)$$

which classifies it. Now one has

1.22. Proposition (Splitting Principle)**.** *The map displayed above induces an injection on cohomology, whose image is the ring of* symmetric *polynomials in $x_1, \ldots, x_k$.*

It is a theorem of algebra [**17**, reference] that the ring of symmetric polynomials in $x_1, \ldots, x_k$ is itself a polynomial ring, generated by the *elementary symmetric polynomials*

$$\begin{aligned} c_1 &= x_1 + \cdots x_k \\ c_2 &= x_1x_2 + \cdots + x_{k-1}x_k \\ &\cdots \\ c_k &= x_1 \cdots x_k \end{aligned}$$

which are defined in general by

$$1 + c_1t + c_2t^2 + \cdots + c_kt^k = \prod_{i=1}^{k}(1 + tx_i).$$

Thus $H^*(BU(k);\mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_k]$ where the generators c_i (of degree $2i$) are called the *i'th Chern classes*. These are the fundamental characteristic classes for k-dimensional complex vector bundles. Notice that the construction immediately gives us the Whitney sum formula for Chern classes,

$$c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2),$$

where the total Chern class is defined by $c(V) = 1 + c_1(V) + c_2(V) + \cdots$.

1.23. Exercise. Show that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ for complex line bundles L_1 and L_2.

1.24. Exercise. The *Chern character* is the characteristic class defined by the sum $e^{x_1} + \cdots + e^{x_k}$ (this is a symmetric formal power series rather than a symmetric polynomial, but things work in the same way). Using the previous exercise, show that the Chern character is a 'homomorphism' in the sense that

$$\mathrm{ch}(E_1 \oplus E_2) = ch(E_1) + \mathrm{ch}(E_2), \quad \mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \cdot \mathrm{ch}(E_2).$$

Now let us think about *real* rather than complex vector bundles. The process of complexifying (tensoring with $\mathbb{C}$) turns real vector bundles into complex ones and therefore provides a map $BO(k) \to BU(k)$. This pulls back the Chern classes to certain characteristic classes in $H^*(BO(k);\mathbb{Z})$. It turns out that the pullbacks of the *odd* Chern classes are 2-torsion elements (this is because the complexification of a real vector bundle is isomorphic to its complex conjugate bundle) but the pullbacks of the *even* Chern classes are significant and up to sign give the *Pontrjagin classes*

$$p_i(V) = (-1)^i c_{2i}(V \otimes \mathbb{C})$$

which generate a polynomial subring $\mathbb{Z}[p_1, p_2, \ldots]$ of $H^*(BO(k);\mathbb{Z})$. Note that p_i has degree $4i$.

1.25. Remark. The odd Chern classes of a complexified real bundle are not necessarily *zero*, but it turns out that they can be computed from in terms of other mod 2 invariants, the *Stiefel-Whitney* classes, which we will discuss later. See the exercises in Chapter 3 of [**?**].

1.26. Remark. When M is a smooth manifold, we refer to the 'Pontrjagin classes of M' instead of the Pontrjagin classes of the tangent bundle of M. By construction, these are diffeomorphism invariants of M.

1.27. Example. Let us calculate the Pontrjagin classes of $M = \mathbb{CP}^n$, considered as a real $2n$-manifold. We recall that the cohomology of $\mathbb{CP}^n$ is a truncated polynomial ring $\mathbb{Z}[x]/(x^{n+1})$, where $x \in H^2(M;\mathbb{Z})$ is the first Chern class of the tautological line bundle L over M.

First we need

1.28. Exercise. Let T be the complex tangent bundle to M. Then one has an isomorphism of bundles $T \oplus \mathbb{C} = (n+1)\bar{L} = \bar{L} \oplus \cdots \oplus \bar{L}$. (Hint: Identify sections of $\bar{L}$ with homogeneous functions on $\mathbb{C}^{n+1}$, that is functions $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ such that $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$ for all $\lambda \in \mathbb{C}$. Identify sections of the bundle $T \oplus \mathbb{C}$ with homogeneous vector fields on $\mathbb{C}^n$. Choose a basis of $\mathbb{C}^n$ to get the desired isomorphism.)

It follows from the Whitney sum formula that $c(T) = (1+x)^{n+1}$. Now the complexification of the *real* tangent bundle to M (which is just the *real* vector-bundle underlying T) is isomorphic (as a *complex* vector-bundle) to $T \oplus \overline{T}$, and thus has total Chern class

$$c(T \oplus \overline{T}) = (1 - x^2)^{n+1}.$$

By definition, then, the k'th Pontrjagin class $p_k(M)$ is equal to $(-1)^k$ times the degree $2k$ term in the above polynomial, so it is equal to $\binom{n+1}{k} x^{2k}$. For instance, $p_1(\mathbb{CP}^2) = 3x^2$, $p_1(\mathbb{CP}^4) = 5x^2$, $p_2(\mathbb{CP}^4) = 10x^4$.

1.29. Exercise. Calculate the Pontrjagin classes of quaternion projective space by a similar method. You should find that the total Pontrjagin class $p(\mathbb{HP}^n)$ equals $(1+x)^{2k+2}(1+4x)^{-1}$, where x is the generator of $H^4(\mathbb{HP}^n;\mathbb{Z})$; in particular, $p_1(\mathbb{HP}^n) = (2n-2)x$. See [**5**, page 519] or [**26**, Problem 20A]. Deduce that if $n > 1$, $\mathbb{HP}^n$ does not admit any orientation-reversing diffeomorphism.

1.3. Cobordism

If M is a *compact, oriented* manifold we define the *Pontrjagin numbers* of M to be the integers obtained by evaluating polynomials in the Pontrjagin classes on the fundamental homology class[4] $[M]$. Thus there is one Pontrjagin number for each polynomial in $\mathbb{Z}[p_1, p_2, \ldots]$ of total degree equal to $\dim M$.

1.30. Lemma. *If the compact, oriented manifold M is the boundary of a compact manifold W, then all its Pontrjagin numbers vanish.*

PROOF. See 7.31, or the reader can do it now as an exercise. □

This simple result shows the connection between Pontrjagin numbers and cobordism.

1.31. Definition. Two compact oriented manifolds M and M' are *cobordant* if $M \sqcup (-M')$ is the boundary of a compact oriented manifold. The *oriented cobordism ring* Ω_* is the graded ring of cobordism classes of compact oriented manifolds: addition is by disjoint union, and multiplication is by Cartesian product.

From lemma 1.30 we see that each Pontrjagin number gives a group homomorphism $\Omega_* \to \mathbb{Z}$. Thom's computations of cobordism [**?**] (which we will review in Chapter 7) showed that the Pontrjagin numbers are sufficiently rich to separate points on $\Omega_* \otimes \mathbb{Q}$. To put this another way, every group homomorphism $\Omega_* \to \mathbb{Z}$ is a Pontrjagin number with rational coefficients (an element of $\mathbb{Q}[p_1, p_2, \ldots]$).

Now there is a completely different way to obtain a homomorphism from Ω_* to $\mathbb{Z}$: make use of Poincaré duality. We've seen above that every compact oriented manifold has a *signature*, defined using the intersection form on middle-dimensional cohomology, and it is not hard to check[5] that this quantity is cobordism invariant, so it defines a functional $\Omega_* \to \mathbb{Z}$. According to Thom's results, then, the signature is a Pontrjagin number. What number is it?

In low dimensions we can do some computations by hand. For instance, in dimension 4, the only Pontrjagin numbers are multiples of p_1. But for $M = \mathbb{CP}^2$ the signature is 1, whereas the Pontrjagin number $p_1(M)$ is 3, by the calculations of Example 1.27. Consequently we obtain

$$\operatorname{Sign}(M) = \frac{1}{3} p_1(M) \tag{1.32}$$

for any compact oriented 4-manifold M.

In dimension 8 there most general Pontrjagin number is $ap_1^2 + bp_2$, for some coefficients $a, b \in \mathbb{Q}$. Using the calculations of Example 1.27 again we obtain the equations

$$25a + 10b = 1, \qquad 18a + 9b = 1$$

by considering the 8-manifolds $M = \mathbb{CP}^4$ and $M = \mathbb{CP}^2 \times \mathbb{CP}^2$ respectively. These equations can be solved to yield $a = -1/45$, $b = 7/45$ and thus the formula

$$\operatorname{Sign}(M) = \frac{1}{45}(7p_2 - p_1^2)[M] \tag{1.33}$$

for any compact oriented 8-manifold.

The general result was found by Hirzebruch — see his own account in [**14**]. The *Hirzebruch Signature Theorem* gives an explicit procedure, in terms of certain power

[4] Since the fundamental homology class depends on the choice of orientation, the Pontrjagin numbers depend on the choice of orientation, even though the Pontrjagin classes do not.

[5] See Proposition 6.29.

series, to build a characteristic class $L(M) = L(p_1, p_2, \ldots)$, which in each degree is a polynomial in the Pontrjagin classes, such that

$$\operatorname{Sign}(M) = \langle L(M), [M] \rangle$$

for any compact oriented manifold M. The signature theorem expresses a deep and unexpected link between the algebra of intersection forms and the geometry of the tangent bundle. As we will see in a moment, it has very strong geometrical consequences.

1.34. Remark. The L-class has components in degrees $0, 4, 8, \ldots$. Moreover, by examining its explicit form one sees that, in rational cohomology $H^*(M; \mathbb{Q})$, the Pontrjagin classes can be recovered from the L-class. On the other hand, by the signature theorem the L-class determines not only the signature of M but also the signature of any submanifold N of M that has trivial normal bundle. (For then the Pontrjagin classes of M restrict to those of N.) Using arguments from homotopy theory (specifically Serre's theorem about the finiteness of the higher homotopy groups of spheres) it can be shown that there is also a converse here: to know the signatures of submanifolds with trivial normal bundle (in M and in certain 'stabilizations' of M) recovers the rational L-class. The conclusion is that the rational Pontrjagin classes determine and are determined by a list of signatures of submanifolds. Many of the deeper properties of Pontrjagin classes in differential topology depend on this fact.

1.35. Remark. Analogous to the connection between oriented cobordism and the Pontrjagin classes, there is a relationship between unoriented cobordism and the *Stiefel-Whitney* classes; these are characteristic classes of real vector bundles which live in cohomology with $\mathbb{Z}_2$ coefficients. However, there is an important distinction to be drawn: as we shall see in Theorem 5.33, the Stiefel-Whitney classes of the tangent bundle of a manifold in fact depend only on its *homotopy type*. By contrast, the Pontrjagin classes reflect the differentiable structure. We shall see an explicit example a little later in this chapter.

1.4. The Poincaré conjecture

Poincaré at first asked whether every *homology sphere* (a manifold having the same homology groups as S^n) is a standard sphere. However, he soon produced an example to show that the answer is 'no' in general [**?**]. Let us look at manifolds of the form $M = S^3/\Gamma$, where Γ is a discrete subgroup of $SO(4)$ acting freely on S^3. Show that if the group Γ is equal to its commutator subgroup $[\Gamma, \Gamma]$ (this is what is called a 'perfect' group), then M is a homology 3-sphere.

To get an explicit example, regard $S^3 = Sp(1)$ as the group of unit quaternions, which is the double cover of $SO(3)$. The inverse image of the symmetry group of the icosahedron, under this double cover, is a subgroup of $Sp(1)$ of order 120, called the *binary icosahedral group*. Show that the binary icosahedral group is perfect (use the fact that the symmetry group of the icosahedron is nonabelian and simple). Thus we obtain a homology sphere by dividing S^3 by Γ acting by group multiplication.

This is an appropriate point to state the Poincaré conjecture.

1.36. Conjecture (Generalized Poincaré Conjecture)**.** *Every smooth homotopy n-sphere (that is, every smooth manifold homotopy equivalent to S^n) is homeomorphic to S^n.*

In order to prove the Poincaré Conjecture one needs some mechanism for recognizing smooth manifolds homeomorphic to S^n. Such a mechanism is provided by the following theorem of Reeb.

1.37. Theorem. *Let M be a compact smooth manifold. Suppose that $f\colon M \to \mathbb{R}$ is a smooth function having no critical points except for a single non-degenerate maximum and a single non-degenerate minimum. Then M is homeomorphic to a sphere.*

A *critical point* of f is a point where its gradient vanishes, and such a critical point is *non-degenerate* if the matrix of *second* derivatives of f has full rank there.

SKETCH PROOF. It is known that around a non-degenerate minimum point one can choose local coordinates so that

$$f(x_1, \dots, x_n) = c + x_1^2 + \cdots + x_n^2$$

where $c = f(0, \dots, 0)$ is the minimum value of f. (This is part of the *Morse Lemma* 7.5.) Consequently, for sufficiently small $\varepsilon > 0$ the region $\{x : f(x) \leqslant c + \varepsilon\}$ is a closed n-disk in M. If we remove from M the interior of this disk, and of the corresponding disk around the maximum point, the part of M that remains can be given the structure of a cylinder $S^{n-1} \times I$ by making use of the gradient flow of f (see Figure 1). Thus M can be obtained by attaching two disks, by diffeomorphisms, to the ends of a cylinder. Since every homeomorphism of the boundary of a closed disk extends, by 'coning', to a homeomorphism of the whole disk, the resulting manifold is homeomorphic to the n-sphere. □

1.38. Remark. The process of extending a homeomorphism of a sphere to a homeomorphism of the disk that it bounds is called the *Alexander trick*. Note carefully that even if we start with a *diffeomorphism* of the sphere, the homeomorphism produced by the Alexander trick need not be smooth at the cone point (though of course it will be smooth everywhere else).

1.39. Exercise. Consider the Milnor 7-manifold M described above. Show that it can be obtained by identifying two copies of $\mathbb{R}^4 \times S^3$ in the following explicit way: the point

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FIGURE 1. The gradient flow provides a diffeomorphism to a cylinder

(u, v) in the first copy of $(\mathbb{R}^4 \setminus \{0\}) \times S^3$ is identified with (u', v') in the second copy, where

$$u' = u/\|u\|^2, \quad v' = u^i v u^j / \|u\|$$

(using quaternion multiplication). Check that, if $i + j = 1$, the function

$$f(u, v) = \Re v / (1 + \|u\|^2)$$

extends smoothly to the whole of M and has precisely two critical points, both non-degenerate. Deduce that M is homeomorphic to S^7.

A few years after Milnor's work, Smale proved the Poincaré conjecture in high dimensions. (His famous remark that the proof occurred to him 'on the beaches of Rio' caused some upset back in the USA, see [**29**].) The idea of the proof is to study manifolds-with-boundary which have the homotopy-theoretic properties of the middle, cylindrical, region in the proof of Reeb's theorem above.

1.40. Definition. Let W be a cobordism, that is, a compact manifold with boundary, whose boundary has two components $\partial_- W$ and $\partial_+ W$. It is said to be an *h-cobordism* if the inclusions $\partial_- W \to W$ and $\partial_+ W \to W$ are both homotopy equivalences.

A simple example of an h-cobordism is $M \times [0, 1]$, where M is compact without boundary. This is called a *product* cobordism. Smale proved

1.41. Theorem (h-cobordism theorem)**.** *Any simply-connected h-cobordism W of dimension $\geqslant 6$ is diffeomorphic to a product. In particular, $\partial_- W$ and $\partial_+ W$ are diffeomorphic to one another.*

The proof works with a smooth real-valued function $f \colon W \to \mathbb{R}$, constant on the two boundary components and having only non-degenerate critical points — a *Morse function*. If f has *no* critical points at all then we can use the gradient flow as in Reeb's proof to show that W is a product; so the idea is to modify f by 'canceling' its critical points until none are left. To give a simple example of how this might work, the cubic function on $\mathbb{R}$ given by $x \mapsto x^3 + 3x^2$ has critical points at 0 and -2; as one varies the function in the family $x^3 + 3x^2 + 3\lambda x$, $\lambda \in [0, 2]$, the two critical points coalesce (at $\lambda = 1$) and then both disappear. In order to carry out this cancellation in general there are some topological necessary conditions that must be satisfied (the h-cobordism condition) and the main part of the proof is to show geometrically that when these necessary conditions are satisfied, cancellation can always be carried out. We shall sketch the proof of the h-cobordism theorem in the appendix.

Granted the h-cobordism theorem, the proof of the Poincaré conjecture, at least in dimensions 6 and above, is easy. We just follow the outline of the proof of Reeb's theorem, above. Let Σ be a homotopy sphere. Remove two small, disjoint disks. The resulting manifold-with-boundary is a simply-connected h-cobordism, hence a product. Gluing the disks back in gives a homeomorphism to the standard sphere, via the Alexander trick.

1.5. Exotic spheres

In the middle 1950s, shortly after the publication of the Hirzebruch signature theorem, Milnor was trying to understand the structure of $(n-1)$-connected manifolds of dimension $2n$. (His paper [**19**] gives some of the history.) Classical examples would be the complex projective plane $\mathbb{CP}^2$ of dimension 4, the quaternionic projective plane $\mathbb{HP}^2$ of dimension 8, and the Cayley projective plane of dimension 16. Each of these has $\pi_n(M) = \mathbb{Z}$, and $\pi_n(M)$ is generated by a single embedded n-sphere S^n in M. In an effort to generalize this construction, Milnor considered n-dimensional real vector bundles V over S^n. Taking the disk bundle of such a V gives a compact $2n$-manifold with boundary, say W; and if the closed $(2n-1)$-manifold ∂W happens to be a sphere, then we can attach a $2n$-disk to it and thus obtain a possibly exotic closed $2n$-manifold M.

The bundles V are classified by their 'clutching functions', which are maps $S^{n-1} \to SO(n)$ (or equivalently maps $S^n \to BSO(n)$, using the theory of classifying spaces discussed in the previous section). To begin his study Milnor asked for what choices of clutching function would the manifold ∂W constructed above have the *homotopy type* of a sphere.

Consider first the case $n = 2$. In this case the 2-plane bundles V over S^2 are completely determined by a single integer k in $\pi_1 SO(2) = \mathbb{Z}$ (that is the Euler class). The manifold ∂W is the total space of an S^1-bundle over S^2, and part of the homotopy exact sequence associated to this is

$$\pi_2(S^2) \xrightarrow{\times k} \pi_1(S^1) \longrightarrow \pi_1(\partial W) \longrightarrow \pi_1(S^2) = 0.$$

We see that ∂W is simply-connected (and thus a homotopy sphere) if and only if $k = \pm 1$. In this case the resulting 4-manifold M is simply $\pm\mathbb{CP}^2$, so the construction yields nothing new.

Look now at the case $n = 4$. The bundles V are 4-plane bundles over S^3, classified up to isomorphism by the homotopy class of the clutching map $S^3 \to SO(4)$, that is an element of the homotopy group $\pi_3(SO(4))$. One knows that the simply connected double cover of $SO(4)$ is $S^3 \times S^3$ (to see how an element of $S^3 \times S^3$ gives rise to a rotation, think of the points of S^3 as unit quaternions and associate to $(u, v) \in S^3 \times S^3$ the rotation $x \mapsto uxv$ of $\mathbb{H} = \mathbb{R}^4$). This gives us the calculation

$$\pi_3(SO(4)) = \pi_3(S^3 \times S^3) = \mathbb{Z} \oplus \mathbb{Z}.$$

So the possible bundles V are classified by *pairs* of integers i, j.

Now investigate what is the condition on i, j for the manifold ∂W constructed as above to be a homotopy sphere. W is the total space of an S^3-bundle over S^4 and part of the homotopy exact sequence associated to this is

$$\pi_4(S^4) \xrightarrow{\times(i+j)} \pi_3(S^3) \longrightarrow \pi_3(\partial W) \longrightarrow \pi_3(S^4) = 0.$$

Thus we conclude that W will be 3-connected (and therefore a homotopy sphere, see Exercise 1.17) if and only if $i + j = \pm 1$. In contrast to the case $n = 2$, this gives infinitely many possibilities. Let us fix $i + j = 1$ and consider the corresponding 8-manifolds W_i and their boundaries, the homotopy 7-spheres ∂W_i.

If $i = 1$, then $\partial W_i = S^7$; in fact, the 8-manifold M obtained by attaching a disk to W_1 is simply quaternion projective space. If $i = 2$, though, something strange happens. To see this, suppose for a moment that ∂W_i is also (diffeomorphic to) the 7-sphere, and let M_i be the closed 8-manifold obtained by attaching a disk. We ask: What are the Pontrjagin classes of M_i? Since the generator of $H_4(M; \mathbb{Z})$ is just the sphere S^4 that we started with, the Pontrjagin class $p_1(M_i)$ can be computed in a neighborhood of S^4, and thus from the data $i, j = 1 - i$ alone.

1.42. Exercise. Show that in the above situation we have $p_1(M_i) = 2(i - j) = 2(2i - 1)$ times the generator of $H^4(M; \mathbb{Z}) = \mathbb{Z}$. Check that this fits with the calculation of Pontrjagin classes for the quaternionic projective plane (Exercise 1.29).

The signature of M must be 1 (if we choose the orientation suitably) so the signature theorem for 8-manifolds, equation 1.33, yields

$$p_2[M] = \frac{p_1^2[M] + 45}{7} = \frac{4(2i-1)^2 + 45}{7}.$$

If i=1 this gives $p_2[M] = 7$, consistent with the calculations earlier (Exercise 1.29) for the quaternionic projective plane. But if $i = 2$ then we get $p_2[M] = 81/7$, which is ridiculous; Pontrjagin numbers are integers! The same integrality problem arises for any i not congruent to 0 or 1 modulo 7.

What can be the problem? The supposed smooth 8-manifold M cannot exist, and this means that the homotopy 7-sphere $\Sigma = \partial W$ cannot, after all, be the standard 7-sphere S^7. At this point two possibilities present themselves:

(a) Perhaps Σ is a homotopy 7-sphere which is not homeomorphic to the standard 7-sphere S^7 (and thus a counterexample to the Poincaré conjecture in dimension 7, see below)?
(b) Or, perhaps Σ is a smooth manifold homeomorphic but not diffeomorphic to S^7 — an 'exotic sphere'?

Milnor has recorded that he at first inclined to the view that (a) was true, but in fact the solution turned out to be (b), a conclusion that he announced in the revolutionary paper [**20**].

1.6. Variation of Pontrjagin classes

The Poincaré conjecture shows that for spheres, homotopy type determines homeomorphism type. This is not always true for more complicated manifolds. In this section we shall construct an example of a homotopy equivalence $f\colon M \to M'$ of smooth manifolds which does not preserve the Pontrjagin classes:

$$f^*(p_1(M')) \neq p_1(M) \in H^4(M; \mathbb{Q}).$$

It follows immediately that f cannot be homotopic to a diffeomorphism.

Once again the construction uses bundle theory. Let us consider 5-dimensional oriented vector bundles V over S^4. These are classified up to isomorphism by the homotopy classes of their clutching maps, which are elements of $\pi_3(SO(5))$. It is known that this group is the integers, $\mathbb{Z}$. Moreover, the integer $k \in \pi_3(SO(5))$ that classifies the bundle is just the Pontrjagin class $p_1(V) \in H^4(S^4)$.

One way to see this is to start with $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ (see previous section) and calculate homotopy groups using the long exact sequence of the fibration $SO(4) \to SO(5) \to S^4$. Alternatively, the result is a special case of the Bott periodicity theorem for the orthogonal group. The statement about the Pontrjagin class follows from the rational Hurewicz isomorphism $\pi_4(BSO) \otimes \mathbb{Q} = H_4(BSO; \mathbb{Q})$.

Taking the boundary of the disk bundle associated to $V \oplus \mathbb{R}$, where $\mathbb{R}$ denotes a 1-dimensional trivial line bundle, we obtain a family M_k of closed 9-manifolds parameterized by integers $k \in \pi_3(SO(5)) = \mathbb{Z}$. The classification of these manifolds M_k up to homotopy type depends on the homotopy class of the clutching map, now considered as a map from $S^3 \times S^5 \to S^5$. Basepoints are preserved (if we take the basepoint in S^5 to be the 'north pole' associated to the added trivial line bundle) so that the clutching map is actually a map from $S^8 = S^3 \wedge S^5$ to S^5, and its homotopy class is an element of $\pi_8(S^5)$. Serre's results show that this group is $\mathbb{Z}/24$, so that if k is divisible by 24 the manifold M_k is homotopy equivalent to $M_0 = S^4 \times S^5$.

1.43. Remark. Lurking just beneath the surface of this discussion is a famous and important construction of homotopy theory, the *J-homomorphism*, which is the map $\pi_k(SO(m)) \to \pi_{m+k}(S^m)$ obtained by making $SO(m)$ act on S^m by rotations about the polar axis.

On the other hand, the trivial bundle factor gives a cross section to the fibration $S^5 \to M \to S^4$. This cross section is a copy N of S^4 which generates $H_4(M)$, and its normal bundle ν_N in M is just the original vector bundle V. Thus, evaluating on $[N]$ and using the Whitney sum formula

$$p_1(M) = p_1(\nu_N) + p_1(TN) = p_1(V) + 0 = k$$

since the tangent bundle to N (as to any sphere) is stably trivial. We conclude that $M_0 = S^4 \times S^5$ and M_{24} are homotopy equivalent, but their first Pontrjagin classes are different. The homotopy equivalence between them therefore cannot be homotopic to a diffeomorphism.

1.44. Remark. We have chosen to work with a particular example here, but it is clear that similar constructions could be based on any element of the kernel $\operatorname{Ker} J$.

As in the previous section, two possibilities now present themselves.

(a) Perhaps M_0 and M_{24} are homotopy equivalent but not homeomorphic?
(b) Or, perhaps M_{24} is a smooth manifold homeomorphic but not diffeomorphic to $S^4 \times S^5$ — an 'exotic product of spheres'?

This time however it is (a) that is the true statement; M_{24} is not even homeomorphic to $S^4 \times S^5$. This follows from a deep theorem of Novikov:

1.45. Theorem ([**27, 28**])**.** *If $f\colon M \to M'$ is a homeomorphism between smooth manifolds, then $f^*(p_i(M')) = p_i(M)$ as elements of the* rational *cohomology groups $H^*(M;\mathbb{Q})$.*

This result, proved in the middle 1960s, lies much deeper than anything else we have mentioned in this introduction. To prove it, Novikov devised an elaborate inductive technique for applying the methods of surgery theory, on non-simply-connected smooth manifolds, to problems about homeomorphisms. We will return to the study of Novikov's theorem in Chapter 19.

CHAPTER 2

Classification of exotic spheres

In our first chapter we saw how Milnor used the Hirzebruch signature theorem to give examples of non-standard smooth structures on the 7-sphere. The surgery method was developed a few years later by Milnor and Kervaire [**16**] who wanted to refine this construction into a complete *classification* of the exotic spheres in any sufficiently high dimension. In order to produce such a classification, what was needed was a sort of 'converse' to the signature theorem, which would say that (subject to certain conditions) two exotic spheres which have the same signature-type invariants are actually diffeomorphic. This is what surgery theory does: it gives a systematic procedure for passing from signature-like algebraic invariants to topological conclusions.

In this chapter we will describe a part of the Kervaire-Milnor classification, and use it as an introduction to the more general ideas of surgery theory. The full story of the exotic spheres will be taken up again in Chapter 16.

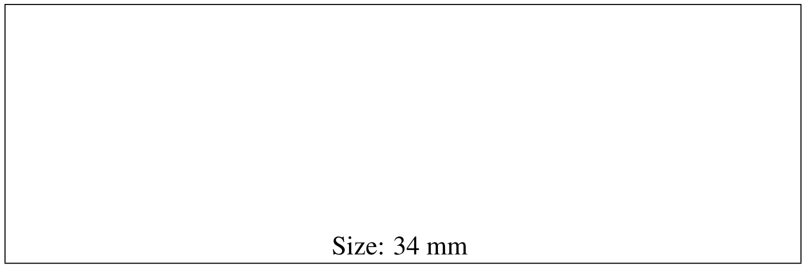

FIGURE 1. Connected sum

2.1. The group of homotopy spheres

Throughout this chapter we will take the dimension n to be large enough for the h-cobordism theorem to apply.

2.1. Definition. A *homotopy n-sphere* is a closed (oriented) n-manifold homotopy equivalent to S^n. It is an *exotic sphere* if it is not diffeomorphic to S^n.

2.2. Definition. Let Θ_n denote the collection of h-cobordism classes of homotopy n-spheres.

The notion of h-cobordism was defined in 1.40. By the h-cobordism theorem, two homotopy spheres are h-cobordant if and only if they are diffeomorphic, so that we could equivalently have defined Θ_n in terms of diffeomorphism classes. However, the definition that we gave fits better with the following more general one, which is central to surgery theory:

2.3. Definition. Let X be a topological space. A *manifold structure* on X is a homotopy equivalence from a closed manifold to X. The *structure set* of X, $\mathcal{S}(X)$, is the collection of h-cobordism classes of manifold structures on X: two manifold structures $f_0\colon M_0 \to X$ and $f_1\colon M_1 \to X$ are *h-cobordant* if there are an h-cobordism M, with $\partial M = M_1 \sqcup (-M_0)$, and a map $F\colon M \to X$ which is a homotopy equivalence and restricts to f_0, f_1 on the ends.

2.4. Exercise. Show that Θ_n is just the structure set $\mathcal{S}(S^n)$.

A special property of Θ_n which is not shared by structure sets in general is

2.5. Theorem. *The operation of connected sum makes Θ_n into an abelian group.*

PROOF. Recall that the *connected sum* of two connected oriented n-manifolds M and M' is defined by removing a small disk from each of M and M', and then joining the boundaries of the resulting manifolds by means of a cylindrical tube $S^{n-1} \times D^1$. See Figure 1. It is not hard to see that this operation is well-defined (up to diffeomorphism), commutative, and associative. Moreover, the connected sum $M \# S^n$ is diffeomorphic to M, so S^n gives an identity element for the connected sum operation. It remains to show that Θ_n has inverses. In fact, the inverse of a homotopy sphere M is the sphere $-M$ with the opposite orientation. To prove this we appeal to the h-cobordism theorem, which implies that M is the union of two n-discs glued along their boundary by some $g \in \mathrm{Diff}(S^{n-1})$. Then $-M$ is the union of two discs glued by g^{-1}, and $M \#(-M) = D^n \cup_g S^{n-1} \times I \cup_{g^{-1}} D^n$ is plainly diffeomorphic to the standard sphere. $\square$

2.6. Exercise. Show directly (without appealing to the h-cobordism theorem) that the inverse operation in Θ_n is given by reversing the orientation. (See Lemma 2.4 of [**16**].)

2.7. Exercise. Show that Θ_n is isomorphic to the quotient of $\mathrm{Diff}(S^{n-1})$ by the subgroup consisting of those diffeomorphisms that extend to diffeomorphisms of the disk D^n.

2.8. Exercise. Think carefully about why the connected sum operation is well-defined. You will need results such as the transitivity of the action of $\mathrm{Diff}(M)$ on M, and the uniqueness of tubular neighborhoods for submanifolds.

2.9. Remark. The operation of connected sum can be described as follows: from the disconnected manifold $M \sqcup M'$ we removed a subset diffeomorphic to $D^n \times S^0$. The boundary of the removed piece, $S^{n-1} \times S^0$, can also be viewed as the boundary of the tube $S^{n-1} \times D^1$. We reinsert this tube, thus obtaining a new closed manifold which has now been made connected.

If we look at connected sums this way it is natural to seek a generalization based on the identity

$$\partial(D^k \times S^{n-k}) = S^{k-1} \times S^{n-k} = \partial(S^{k-1} \times D^{n-k+1}).$$

This generalization is the procedure of *surgery*.

2.2. Spheres that bound parallelizable manifolds

2.10. Definition. A manifold M is *parallelizable* if its tangent bundle is trivial.

Milnor and Kervaire's analysis begins by singling out a certain subgroup of Θ_n.

2.11. Definition. The subgroup $bP_{n+1} \subseteq \Theta_n$ consists of those homotopy spheres which are the boundaries of parallelizable manifolds.

To motivate the definition, observe that a contractible manifold is certainly parallelizable; and, if a homotopy sphere bounds a contractible manifold M, then removing a small disk from M gives an h-cobordism to the standard sphere.

Implicit in this definition is the assertion that bP_{n+1} actually *is* a subgroup. Suppose M_1 and M_2 are homotopy spheres, bounding parallelizable manifolds M_1 and M_2 respectively. Join a disk in ∂M_1 to a disk in ∂M_2 by a tube $D^{4k-1} \times D^1$. You obtain a parallelizable manifold M whose boundary is $M_1 \# M_2$. Thus bP_{n+1} is closed under the group operation.

Remember our mnemonic for surgery theory: manifolds = bundles + handles? The exact sequence

$$0 \to bP_{n+1} \to \Theta_n \to \Theta_n/bP_{n+1} \to 0 \tag{2.12}$$

displays two reasons why a homotopy sphere Σ might be exotic. Firstly, Σ might not bound any parallelizable manifold at all. This phenomenon is related to the tangent *bundle* of Σ and ultimately to the J-homomorphism (Remark 1.43); it is measured by the quotient group Θ_n/bP_{n+1}. We shall take this part of the story up again in Chapter 7 and in particular we shall show that Θ_n/bP_{n+1} is a finite group. Secondly, however, Σ might bound a parallelizable manifold but not bound a contractible one. The obstructions here have to do with signatures, and can be explored by attaching and detaching *handles*, using surgery. For the rest of this chapter we will be concerned with this calculation. Its conclusion is that bP_{n+1} is a finite cyclic group; moreover, its order is 1 or 2 except in the case $n = 4k - 1$, when the order can be large. We will therefore focus attention on the group bP_{4k}.

2.3. A signature invariant

Suppose that M is a parallelizable $4k$-manifold with boundary $\partial M = \Sigma$ a homotopy $(4k-1)$-sphere, $k \geqslant 2$. We know from the Poincaré conjecture that Σ is homeomorphic to S^{4k-1} and therefore that we can build a closed topological manifold M^* by attaching a disk D^{4k} to Σ via this homeomorphism. By definition, the *signature of* M is the signature of the closed manifold M^*.

2.13. Exercise. Show that the signature could have been defined directly in terms of Poincaré-Lefschetz duality for the pair $(M, \partial M)$ (so we did not really need to appeal to the Poincaré conjecture here.)

We are going to prove two results.

2.14. Proposition. *The signature of M is always a multiple of 8.*

2.15. Proposition. *If $\partial M = \Sigma$ is a* standard *$4k-1$-sphere, the signature of M is a multiple of t_k, where*

$$t_k = 2^{2k-1}(2^{2k-1}-1)\frac{3-(-1)^k}{2k}B_k|\operatorname{Im} J_{4k-1}|$$

where B_k denotes the k'th Bernoulli number and J is the stable J-homomorphism. Moreover, any multiple of t_k can occur as the signature of such a M.

Here is a table of the numbers t_k for some small values of k.

k	2	3	4	5
t_k	224	7936	65024	1046528

Adams calculated the size of $\operatorname{Im} J$ in terms of Bernoulli numbers, so that all terms in the expression for t_k are known.

Together, these results give a homomorphism (one-eighth of the signature) from bP_{4k} to a cyclic group of order $t_k/8$. In the next section, we shall construct a specific example of a manifold M with signature 8; this will show that our homomorphism is surjective. In the section after that,we shall use surgery to show that if Σ bounds a parallelizable manifold M with signature a multiple of t_k, then it is standard; this will show that our homomorphism is injective. Thus, the final conclusion will be that bP_{4k} is cyclic of order $t_k/8$.

2.16. Exercise. Check that the signature does, as stated above, give a *homomorphism*.

PROOF OF PROPOSITION 2.14. We are going to show that the intersection form of M is *even* in the sense of Remark 1.8. A result on integral quadratic forms due to van der Blij [**31**] then implies that its signature is divisible by 8. We shall give a proof of van der Blij's lemma in Proposition 8.47.

To show that the intersection form is even it is enough, of course, to show that for every $x \in H^{2k}(M;\mathbb{Z}_2) = H^{2k}(M,\partial M;\mathbb{Z}_2)$ the cup-square $x \smile x$ vanishes in $H^{4k}(M,\partial M;\mathbb{Z}_2) = \mathbb{Z}_2$. Recall now that squaring is a *linear* operation over the field of 2 elements (the Frobenius map!); the map $x \mapsto x \smile x$ can therefore be considered as a linear functional on $H^{2k}(M;\mathbb{Z}_2)$, and therefore there exists some $y \in H^{2k}(M;\mathbb{Z}_2)$ such that

$$x \smile x = x \smile y$$

for all $x \in H^{2k}(M;\mathbb{Z}_2)$.

The class $y \in H^{2k}(M;\mathbb{Z}_2)$ is called the $(2k$'th) *Wu class* of M. In Chapter 5 we shall see that it is a characteristic class for the tangent bundle of M, expressible as a certain combination of Stiefel-Whitney classes (Theorem 5.33). But the tangent bundle of M is trivial by assumption, so $y = 0$ and thus $x \smile x = 0$ mod 2 for all x and the intersection form is even. $\square$

In the next proof we will use the following notion.

2.17. Definition. Let V be an n-dimensional vector bundle over a space X. A *framing* of V is a set of n continuous sections that form a basis for the fiber at every point. A *stable framing* is a framing for $V \oplus \varepsilon^m$ for some trivial bundle ε^m.

Thus TM admits a framing if and only if M is parallelizable. We will usually assume that framings are orthonormal with respect to some metric.

PROOF OF PROPOSITION 2.15. Let M have boundary the standard sphere S^{4k-1}. Since M is parallelizable, we can find a stable framing of the tangent bundle TS^{4k-1} which is compatible with the framing of TM. Such a stable framing may not be the same as the usual stable framing coming from the embedding $S^{4k-1} \subseteq \mathbb{R}^{4k}$. The difference between the framings is measured by an element u of the homotopy group $\pi_{4k-1}(SO)$.

The homotopy groups of the stable orthogonal groups were calculated by Bott at the end of the 1950s. The *Bott periodicity theorem* states that the groups $\pi_{r-1}(SO)$ are 8-periodic in r and are given by the following table

(2.18)

r modulo 8	0	1	2	3	4	5	6	7
$\pi_{r-1}(SO)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

In particular, $\pi_{4k-1}(SO)$ is infinite cyclic. The generator b of this group (the *Bott generator*) defines a vector bundle over S^{4k} and its topmost Pontrjagin class $p_k \in H^{4k}(S^{4k}) = \mathbb{Z}$ can be calculated: the result is

$$p_k(b) = (2k-1)!(3-(-1)^k)/2. \tag{2.19}$$

Return now to our parallelizable M with $\partial M = S^{4k-1}$. Let $u = mb \in \pi_{4k-1}(SO) = \mathbb{Z}b$ be the element defined above. The manifold M^* obtained by attaching a disk to ∂M is now *smooth*. The (stable) tangent bundle of M^* is obtained by clutching together two trivial bundles, one over M and one over D^{4k}, by means of the element u. Therefore, we have the equation $T(M^*) = f^*(mb)$ of stable tangent bundles, where $f\colon M^* \to S^{4k}$ is the degree one map defined by crushing M to a point and b is the Bott generator.

The number m is not arbitrary. Consider the following geometrical construction: regard S^{4k-1} as an equatorial sphere in some S^N, for N large. Since the tangent bundle to S^N has a canonical (stable) framing, the stable framing u of TS^{4k-1} gives rise to a stable framing of the normal bundle of S^{4k-1} in S^N, and therefore to a product structure on a tubular neighborhood U of S^{4k-1} in S^N. Using this product structure we can identify the one-point compactification U^+ as

$$U^+ = (S^{4k-1} \times \mathbb{R}^{N-4k+1})^+ = \Sigma^{N-4k+1}(S^{4k-1} \sqcup \bullet).$$

Crushing the exterior of U to a point, and then crushing S^{4k-1} to another point, gives a composite map

$$S^N \to U^+ = \Sigma^{N-4k+1}(S^{4k-1} \sqcup \bullet) \to \Sigma^{N-4k+1}(S^0) = S^{N-4k+1} \tag{2.20}$$

which defines an element of the stable homotopy group $\pi^s_{4k-1}(S^0)$. In fact, it is not hard to identify this element; it is simply the image of u under the stable J-homomorphism $\pi_{4k-1}(SO) \to \pi^s_{4k-1}(S^0)$. The point about this way of realizing it is that if N is large enough we may assume that not only is S^{4k-1} embedded (with framed normal bundle) in S^N but that M is similarly embedded in D^{N+1}. (This is a consequence of the embedding theorems that we will discuss in Chapter 4.) Applying the construction to M then shows us that $J(u) = 0$. In other words, u is in the kernel of the stable J-homomorphism. Since the domain of this J-homomorphism is infinite cyclic generated by b, it follows that $u = mb$ where m is a multiple of the order $|\operatorname{Im}(J)|$. Moreover, a more detailed analysis of this *Pontrjagin-Thom construction*, which we shall carry out in Chapter 7, will show[1] that *any* element of $\operatorname{Ker} J$ can be realized by a framed manifold M in this way. Thus, any multiple of t_k can arise as a signature.

The final step in the argument is to apply the Hirzebruch signature theorem to the smooth manifold M^*. One needs to know in detail what are the coefficients of the L-classes appearing in that theorem; this will be investigated in Chapter 7. The result, Proposition 7.40, is

$$\operatorname{Sign}(M) = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!} p_k(u)$$

(note that the lower Pontrjagin classes of u are all zero.) Combining this with Equation 2.19 and the fact that m is a multiple of $|\operatorname{Im}(J)|$, we get the desired result. □

2.21. Remark. For the original proof of Bott periodicity, using Morse theory, see [**22**] as well as the wonderful overview in [**6**]. The connections between K-theory and elliptic operator theory gave rise to a variety of new proofs of periodicity using various kinds of analysis, see [**3**] for the most elegant formulation and [**18**] for a full-scale account of the connections between periodicity, spin geometry, Clifford algebras, K-theory, and index theorems for the Dirac operator.

2.22. Exercise. Verify equation 2.19.

2.23. Remark. The theme of this section is that if Σ is standard, M^* is a closed $4k$-dimensional *smooth* manifold, and in that case there are stricter constraints on its signature than if it were merely a *topological* manifold. This idea is still very interesting for $k = 1$, although the dimension is now too low for surgery to function smoothly. If M^4 is a closed 4-dimensional smooth manifold whose first two Stiefel-Whitney classes vanish[2] then the Wu class vanishes also and thus the signature is a multiple of 8, by the argument of Proposition 2.14. But Rochlin drew the sharper conclusion that in this case the signature is actually a multiple of 16. One can see this as an application of the Atiyah-Singer index theorem [**4**]; the condition $w_1 = w_2 = 0$ implies that M admits a spin structure; an application of the index theorem shows that for a spin 4-manifold the signature is 8 times the index of the Dirac operator associated to the spin structure; the Dirac index is even because of the quaternionic structure on spinor bundles in dimensions congruent to 4 mod 8. All this is discussed in detail in [**18**]. It was for long an open question whether the signature of a *topological* 4-manifold with $w_1 = w_2 = 0$ must be a multiple of 16. This question was answered negatively by an example of Freedman, see [**11**] and the book [**12**]. As this example suggests, four-dimensional topology is extremely subtle. We won't discuss it in any further detail in this book.

[1] This is a transversality argument like that in Theorem 2.57.

[2] This hypothesis substitutes for the parallelizability of M.

2.4. The Milnor manifold

In this section we shall explain Milnor's process of *plumbing*, which allows one to construct parallelizable manifolds with homotopy sphere boundary and with suitably prescribed intersection form.

Suppose given a finite graph Γ (for us, it will usually be a tree) each of whose vertices is labelled by an integer. We can regard this as a prescription for defining a symmetric bilinear form on $\mathbb{Z}^p$, p being the number of vertices, as follows: the (i, j) entry in the matrix defining our form is the label on the i'th vertex if $i = j$, and is $i \neq j$ the entry is the number (0 or 1) of edges joining the i'th to the j'th vertex. Thus for instance the graph

$$\begin{tikzcd}[arrows=dash]
2 \arrow[r] & 2 \arrow[r] & 2 \arrow[r] & 2 \arrow[r] & 2 \arrow[r] \arrow[d] & 2 \arrow[r] & 2 \\
& & & & 2 & &
\end{tikzcd}$$

corresponds to the 'E_8 matrix'

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{2.24}$$

The importance of this matrix is that the associated symmetric bilinear form is even, positive definite and unimodular (determinant is $+1$).

2.25. Exercise. Prove these facts using elementary row and column operations. (Browder [**9**] devotes several pages of his surgery book to this calculation.)

The building blocks for plumbing are *disk bundles* over spheres. The *disk bundle* associated to a vector bundle over a compact base is the bundle of vectors of length $\leqslant 1$ in some metric. If the base is a manifold, then the disk bundle may be considered to be a manifold with boundary.

Now let V be a m-dimensional oriented vector bundle over S^m. It is classified up to isomorphism by its clutching function, which is an element of $\pi_{m-1}(SO(m))$. The fibration

$$SO(m-1) \to SO(m) \to S^{m-1}$$

gives a homomorphism $\pi_{m-1}(SO(m)) \to \pi_{m-1}(S^{m-1}) = \mathbb{Z}$.

2.26. Definition. The integer invariant associated to V by this construction is the *Euler number* $e(V)$ of V.

By construction, $e(V)$ vanishes if and only if V has a nowhere vanishing section. In fact, the Euler number $e(V)$ is precisely the number of zeroes (counted according to sign) of a 'generic' section of V. See Proposition 3.20. This leads to an important relation between the Euler number and *self-intersections*.

2.27. Lemma. *Let M be the disk bundle associated to an m-dimensional oriented vector bundle over S^m. Let $x \in H_m(M; \mathbb{Z})$ be the homology class associated to the zero-section $S^m \subseteq M$. Then the self-intersection number $x \cdot x$ is equal to the Euler number of the bundle.*

PROOF. To define the self-intersection number of a submanifold N of M we take a small perturbation N' of N, in the same isotopy class, and count the points of intersection of N' and N. Here we may take N' to be a generic section of the bundle, and our count of intersection points is precisely the count of the zeroes of this section. $\square$

The Euler number plays a double rôle in surgery theory: as well as its connection with self-intersections it is also relevant to the problem of *destabilizing* a stable framing of a vector bundle over a sphere. Suppose that V is a stably framed vector bundle (that is, $V \oplus \varepsilon^k$ is framed for some k). A framing of V is *compatible* with the given stable framing if the two framings that can be constructed on $V \oplus \varepsilon^k$ are homotopic; if V admits a compatible framing we shall say that the given stable framing of V is *destabilized*.

2.28. Proposition. *Suppose that V is an n-dimensional vector bundle over a k-dimensional CM-complex X, $k < n$. and that $V \oplus \varepsilon^1$ admits a framing. Then any stable framing of V admits a destabilization.*

PROOF. Inductively we may assume that $V \oplus \varepsilon^1$ is framed. We try to construct the desired framing of V by induction over the cells. The inductive step is then this: given an i-cell $(D^i, \partial D^i)$, and a framing of $V \oplus \varepsilon^1$ over D^i which arises on ∂D^i from a framing of V, deform rel boundary to obtain a framing of V on D^i. If we trivialize V over D^i the problem becomes one of filling in the dotted arrow in the diagram

$$\begin{tikzcd} \partial D^i \arrow[r] \arrow[d] & SO(n) \arrow[d] \\ D^i \arrow[r] \arrow[ur, dotted] & SO(n+1) \end{tikzcd}$$

The possibility of doing this is controlled by the relative homotopy group $\pi_i(SO(n+1), SO(n))$; however, the homotopy sequence of the fibration $SO(n) \to SO(n+1) \to S^n$ shows that this group is $\mathbb{Z}$ if $i = n$ and 0 if $i < n$. Since $i \leqslant k < n$ there is no obstruction. $\square$

In Chapter 3 we shall prove the following result for even-dimensional spheres.

2.29. Proposition. *Let V be an m-dimensional oriented vector bundle over S^m, m even. Then*

(a) *If V admits a stable framing then its Euler number is even.*
(b) *Stably framed bundles exist realizing all even Euler numbers.*
(c) *Two stably framed bundles are isomorphic (respecting the stable framing) if and only if their Euler numbers are the same. In particular, a stable framing for V can be destabilized if and only if the Euler number of V is zero.* $\square$

The tangent bundle TS^m, which is stably trivial and has Euler number 2, is an important example.

2.30. Exercise. Give an example of an oriented 4-dimensional vector bundle over S^4 which has Euler number 0 but is not framed (even stably).

Suppose now that we are given two m-disk bundles M_1 and M_2 over m-spheres S_1 and S_2. Pick points $p_1 \in S_1$ and $p_2 \in S_2$. Then in each M_i there is a product neighborhood of p_i diffeomorphic to $D^m \times D^m$.

Size: 49 mm

FIGURE 2. Plumbing

2.31. Definition. The *plumbing* $M = M_1 \diamond M_2$ of M_1 to M_2 is obtained by identifying the product neighborhoods of the p_i in M_i, in such a way that 'fiber disks' in M_1 are identified with 'base disks' in M_2 and vice versa.

See Figure 2 for a graphical representation of plumbing. The plumbing $M_1 \diamond M_2$ is a manifold with 'corners', but there is a natural way of blowing up the corner points so as to regard it as, in fact, a manifold with boundary. (Similar issues arise in defining the connected sum, and many other important operations of surgery theory.)

2.32. Remark. A *manifold with corners* is a space locally modeled on an open subset of $(\mathbb{R}^+)^n$; the *corner set* is the set of points where two or more coordinates are zero in the local model. To be more precise, M has 'boundaries' $\partial_S M$, possibly empty, for each subset S of $\{1, \ldots, n\}$, where $\partial_S M$ is modeled locally by $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = 0 \Leftarrow i \in S\}$. The transition functions are of course required to be smooth. Apart from their use in defining plumbing, connected sum, and so on, manifolds with corners will also have to be considered when we deal with cobordisms of manifolds with boundary.

Corners are a nuisance of the smooth category: in the topological or piecewise-linear categories, $(\mathbb{R}^+)^n$ is homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}^+$, so that manifolds with corners are the same thing as manifolds with boundary. We won't have occasion to consider manifolds with worse than second-order corners, so that $\partial_S M = \emptyset$ whenever $|S| \geqslant 3$. The corner set of such a manifold is then itself a closed manifold, and it has a tubular neighborhood which is fibered (trivially) by quarter-spaces. We may turn such a manifold into an ordinary manifold with boundary by excising the tubular neighborhood of the corners, doubling all the angles (thus turning the quarter-space into a half-space), and re-attaching the resulting bundle of half-spaces. This process is called *unbending the corners*. Conversely, suppose that we are given an ordinary manifold M with boundary, and a codimension zero submanifold X (with boundary) of the closed manifold ∂M. Then we may *bend* the manifold M along M, obtaining a manifold M_c with corners such that $\partial_1 M_c = X$ and $\partial_2 M_c = \partial M \setminus X^\circ$.

It is a non-trivial matter to make this precise (in particular to prove the required tubular neighborhood theorems) and to show that 'unbending' and 'bending' are well-defined operations (up to the appropriate notion of diffeomorphism in each case). For a detailed treatment, see the lecture notes of Wall [**32**].

2.33. Exercise. In plumbing one must take account of the orientations. Show that the plumbing operation is symmetric if m is even ($M_1 \diamond M_2$ has the same orientation as $M_2 \diamond M_1$), but skew-symmetric if m is odd.

2.34. Definition. Let Γ be a labeled graph, as was considered above, all of whose vertex labels are even, and let $k \geqslant 2$. The *Milnor manifold* M^{4k}_Γ is obtained as follows: to each vertex of Γ associate the unique stably framed disk bundle with the prescribed Euler number (Proposition 2.29) and plumb them together as prescribed by the edges of Γ (two disk bundles are plumbed if and only if the corresponding vertices are joined by an edge.)

2.35. Theorem (Plumbing Theorem)**.** *Suppose that the graph Γ is a tree. Then the plumbed manifold M built from Γ and $k \geqslant 2$ has the following properties:*

(i) *Both M and its boundary ∂M are simply connected;*
(ii) *The middle-dimensional cohomology $H^{2k}(M, \partial M) = \mathbb{Z}^p$, where p is the number of vertices of Γ, and the symmetric form given by the cup-product is the one associated to Γ;*
(iii) *M is parallelizable;*
(iv) *If Γ defines a unimodular form, then ∂M is a homotopy sphere.*

PROOF. (i) Use the van Kampen theorem.

(ii) By Poincaré duality, the cohomology group $H^{2k}(M, \partial M)$ is isomorphic to $H_{2k}(M)$. Now each disk bundle over S^{2k} deformation retracts onto S^{2k}, so it is easy to see that M has the homotopy type of a wedge $\bigvee^p S^{2k}$, and thus its $2k$-dimensional homology is $\mathbb{Z}^p$ generated by the base spheres of the plumbed disk bundles. Each homology generator has self-intersection given by the associated Euler number, by lemma 2.27; and, by the construction, distinct homology generators have intersection $+1$ if the corresponding disk bundles have been plumbed together, and 0 otherwise.

(iii) All of the disk bundles used in the plumbing are themselves stably trivial. Since the tangent bundle to a sphere is stably trivial, one easily deduces that the tangent bundle to each disk bundle, and therefore the tangent bundle to M itself, are stably trivial. But M has the homotopy type of a $2k$-dimensional CM-complex and therefore stable triviality of the tangent bundle implies triviality (by Proposition 2.28).

(iv) M is $(2k-1)$-connected, because it is homotopy equivalent to a wedge of $2k$-spheres, and similarly it is not hard to see that ∂M is $(2k-2)$-connected. Using Poincaré duality one sees that the only non-trivial part of the long exact homology sequence of the pair $(M, \partial M)$ is

$$\begin{tikzcd} 0 \arrow[r] & H_{2k}(\partial M) \arrow[r] & H_{2k}(M) \arrow[r] & H_{2k}(M, \partial M) \arrow[r] \arrow[d, equal] & H_{2k-1}(\partial M) \arrow[r] & 0 \\ & & & \mathrm{Hom}(H_{2k}(M), \mathbb{Z}) & & \end{tikzcd}$$

The vertical isomorphism uses $H_{2k}(M, \partial M) \cong H^{2k}(M)$ (Poincaré duality) together with $H^{2k}(M) \cong \mathrm{Hom}(H_{2k}(M), \mathbb{Z})$ (the universal coefficient theorem using the fact that $H_{2k}(M)$ is free abelian). The middle arrow, considered as a bilinear form on $H_{2k}(M)$, is just the intersection form; so, if the intersection form is unimodular, this arrow is a bijection and $H_{2k}(\partial M) = H_{2k-1}(\partial M) = 0$. Thus ∂M is a homology sphere. Since it is simply connected, it is a homotopy sphere. $\square$

2.36. Remark.

2.5. Surgery and the calculation of bP_{4k}

In this section we shall use surgery to prove the following result.

2.37. Proposition. *Let Σ be a homotopy sphere bounding a parallelizable manifold M^{4k}, $k \geqslant 2$. If M has signature zero, then Σ is standard.*

The idea of the proof is to modify M (keeping the boundary fixed) by a sequence of *surgeries* until it becomes contractible. If Σ bounds a contractible manifold, then it will certainly be h-cobordant to a standard sphere.

2.38. Definition. Let M^n be a smooth manifold and let $i\colon S^m \to M$ be a framed embedding of a m-sphere in M (that is, an embedding together with a specific framing of its framed normal bundle). The operation of *surgery on i* is defined as follows: identify a closed tubular neighborhood U of $i(S^m)$ with $S^k \times D^{n-m}$ (this uses the given framing); remove this tubular neighborhood and replace it with $D^{m+1} \times S^{n-m-1}$ (which has the same boundary), thus obtaining a new smooth n-manifold M' called the *effect* of the surgery.

Compare our discussion of connected sums, Remark 2.9. In fact, a connected sum is just a surgery on an embedded 0-sphere.

If M is a manifold with boundary, we can still do surgery on framed embeddings of spheres in the *interior* of M. The boundary makes no difference.

2.39. Proposition. *Let M' be obtained from M by a surgery. There is a natural cobordism W' from M to M' (which we call the* trace *of the surgery). If M has a boundary, the trace of the surgery is a product along the boundary.*

PROOF. The cobordism is constructed by attaching $D^{m+1} \times D^{n-m}$ to $W = M \times [0,1]$ along $S^m \times D^{n-m} \subseteq M \times \{1\}$. See figure 3. $\square$

The process of attaching $D^{m+1} \times D^{n-m}$ to a framed sphere in the boundary (of $M \times [0,1]$ in this instance) is called *handle attachment*, and the product $D^{m+1} \times D^{n-m}$ itself is a *handle*.

Some bending and unbending (compare Remark 2.32) is needed to accomplish handle attachment smoothly. Suppose that W is a manifold with boundary, and that ∂W contains an embedded sphere S^q with trivial normal bundle. Let $X = D^p \times S^q$ be a tubular neighborhood of S^q in ∂W; it is a codimension-zero submanifold with boundary. Bend along ∂X, to obtain a manifold W_c with corners, having $\partial_1 W_c = D^p \times S^q$. Now consider the product $H = D^p \times D^{q+1}$; it is in a natural way a manifold with corners, having $\partial_1 H = D^p \times S^q$ and $\partial_2 H = S^{p-1} \times D^{q+1}$. Glue H to W_c along ∂_1; the corners fit together smoothly and we obtain a new manifold with boundary $W' = W \cup_X H$. The trace of the surgery in Proposition 2.39 above is obtained in this way from $W = M \times [0,1]$.

2.40. Remark. It follows that the signature of M' is equal to the signature of M. (This will also follow from our calculations of the effect of surgery on homology.)

2.41. Remark. It is important to observe that the surgery process is reversible. If M' is obtained from M by surgery on a framed embedding of S^m, then M' is provided (by construction) with a framed embedding of S^{n-m-1}. Carrying out the *dual surgery* on this embedding recovers M once more.

Suppose now that M is parallelizable, and choose a framing of its tangent bundle. An embedding $i\colon S^m \to M$ is *compatibly framed* if its normal bundle is framed and the direct sum of this framing of the normal bundle and the canonical stable framing of $T(S^m)$ is compatible with the chosen framing on $TM_{|i(S^m)}$.

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FIGURE 3. Trace of a surgery

2.42. Proposition. *Suppose that M is parallelizable. If we do surgery on a compatibly framed embedding $i\colon S^m \to M$, then the effect of the surgery, M', is also parallelizable. Furthermore, the trace of the surgery is parallelizable (compatibly with M and M').*

PROOF. $\square$

In the proof of Proposition 2.37 one carries out a sequence of surgeries on stably framed embeddings $S^m \to M^{4k}$, hoping that the final effect of all these surgeries will be contractible. The process falls naturally into two stages:

(a) Surgery below the middle dimension: we carry out successive surgeries on compatibly framed embeddings of S^m for $m = 0, 1, \ldots, 2k-1$. The overall effect of these surgeries is to produce a $(2k-1)$-connected M', parallelizable and with Σ as boundary. This process can always be carried out whatever the signature of M.

(b) Surgery in the middle dimension: we carry out surgeries on compatibly framed $2k$-spheres whose effect is to produce a $2k$-connected parallelizable manifold M'' with boundary Σ. A Poincaré duality argument shows that M'' is contractible.

2.43. Exercise. Show that, as asserted above, a $2k$-connected M^{4k} with homotopy sphere boundary is contractible. (Modify the idea of Exercise 1.17 to take into account the presence of the boundary.)

Surgery below the middle dimension can always be carried out; this is proved in Chapter 13. To simplify our discussion we shall *assume* this result for now, which is to say that we shall assume that the M appearing in Proposition 2.37 is already $(2k-1)$-connected. We ask the reader to take on trust for now that this is the 'easy part' of surgery theory, in which no obstructions occur, and that the main point will still be clear when we think about the final stage: passing from $(2k-1)$-connectivity to $2k$-connectivity.

For the rest of this section let us therefore assume that M^{4k} is $(2k-1)$-connected, parallelizable, and has homotopy sphere boundary Σ and signature 0.

2.44. Lemma. *The middle-dimensional homology group $H_{2k}(M) = H_{2k}(M, \partial M)$ is free abelian.*

PROOF. We have

$$H_{2k}(M) \cong H^{2k}(M) \cong \mathrm{Hom}(H_{2k}(M), \mathbb{Z}),$$

the first isomorphism by Poincaré duality and the second by the Universal Coefficient Theorem. But the group on the right is free. $\square$

The intersection form B of M is therefore a symmetric bilinear form on a free $\mathbb{Z}$-module N. Moreover, the intersection form is *even* (see the proof of Proposition 2.14.)

2.45. Proposition. *There is a basis for N such that the matrix of B with respect to this basis is a direct sum of copies of the 2×2 matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The assumption that the signature is zero is crucial here, of course! The 2×2 matrix shown above is said to define a *hyperbolic* form.

PROOF. We need a deep fact from number theory. Since the intersection form has signature 0, it must *represent zero* over the reals: there must be a nonzero real vector v such that $B(v, v) = 0$. We use

2.46. Theorem. *If a unimodular integral quadratic form represents zero over the reals, then it represents zero over the integers.* $\square$

What this tells us is that there exists an $x \in N$ (an integer vector) such that $B(x, x) = 0$. Clearly there is no loss of generality in assuming that x is *indivisible* (unable to be written as a nontrivial integer multiple of some other vector), and it is easy to see that if x is indivisible then $\mathbb{Z}x$ is a direct summand in N. Since B is unimodular there is now $y' \in M$ such that $B(x, y') = 1$. Consider $B(y', y') = 2p$ (since B is even); replacing y' by $y = y' - px$ gives two elements $x, y \in M$ such that $B(x, x) = B(y, y) = 0$ and $B(x, y) = 1$. Now it is easy to see that $N = N_1 \oplus N_2$, where N_1 is the submodule spanned by x and y and N_2 is its orthogonal complement with respect to the form B; the result then follows by induction. $\square$

2.47. Remark. Here are a few indications about the proof of Theorem 2.46. If the rank of N is $\leqslant 4$ one can work 'by hand' to classify all unimodular integral quadratic forms (the condition on the rank implies that there is a vector $x \in N$ with $|B(x, x)| < 2$). On the other hand, for rank $\geqslant 5$ one can appeal to a result of algebraic number theory, the Hasse-Minkowski theorem, which states that a quadratic form with rational coefficients represents zero over $\mathbb{Q}$ if and only if it represents zero over $\mathbb{R}$ and over each p-adic completion $\mathbb{Q}_p$. In rank $\geqslant 5$ the p-adic condition is satisfied automatically, so B represents zero over the rationals and thus by clearing denominators it represents zero over the integers. Notice that we did not use unimodularity in this argument; on the other hand unimodularity is essential for small ranks since, for example, the form $x^2 - 2y^2$ certainly does not represent zero over $\mathbb{Z}$.

For more about these matters, see the book of Milnor and Husemoller [**24**]. One can also find in this book a proof of Theorem 2.46 which avoids algebraic number theory — see Corollary 2.6 in Chapter IV of [**24**].

So far we have shown that, relative to a suitable basis, the middle-dimensional homology of M is a direct sum of 2-dimensional pieces on each of which the intersection form is hyperbolic. We now need to understand geometrically how this can have come about.

An example of a manifold with hyperbolic intersection form is the product $S^{2k} \times S^{2k}$ (see Figure 4). Thus, if M is any manifold, the intersection form of the connected sum $M\#(S^{2k} \times S^{2k})$ has a hyperbolic direct summand. We aim to show that this is the *only* way

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FIGURE 4. Realizing a hyperbolic form

that a hyperbolic direct summand can arise in the intersection form of M (parallelizable, $(2k-1)$-connected, with homotopy sphere boundary).

2.48. Exercise. Show that connected sum with $S^{2k} \times S^{2k}$ is just the effect of surgery on a trivial S^{2k-1} in M (that is, an S^{2k-1} that bounds an embedded disk D^{2k}, and is framed in a way that extends to a framing of that disk).

2.49. Exercise. In the situation of the preceding exercise, suppose that we carry out surgery on an S^{2k-1} that bounds an embedded disk, but that we use a possibly nontrivial framing on the S^{2k-1}. Show that the effect of this surgery is a connected sum with the total space of a possibly nontrivial S^{2k}-bundle over S^{2k}.

The reverse process (removing the $S^{2k} \times S^{2k}$) can therefore be accomplished by a dual surgery (Remark 2.41). That is the idea of the following proof.

2.50. Proposition. *Let M^{4k} be parallelizable, $(2k-1)$-connected, and have homotopy sphere boundary. Suppose that the intersection form of M has a hyperbolic direct summand. Then one can find a compatibly framed embedding $S^{2k} \to M$ such that, if M' denotes the effect of surgery on this embedding, then*

(a) *the intersection form of M' is the complement of the hyperbolic summand in the intersection form of M, and*
(b) *M is diffeomorphic to $M' \# (S^{2k} \times S^{2k})$.*

PROOF. Let $x, y \in H_{2k}(M)$ span the hyperbolic direct summand that we are considering. Since M is $(2k-1)$-connected, the Hurewicz theorem gives a homotopy class of maps $S^{2k} \to M$ which represents the homology class x.

An important and subtle theorem of Whitney states that every homotopy class of maps from an m-sphere to a simply-connected $2m$-manifold, $m \geqslant 3$, contains an embedding[3]. Apply this theorem to x to get an embedding $i\colon S^{2k} \to M$ representing it. Since the Euler number of the normal bundle is equal to the self-intersection $x \cdot x$ and is therefore zero, is zero, this embedding can be compatibly framed (Proposition 2.29). Carry out surgery on this framed embedding, with trace W and effect M'.

[3]We shall prove this in Chapter 10. The condition $m \geqslant 3$ is one of the places where the requirement of high-dimensionality enters into surgery theory.

To compute the homology of M', consider the following braid diagram which displays various homology exact sequences associated to the triple $(T; M, M')$.

$$
\begin{tikzcd}
H_{2k+1}(W,M) \arrow[rr, bend left, "x"] \arrow[dr] & & H_{2k}(M) \arrow[rr, bend left, "x^*"] \arrow[dr] & & H_{2k}(W,M') \\
& H_{2k+1}(W, M\cup M') \arrow[ur] \arrow[dr] & & H_k(W) \arrow[ur] \arrow[dr] & \\
H_{2k+1}(W,M') \arrow[ur] \arrow[rr, bend right] & & H_{2k}(M') \arrow[ur] \arrow[rr, bend right] & & H_{2k}(W,M)
\end{tikzcd}
$$

We claim that

(i) All of the groups $H_j(W, M)$ are zero except for $H_{2k+1}(W, M)$ which is $\mathbb{Z}$;
(ii) All of the groups $H_j(W, M')$ are zero except for $H_{2k}(W, M')$ which is $\mathbb{Z}$;
(iii) The map displayed as x above sends the generator of $H_{2k+1}(W, M)$ to the homology class x;
(iv) The map displayed as x^* above sends a homology class to its intersection product with x.

All of these are clear except perhaps (iv). Let z be another element of $H_{2k}(M)$, represented as above by an embedding $S^{2k} \to M$. By construction, the integer $x^*(z)$ is the degree of the composite map

$$S^{2k} \to M \to U^+ = \Sigma^{2k}(S^{2k} \sqcup \bullet) \to S^{2k}$$

obtained by performing the Pontrjagin-Thom construction (see Equation 2.20) on a tubular neighborhood U of the original framed embedding $i\colon S^{2k} \to M$. The result (d) now follows from the familiar fact that the degree of a map between spheres is the number of preimages of a generic point.

Now it is a general fact about exact braid diagrams of the above sort that the top row and the bottom row are chain complexes with (naturally) isomorphic homology. Thus in our situation we get a natural isomorphism

$$H_{2k}(M') \cong \mathrm{Ker}(x^*)/\,\mathrm{Im}(x) = \langle x\rangle^{\perp}/\langle x\rangle$$

and this can be naturally identified with the orthogonal complement to the hyperbolic subspace $\langle x, y\rangle$ in $H_{2k}(M)$. This proves part (a) of the proposition.

To prove part (b) we need to know that Whitney's ideas can also be used to study the *intersections* of embedded spheres[4] The homology classes x and y can be represented by embedded spheres, and their 'algebraic' intersection number is 1. Since the algebraic intersection number is the *signed* count of the geometric intersection points, there might in principle be any odd number of *geometric* intersections between the representative spheres. The specific fact that we need from Whitney's theory is that we can choose embeddings i_x and i_y representing x and y which intersect (transversely) in exactly one point. (This follows from Theorem 6.31.)

[4] This will be a central theme in Chapters 6 and 9.

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FIGURE 5. Surgery on a hyperbolic form

Let U be a tubular neighborhood of $i_x(S^{2k})$. Then $i_y(S^{2k})$ meets ∂U in a 'dual sphere' S^{2k-1}, which bounds two disks, one contained in U and one contained in its complement (see Figure 5). Now the parts of M and M' outside the surgery region are identical. Thus the dual sphere bounds a disk in M'. However, M is obtained from M' by surgery on this dual sphere. Since this sphere bounds a disk, and has a framing which extends to the disk, $M = M' \# (S^{2k} \times S^{2k})$, as we observed above (Exercise 2.48). This completes the proof. $\square$

2.51. Exercise. Prove the algebraic fact about braid diagrams used above, that the top and bottom rows have isomorphic homology. Also, prove that in the situation above the isomorphism respects the intersection form.

PROOF OF PROPOSITION 2.37. Given M as in the statement of the proposition, we may assume by surgery below the middle dimension that it is $(2k-1)$-connected. According to Proposition 2.45, the intersection form on the free abelian group $H_{2k}(M)$ is a direct sum of hyperbolic pieces. According to Proposition 2.50, we can carry out a succession of compatibly framed surgeries which remove these hyperbolic pieces. The final effect of these surgeries is a M' which is framed, has the same boundary as M, and is $2k$-connected; therefore it is contractible (Exercise 2.43). Removing a small disk from M' now gives an h-cobordism between Σ and a standard sphere. $\square$

We can now complete the calculation of the group bP_{4k}.

2.52. Theorem. *The group bP_{4k} of exotic $(4k-1)$-spheres that bound parallelizable manifolds is cyclic and of order $t_k/8$, where t_k is defined in Proposition 2.15.*

PROOF. Propositions 2.14 and 2.15 tell us that one-eighth of the signature gives a homomorphism φ from bP_{4k} to the cyclic group of order $t_k/8$.

By applying the Plumbing Theorem 2.35 to the E_8 matrix of Equation 2.24, we see that bP_{4k} contains a homotopy sphere of signature 8. Thus the homomorphism φ is surjective.

Suppose that Σ is in the kernel of φ, so that it bounds a parallelizable manifold with signature a multiple of t_k. By the existence part of Proposition 2.15, there exists a parallelizable manifold with the same signature bounding a standard sphere. By taking the difference of these in the group bP_{4k}, we see that there is no loss of generality in assuming that Σ bounds a parallelizable manifold of signature 0. But now Σ is standard by surgery (Proposition 2.37). Thus φ is injective, and hence an isomorphism. $\square$

2.6. Overview of surgery theory

The classification of exotic spheres provides a model of how surgery can be applied to compute other structure sets $\mathcal{S}(X)$. In general it is necessary to assume only that X is a compact *Poincaré duality space*, that is, it has a 'fundamental class' which induces appropriate isomorphisms between cohomology and homology. Surgery theory then gives a uniform treatment both of the existence question — does X have a manifold structure at all? — and of the uniqueness question — how many such structures are there? The classification of exotic spheres focuses on the uniqueness question, but it was soon observed that the same methods could be applied to the existence question as well (see Browder [**10**]; this manuscript, written in 1962, was finally published more than thirty years later!).

The classification of exotic spheres is a two-stage process, as displayed in the sequence 2.12: one stage having to do with surgery, and the other with bundle theory. The same is true for the calculation of more general structure sets. We begin with the bundle-theoretic side, which is based on Browder's notion of a *normal map*. This theory will be developed in detail in Chapter 11.

If X is to have the structure of a smooth manifold then there must be a vector bundle that will serve as its tangent bundle. In fact, it is often geometrically more convenient to think about the *stable normal bundle*. If X were a manifold and embedded in $\mathbb{R}^{n+k}$, k large, then it would have a tubular neighborhood diffeomorphic to the total space of a vector bundle V, the *normal bundle* of the embedding. Once k is big enough, making it bigger just changes V by the addition of a trivial bundle, so it makes sense to speak of a well-defined stable normal bundle. By construction, the tangent bundle plus the stable normal bundle is (stably) trivial.

The stable normal bundle cannot be prescribed arbitrarily. Its twofold interpretation (as a vector bundle over X and as a neighborhood of X in Euclidean space) gives a compatibility condition between Poincaré duality and the *Thom isomorphism* of the bundle V. This condition is necessary (but not sufficient) for V to be the stable normal bundle of a manifold structure on X.

2.53. Claim. In order that the Poincaré duality space X admit a manifold structure, it is necessary that there exist a k-vector bundle V over X, having the following property: if $\Phi\colon H_{r+k}(\mathrm{Th}\,V,\infty) \to H_r(X)$ denotes the Thom isomorphism, then $\Phi^{-1}([X])$ is a *spherical* class.

Here $\mathrm{Th}\,V$ denotes the *Thom space* of V, that is the one-point compactification of the total space of V. By definition, a *spherical* homology class is one in the image of the Hurewicz homomorphism — that is, one arising from a map $S^{r+k} \to \mathrm{Th}\,V$. To see how the condition arises, think of V as the normal bundle of an embedding of X in S^{r+k}.

2.54. Definition. A *normal invariant* for the Poincaré duality space X is a stable isomorphism class of vector bundles V as described in the Claim above. The pair (X, V) will be referred to as *normal data*.

2.55. Remark. We shall see in Chapter 11 that this notion can be expressed in homotopy-theoretic terms. In fact, any Poincaré duality space possesses a *Spivak normal bundle*, a spherical fibration canonically determined by the Poincaré duality structure. There is a forgetful functor from (stable) vector bundles to (stable) spherical fibrations, corresponding to a map of classifying spaces $BO \to BG$; a normal invariant is just a *reduction of structure* of the Spivak normal bundle to a vector bundle, or equivalently, a factorization of its classifying map $X \to BG$ through the space BO.

2.56. Definition. A *normal map* associated to given normal data (X, V) is a pair (f, b) where f is a degree-one map from an oriented smooth manifold M to X, and b is a (stable) isomorphism from $f^*(V)$ to the (stable) normal bundle of M.

We say that $f\colon M \to X$ is of *degree one* if $f_*[M] = [X]$, where the square brackets denote the fundamental homology classes; in particular the dimension of M equals the 'formal dimension' of X (the degree in which its fundamental class appears).

Surgery theory regards a normal map as a 'first approximation' to a homotopy equivalence from a manifold to X. The following theorem is therefore fundamental to the subject.

2.57. Theorem. *Given normal data (X, V), there exist normal maps $f\colon M \to V$ associated to it.*

SKETCH OF PROOF. How shall we obtain a manifold M from normal data? The answer is to apply *transversality theory*. This theory — one of Thom's beautiful ideas — is about the 'generic' behavior of smooth maps. In its simplest form it concerns a smooth map between manifolds, $g\colon M^{n+k} \to N^n$. It is easy to see that any closed subset of M can appear as such an inverse image. Nevertheless, 'generically' the inverse image $g^{-1}\{p\}$ of a point $p \in N$ is a smooth k-dimensional submanifold. Notice that in linear algebra, 'generically' a linear map from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$ is surjective with k-dimensional kernel. The basic idea of transversality theory is that the generic behavior of *smooth* maps is modeled by the generic behavior of *linear* maps (which of course appear as the tangent maps to the smooth maps in question).

We will leave until later the question of what exactly is meant by 'generic'. Let us suppose that normal data (X, V) are given, and apply transversality to the map $g\colon S^{n+k} \to \operatorname{Th} V$ which is given to us by the assumption that the Thom class of V is spherical. What we mean by this is the following. Locally V is a product, so (away from the point at infinity) g looks like $(g_1, g_2)\colon U \to Y \times \mathbb{R}^k$, $Y \subseteq X$, $U \subseteq S^{n+k}$ and we may assume without loss of generality that g_2 is smooth. Transversality theory tells us that the 'generic' behavior of g_2 is as described above: the inverse image of a point (say the origin) is an n-dimensional submanifold. But the inverse image of a point under g_2 is just the inverse image of the *zero-section* of the bundle V under f. We therefore expect, and Thom's transversality theorem verifies, that 'generically' the inverse image of the zero section of $\operatorname{Th} V$ will be an n-dimensional submanifold of S^{n+k}, equipped with a map $f\colon M \to X$ (just the restriction of g) which pulls back V to the normal bundle of M. $\square$

It should be underlined that this is a very *non-constructive* way to obtain the manifold M. The 'generic' perturbation of g cannot be precisely specified in advance.

We have seen that from normal data (X, V) it is always possible to construct normal maps $M \to V$ — in fact, one can show that the normal data determine an entire *normal bordism* class of normal maps. Now, following Browder, we ask: Does this normal bordism class contain at least one manifold structure, that is, a normal map which is actually a homotopy equivalence? This is where surgery proper enters the picture. Surgery gives a means of constructing normal bordisms, and conversely any normal bordism can be analyzed into a sequence of surgeries. The question is, therefore, whether starting with a 'generic' normal map produced by transversality, one can improve it by a sequence of surgeries until at last one obtains a homotopy equivalence. The key result is

2.58. Theorem (Fundamental Surgery Theorem)**.** *Let X be a Poincaré duality space with formal dimension $n \geqslant 5$ and fundamental group π. There is a* surgery obstruction group *$L_n(\pi)$, an abelian group depending only on n and π, and for each normal map $f\colon M \to X$*

there is a surgery obstruction $\sigma(f) \in L_n(\pi)$*, such that* f *is normal bordant to a homotopy equivalence if and only if* $\sigma(f) = 0$.

This will be proved in Chapter 15. Proposition 2.37 is the special case $n = 4k$, π trivial. As one can see in that example, the fundamental surgery theorem also applies to manifolds with boundary, provided that the boundary is 'left alone' during the surgery process. What this turns out to mean is that $\partial X \to X$ should induce an isomorphism on fundamental groups and that normal maps should already be homotopy equivalences on the boundary.

We can express the fundamental theorem formally as an 'exact sequence' which answers the existence question, relating the L-group, the structure set, and the *normal bordism set* $\mathcal{N}(X)$ of normal bordism classes of degree one normal maps to X. The sequence is

$$\mathcal{S}(X) \longrightarrow \mathcal{N}(X) \xrightarrow{\sigma} L_n(\pi)$$

When $X = S^n$ this corresponds to the right-hand half of the sequence 2.12: Θ_n/bP_{n+1} is the kernel of $\sigma\colon \mathcal{N}(S^n) \to L_n(e)$.

To extend the sequence to the left and so answer the uniqueness question as well, suppose that $f\colon M \to X$ and $f'\colon M' \to X$ are two manifold structures which have the same normal invariant. Then there is a normal bordism M between M and M', equipped with a normal map to $X \times [0,1]$. We want to know whether M can be made into an h-cobordism, and that is the same question as asking whether the normal map $M \to X \times [0,1]$ can be modified, leaving the boundary fixed, to make it into a homotopy equivalence. Once again we try to do this by means of surgery. The surgery obstruction apparently lies in $L_{n+1}(\pi)$, but now there is an ambiguity coming from the surgery obstructions of normal bordisms from f to itself, so the correct place to measure is in the cokernel of $\sigma_{n+1}\colon \mathcal{N}(X \times [0,1]; \partial) \to L_{n+1}(\pi)$. We see that there is a natural map which assigns to a pair (f, f') of structures having the same normal invariant an element of $\operatorname{Coker} \sigma_{n+1}$, which is zero if and only if the two structures are h-cobordant. This is the map $bP_{4k} \to \mathbb{Z}/(t_k/8)$ in our discussion of exotic spheres; the group $L_{4k}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, and Proposition 2.15 says that the image of σ_{4k} is generated by $t_k/8$.

As in the discussion of exotic spheres, the final piece of the puzzle is a geometric construction that says that any element of $L_{n+1}(\pi)$ can be *realized* as the surgery obstruction of a normal bordism. This construction, a generalization of Milnor's plumbing (Theorem 2.35), will be discussed in Section 16.1. It gives us an *action* of $L_{n+1}(\pi)$ on the structure set $\mathcal{S}(X)$: given a structure f and $x \in L_{n+1}(\pi)$, realize x as the surgery obstruction of a normal bordism one of whose ends is f, and let $x \cdot f$ be the other end of the normal bordism. In terms of this action we can extend our exact sequence to the left

$$\mathcal{N}(X \times [0,1]; \partial) \xrightarrow{\sigma} L_{n+1}(\pi) \dashrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \xrightarrow{\sigma} L_n(\pi)$$

The dotted arrow denotes the group action above, and exactness at $\mathcal{S}(X)$ means that the orbits of this action are precisely the inverse images of elements of $\mathcal{N}(X)$. This *surgery exact sequence* is a succinct formulation, due to Sullivan, of all the main results in the surgery classification of manifold structures.

2.59. Remark. To end this section, let's look at what we are going to have to do to set up the theory that has just been sketched. The first requirement is to define the L-groups. At least for $n = 4k$, the discussion of Section 2.5 gives a good hint as to how this is to be done: the L-groups will be stable isomorphism classes of quadratic forms, modulo hyperbolic ones. Remember though that the geometrical part of Section 2.5 depended heavily on

Whitney's theory of embeddings. In turn, this theory depends on a geometrical device, the *Whitney lemma* (Lemma 4.26) which allows one to 'cancel' superfluous intersection points of middle-dimensional submanifolds. The Whitney lemma requires that a certain circle in M can be spanned by a 2-disk; this is automatic for a simply-connected manifold, but in general there is an obstruction in the fundamental group. The upshot of this is that we have to study intersection theory, Poincaré duality , embeddings and immersions, and so on, not just in M but in the universal cover $\widetilde{M}$, equivariantly with respect to the fundamental group, and that the L-groups will involve quadratic forms, not on abelian groups, but on modules over the (noncommutative) group ring $\mathbb{Z}[\pi]$.

CHAPTER 3

Bundles and the Thom isomorphism

3.1. Orientations and the Thom isomorphism

In this section we will work with de Rham cohomology for smooth manifolds M. On such a manifold we have the familiar de Rham complex of differential forms

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

which computes the cohomology $H^*(M;\mathbb{R})$. If M is not compact, there is also the important subcomplex of *compactly supported* forms

$$\Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_c(M);$$

its cohomology is the *compactly supported cohomology* $H^*_c(M;\mathbb{R})$. We will need to work with both of these cohomology theories.

3.1. Remark. Any definition of cohomology (singular, cellular, or even a generalized cohomology theory) has a 'compactly supported' variant defined for locally compact spaces X: it is the direct limit $\varinjlim H^*(X, X\setminus K)$ taken over the direct system of compact subsets of K of X ordered by inclusion. It is functorial for *proper* maps (a map is proper if the inverse image of any compact set is compact). The most familiar example of a compactly supported generalized cohomology theory is Atiyah-Hirzebruch K-theory.

3.2. Exercise. Show that our definition of compactly supported de Rham cohomology is consistent with the general definition in terms of relative groups given in the remark above. (Relative de Rham theory can be defined in terms of mapping cone complexes; see [**7**, page 78].)

Any calculation of de Rham cohomology begins with the *Poincaré lemma*, which gives the cohomology of Euclidean space. The result is:

3.3. Lemma. *Let M be diffeomorphic to Euclidean space $\mathbb{R}^n$. Then*

(a) $H^k(M;\mathbb{R})$ *is isomorphic to* $\mathbb{R}$ *when* $k=0$, *to* 0 *otherwise; the generator is the cohomology class of the constant function 1;*
(b) $H^k_c(M;\mathbb{R})$ *is isomorphic to* $\mathbb{R}$ *when* $k=n$, *to* 0 *otherwise; the generator is the cohomology class of a 'bump n-form' ω, compactly supported and with* $\int_M \omega = 1$.

Notice in (b) that the operation $\int$, and therefore the normalization of the generator, depend on the choice of an orientation of M.

Having understood the de Rham cohomology of a single Euclidean space, the next thing to understand is a smoothly varying collection of such spaces — that is, a vector bundle.

3.4. Definition. Let V be an oriented ℓ-dimensional vector bundle over a compact manifold M. A *Thom form* for V is a closed ℓ-form α on the total space of V (considered as a non-compact manifold in its own right) such that

(a) α is compactly supported,
(b) α is closed, that is, $d\alpha = 0$, and
(c) for each fiber F of V (oriented by the orientation of V) we have $\int_F \alpha = 1$.

The cohomology class (in $H^\ell_c(V;\mathbb{R})$) of a Thom form is called a *Thom class* for V.

3.5. Exercise. Show that Thom forms exist. (Use a partition of unity to glue together local Thom forms.) FIX THIS — IT'S NOT THAT EASY.

3.6. Theorem. *(Thom Isomorphism Theorem) Let V be an oriented ℓ-dimensional vector bundle over a compact manifold M. All Thom forms α for V define the same Thom class in $H^\ell_c(V;\mathbb{R})$. If π denotes the projection of the vector bundle V, then the map $\beta \mapsto \pi^*\beta \wedge \alpha$ gives an isomorphism (the* Thom isomorphism*) $\Phi\colon H^*(X;\mathbb{R}) \to H^{*+\ell}_c(V;\mathbb{R})$.*

SKETCH PROOF. Leave to one side for now the question of the uniqueness of the Thom classes; just choose a particular Thom form. Cap-product with it defines Thom maps Φ not just for M itself but for any open subset U: we have

$$\Phi_U\colon \Omega^*_c(U) \to \Omega^{*+\ell}_c(\pi^{-1}(U))$$

on the level of differential forms, and

$$\Phi_U\colon H^*_c(U;\mathbb{R}) \to H^{*+\ell}_c(\pi^{-1}(U);\mathbb{R})$$

on the level of cohomology. If U is a small open ball in a coordinate chart (so that the restriction of V to U is a trivial bundle) then both U and $\pi^{-1}(U)$ are diffeomorphic to Euclidean spaces and Φ_U is an isomorphism on cohomology by Lemma 3.3. We now piece these isomorphisms together using a Mayer-Vietoris argument. Suppose that U_1 and U_2 are open subsets of M and that it is known that Φ_{U_1}, Φ_{U_2}, and $\Phi_{U_1\cap U_2}$ are all isomorphisms. There is a commutative diagram of complexes and chain maps with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega^*_c(U_1\cap U_2) \arrow[r, "\Phi"] \arrow[d] & \Omega^{\ell+*}_c(\pi^{-1}(U_1)\cap\pi^{-1}(U_2)) \arrow[d] \\
\Omega^*_c(U_1)\oplus\Omega^*_c(U_2) \arrow[r, "\Phi"] \arrow[d] & \Omega^{\ell+*}_c(\pi^{-1}(U_1))\oplus\Omega^{\ell+*}_c(\pi^{-1}(U_2)) \arrow[d] \\
\Omega^*_c(U_1\cup U_2) \arrow[r, "\Phi"] \arrow[d] & \Omega^{\ell+*}_c(\pi^{-1}(U_1)\cup\pi^{-1}(U_2)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

which gives rise to a commutative diagram of Mayer-Vietoris sequences in compactly supported cohomology. (See **[7]** for details of the construction of these Mayer-Vietoris sequences.) By our supposition, two out of the three horizontal maps give rise to cohomology isomorphisms; so the five lemma tells us that the third one, $\Phi_{U_1\cup U_2}$, will do so also. The proof that the Thom map for the compact manifold M is an isomorphism is now completed by an induction on the number of sets in a 'good' open covering[1].

Finally, let us return to the question of the uniqueness of Thom classes. We may assume that M is connected. Then, by the isomorphism result that we have just proved, $H^\ell_c(V;\mathbb{R}) \cong H^0(M;\mathbb{R})$ is one-dimensional. Thus, any two Thom classes are scalar multiples of one another. The normalization condition (c) in the definition of a Thom class ensures that the multiple is 1. $\square$

[1]That is, a covering by open sets such that every intersection of these sets is either empty or diffeomorphic to $\mathbb{R}^n$.

3.7. Remark. In particular note that $H^q_c(V;\mathbb{R}) = 0$ for $q < \ell$. The Mayer-Vietoris type argument used in this proof will recur frequently. We refer to it as as an *assembly* construction; it 'assembles' the local Thom isomorphisms given by the Poincaré lemma into a global isomorphism.

3.8. Remark. The inverse of the Thom isomorphism can be described in de Rham theory as the operation of *integration along the fiber*. This process defines a 'wrong way' map on the complexes of differential forms, $\pi_*\colon \Omega^*_c(V) \to \Omega^{*-\ell}(M)$; one uses the local product structure to express a form as a product of terms coming from the root and from the fiber, and then integrates out the top-dimensional fiber component using the orientation. See [**7**, pages 61–63] for details. We will need to make use of the 'Fubini principle for integration along the fiber'

$$\int_V \pi^*\theta \wedge \varphi = \int_M \theta \wedge \pi_*\varphi,$$

where $\theta \in \Omega^*(M)$, $\varphi \in \Omega^*_c(V)$. This is proved by using a partition of unity to work in product neighborhoods.

Imagine now that the closed manifold M^m is embedded as a submanifold of the closed manifold W^n, and that both W and M are oriented. Then the normal bundle V of M in W is oriented also, and so it possesses a Thom class $[\alpha] \in H^{n-m}_c(V;\mathbb{R})$. Now, by the tubular neighborhood theorem (see Appendix ?), the total space of V may be identified with an open subset of W, and so there is a map on cohomology $H^*_c(V;\mathbb{R}) \to H^*(W;\mathbb{R})$ — in terms of differential forms, this is the operation of 'extension by zero' of a compactly supported form. Applying this map to $[\alpha]$ we see that each oriented submanifold M of W gives rise to a cohomology class

$$[\alpha_M] \in H^{n-m}(W;\mathbb{R}).$$

3.9. Definition. The cohomology class α_M defined above is called the *dual cohomology class* to M.

3.10. Proposition. *With M and W as above, the dual cohomology class $[\alpha_M]$ has the following property: for every closed form $\beta \in \Omega^m(W)$, we have*

$$\int_M \beta = \int_W \beta \wedge \alpha_M = \langle [\beta] \smile [\alpha_M], [W] \rangle.$$

Thus, in terms of the Poincaré duality map D of Remark 1.3, $D[\alpha_M]$ is the homology class $[M] \in H_m(W;\mathbb{R})$.

PROOF. Denote by $i\colon M \to V$ the inclusion of the zero-section. Then

$$\int_W \beta \wedge \alpha_M = \int_V \beta \wedge \alpha_M = \int_V \pi^* i^* \beta \wedge \alpha_M,$$

the first equality by restriction and the second because i and π are mutually inverse homotopy equivalences between M and V. By the Fubini principle for integration along the fiber (Remark 3.8),

$$\int_V \pi^* i^* \beta \wedge \alpha_M = \int_M i^*\beta \wedge \pi_*(\alpha_M) = \int_M \beta,$$

since $\pi_*(\alpha_M) = 1$ by definition of the Thom form. $\square$

Suppose now that M_1 and M_2 are two submanifolds of W^n, of dimensions m_1 and m_2 respectively, $m_1 + m_2 \geqslant n$. One says that M_1 and M_2 *intersect transversely* if, near any point of their intersection, there is a coordinate chart in which M_1 is represented by the subspace spanned by the first m_1 basis vectors of $\mathbb{R}^n$ and M_2 is represented by the subspace spanned by the last m_2 basis vectors.

This is a special case of the general notions of transversality that we will investigate in Chapter 4. There we will see that given any two submanifolds it is possible to deform one of them slightly so as to make their intersection transverse.

In particular, the intersection $M_1 \cap M_2$ is a submanifold of dimension $m_1 + m_2 - n$. Moreover, the normal bundles are related by

$$\nu_{M_1 \cap M_2} = \nu_{M_1} \oplus \nu_{M_2}.$$

Since the Thom class of a direct sum of vector bundles is easily seen to be the product of the Thom classes of the summands, we have

3.11. Proposition. *If M_1 and M_2 intersect transversely, then the dual cohomology classes are related by*

$$\alpha_{M_1 \cap M_2} = \alpha_{M_1} \wedge \alpha_{M_2}.$$

3.12. Example. In particular suppose that M_1 and M_2 intersect transversely and have complementary dimensions, $m_1 + m_2 = n$. The intersection $M_1 \cap M_2$ is then just a finite set of points p, each of which acquires a sign $\varepsilon(p) \in \{\pm 1\}$ according to whether or not the orientations of M_1 and M_2 at that point combine to yield the orientation of W. The signed count $\sum_{p \in M_1 \cap M_2} \varepsilon(p)$ of these points is called the *intersection number* $\lambda(M_1, M_2)$ of the two submanifolds. Plainly, this is just the integral over W of the dual class to the oriented 0-dimensional manifold $M_1 \cap M_2$. Thus, from Proposition 3.11 we obtain

$$\lambda(M_1, M_2) = \int_M \alpha_{M_1} \wedge \alpha_{M_2} \tag{3.13}$$

Thus the intersection numbers of submanifolds are given by the cohomological intersection form applied to their dual cohomology classes.

Notice the important symmetry property

$$\lambda(M_2, M_1) = (-1)^{m_1 m_2} \lambda(M_1, M_2) \tag{3.14}$$

which may be derived either by considering the orientation of the intersection points, or from the graded commutativity of the wedge product.

3.15. Remark. When M_1 and M_2 are *not* transverse, we may use the homological formula as the *definition* of the intersection number; this will then be equal to the 'geometric' intersection number of M_1' and M_2, where M_1' is a small deformation of M_1 in the same homology class, and is transverse to M_2.

Although we have worked in this section with de Rham cohomology, and therefore with *real* coefficients, the Thom isomorphism actually holds good with coefficients in the *integers*, and also for vector bundles over an *arbitrary* compact base (not just a manifold). We shall prove this in a more general context in Section 3.4 below.

3.16. Remark. It is traditional, though not strictly necessary, to express the Thom isomorphism for a vector bundle V over a compact base X in terms of the *Thom space* of V. To define it, first notice that by giving V a Euclidean metric we can define the *disk bundle* $D(V)$ and *sphere bundle* $S(V)$ of V; these are the spaces of vectors of length $\leqslant 1$ and length exactly 1, respectively. Up to homeomorphism they are independent of the choice of Euclidean structure, and $D(V) \setminus S(V)$ is naturally identified with V itself. The *Thom space* $\mathrm{Th}(V)$ is the identification space $D(V)/S(V)$; we denote the point in the Thom space corresponding to $S(V)$ by ∞. Then we have

$$H_c^*(V) \cong H^*(D(V), S(V)) \cong H^*(\mathrm{Th}(V), \infty) = \tilde{H}^*(V),$$

so the Thom isomorphism may be expressed in terms of the (reduced) cohomology of the Thom space.

3.17. Exercise. Let V be an oriented q-dimensional vector bundle over S^M, classified by an element $\alpha \in \pi_m(BSO(q)) = \pi_{m-1}(SO(q))$. Show that the Thom space of V has the structure of a CW-complex with a single cell in dimensions 0, q, and $m+q$; in fact

$$\mathrm{Th}(V) \cong S^q \cup_{J(\alpha)} D^{m+q},$$

where $J\colon \pi_{m-1}(SO(q)) \to \pi_{m+q-1}(S^q)$ is the usual J-homomorphism.. See [**21**].

3.2. The Euler class

Let V be an oriented ℓ-dimensional vector bundle over a compact base X. The zero-section of V defines a proper map $i\colon X \to V$, which induces $i^*\colon H_c^*(V;\mathbb{R}) \to H^*(X;\mathbb{R})$ on (compactly supported) cohomology.

3.18. Definition. Let V be as above. The image $i^*(\alpha_V)$ of the Thom class of V under the map induced by the zero-section is called the *Euler class* of V, $e(V) \in H^\ell(X)$.

Because the Thom isomorphism is natural, the Euler class is a *characteristic class* for oriented ℓ-dimensional bundles, i.e. an element of $H^\ell(BSO(\ell))$.

3.19. Proposition. *If V admits a nowhere vanishing section, then $e(V) = 0$.*

PROOF. Let $S(V)$ denote the $(\ell-1)$-sphere bundle associated to V, and $D(V)$ the corresponding ℓ-disk bundle, so that $V \cong D(V) \setminus S(V)$. If V admits a nowhere vanishing section, then the zero-section of $D(V)$ is homotopic to a map $X \to S(V)$. By the exact sequence of the pair $(D(V), S(V))$, the zero-section therefore induces the 0 map on $H^*(D(V), S(V)) = H_c^*(V)$. $\square$

Thus the Euler class is an *unstable* characteristic class — it vanishes when we add a trivial summand. (Contrast this with the behavior of the Pontrjagin and Stiefel-Whitney classes.)

We shall be particularly interested in the case where the base is an oriented manifold and the fiber dimension of V is equal to the dimension of the manifold. In this situation we can make the preceding proposition more precise.

3.20. Proposition. *Let V be an oriented m-dimensional vector bundle over a closed m-dimensional manifold M. Then the* Euler number $\langle e(V), [X]\rangle$ *is equal to the signed count of the number of zeroes of a generic section of V. It is also equal to the self-intersection number $\lambda([X],[X])$ of the zero-section of V, considered as a submanifold of the oriented $2m$-manifold $D(V)$ with boundary $S(V)$.*

Locally, a section of V is the graph of a function $f\colon \mathbb{R}^m \to \mathbb{R}^m$, and the genericity condition is that $Df(p)$ should be invertible for all points p such that $f(p) = 0$. The sign that we associate to the point p is then the sign of the determinant $Df(p)$.

PROOF. By definition of the Euler class and the Fubini principle of Remark 3.8, we have

$$\langle e(V), [M]\rangle = \langle \alpha_V \smile \alpha_V, [V]\rangle.$$

This is the self-intersection of the zero section M of V, by Equation 3.13. But the geometric definition of this self-intersection requires that we perturb the zero section within its isotopy class, to be transverse to M, and then count the intersections of this perturbation M' with M. We can take M' to be the graph of a generic section of V, and then the intersections of M' with M are precisely the zeroes of that section. $\square$

The proposition still holds for manifolds M with boundary, so long as we consider sections of V that do not vanish on the boundary.

An important example of the above situation occurs when V is the tangent bundle to the oriented manifold M. Here we have

3.21. Proposition. *The Euler number $\langle e(TM), [M]\rangle$ of the tangent bundle to a closed oriented manifold M is equal to the Euler characteristic*

$$\chi(M) = \sum (-1)^i \dim H^i(M;\mathbb{R}).$$

OUTLINE PROOF. Consider the diagonal embedding of M in $M \times M$. Show that the normal bundle to this embedding is just the tangent bundle to M. Work out the cohomology class dual to the diagonal in terms of the decomposition $H^*(M \times M; \mathbb{R}) = H^*(M; \mathbb{R}) \otimes H^*(M; \mathbb{R})$ given by the Künneth theorem. Restrict to the diagonal and evaluate on $[M]$ to get the result. For details, see [**26**]. $\square$

3.3. Framings and stable framings

Recall (Definition 2.17) that a *framing* for an ℓ-dimensional vector bundle V is an isomorphism from V to the standard trivial bundle ε^ℓ. A *stable framing* is a framing of $V \oplus \varepsilon^m$ for some m. We say V is *stably trivial* if it admits a stable framing.

3.22. Proposition. *Let V be an ℓ-dimensional oriented vector bundle over a closed oriented ℓ-manifold M. If V admits a stable framing, then the Euler number $\langle e(V), [M]\rangle$ is even.*

PROOF. We need to know that the Euler class is in fact an *integral* cohomology class.

From the proof of Proposition 3.20, the Euler number equals the self-intersection $\langle \alpha_V \smile \alpha_V, [V]\rangle$, and its mod 2 reduction can therefore be written in terms of Steenrod squares as $\langle \mathrm{Sq}^\ell \alpha_V, [V]\rangle$.

Let $V' = V \oplus \varepsilon^m$. Because of the naturality of the Steenrod squares, the Euler number mod 2 is also equal to $\langle \mathrm{Sq}^\ell \alpha_{V'}, [V']\rangle$. But since V' is trivial for sufficiently large m, this expression is zero (use the naturality of the Steenrod squares again, and the Künneth decomposition of $H^*_c(V') = H^*(M) \otimes H^*_c(\mathbb{R}^{\ell+m})$.) $\square$

It amounts to the same thing to say that the mod 2 reduction of the Euler class of V is the ℓth Stiefel-Whitney class [**26**, Theorem ?], which vanishes for stably trivial bundles.

3.23. Remark. This simple argument leads us into one of the most delicate parts of surgery theory: correctly accounting for the *self*-intersections of middle dimensional submanifolds. It is telling us that the simple 'symmetric' self-intersection of such a manifold (in this case the zero-section of V), given by the intersection form λ, should be realized as *twice* some more refined 'quadratic' self-intersection invariant. We are going to need to work over integral group rings (Remark 2.59) where multiplication by 2 is not necessarily an injective map, and in such circumstances the quadratic invariant will contain strictly more information than the symmetric one. We will develop the general theory of quadratic self-intersections when we study immersions, in Chapter 10.

We want to investigate when an m-dimensional, stably framed vector bundle over S^m has a compatible (unstable) framing. In particular we want to prove Proposition 2.29, which was important in our study of the exotic spheres.

3.24. Exercise. Recall that the group $\pi_{m-1}(SO(k)) = \pi_m(BSO(k))$ classifies k-dimensional oriented vector bundles over the m-sphere. Show that the natural homomorphism

$$\pi_{m-1}(SO(m)) \to \pi_{m-1}(SO(m), SO(m-1)) \cong \pi_{m-1}(S^{m-1}) \cong \mathbb{Z}$$

sends an oriented m-dimensional vector bundle over S^m to its Euler number. (Hint: Write $S^m = D^m \cup_{S^{m-1}} D^m$. Think of a map $S^{m-1} \to SO(m)$ as the clutching function joining two trivial bundles on the disks to make a non-trivial bundle on the sphere. Try to construct a section which is equal to the vector $(1, 0, 0, \ldots)$ on one of the disks. Use degree theory to count its zeroes.)

This exercise reconciles the definition of the Euler number in Proposition 3.20 with that given in Definition 2.26.

3.25. Exercise. Show that the boundary map in the homotopy exact sequence associated to the fibration $SO(m) \to SO(m+1) \to S^m$, namely

$$\mathbb{Z} \cong \pi_m(S^m) \cong \pi_m(SO(m+1), SO(m)) \to \pi_{m-1}(SO(m)),$$

sends the generator $1 \in \mathbb{Z}$ (corresponding to the identity map $S^m \to S^m$) to the class of the tangent bundle TS^m in $\pi_{m-1}(SO(m))$.

Let V be an oriented m-dimensional vector bundle over S^m with a stable framing $\mathfrak{f}$. Thus $(V, \mathfrak{f})$ defines an element of $\pi_{m-1}(SO(m))$ together with a nullhomotopy of its image in $\pi_{m-1}(SO(m+k))$, for some large k. As in the proof of Proposition 2.28, we see we can destabilize the given stable framing if we can fill in the dotted arrow in

$$\begin{tikzcd} S^{m-1} \arrow[r] \arrow[d] & SO(m) \arrow[d] \\ D^m \arrow[r] \arrow[ur, dotted] & SO(m+k); \end{tikzcd}$$

that is, if a certain element of the relative homotopy group $\pi_m(SO(m+k), SO(m))$ vanishes. These groups stabilize for $k \geqslant 2$, so it is enough to take $k = 2$.

Introduce the following notation[2]: for $\varepsilon \in \{\pm 1\}$, let $Q_\varepsilon(\mathbb{Z})$ denote the quotient $\mathbb{Z}/(1-\varepsilon)\mathbb{Z}$, that is either $\mathbb{Z}$ or $\mathbb{Z}/2$ according to the sign of ε.

3.26. Proposition. *The relative homotopy group $\pi_m(SO(m+2), SO(m))$ is isomorphic to $Q_{(-1)^m}(\mathbb{Z})$.*

PROOF. Because we are dealing with Lie groups, there is a fibration $SO(m) \to SO(m+2) \to SO(m+2)/SO(m)$, and the relative homotopy group appearing in the proposition is just the group $\pi_m(SO(m+2)/SO(m))$. Now one can identify $SO(m+2)/SO(m)$ with $S(T(S^{m+1}))$, the sphere bundle of the tangent bundle to S^{m+1}. (To see this, notice that $SO(m+2)/SO(m)$ is the space of orthogonal 2-frames in $\mathbb{R}^{m+2}$; the first vector of such a 2-frame gives a point in S^{m+1}, the second gives a tangent vector to S^{m+1} at that point.) Thus there is a fibration[3]

$$S^m \to SO(m+2)/SO(m) \to S^{m+1}.$$

The associated long exact sequence gives

$$\begin{tikzcd} \pi_{m+1}(S^{m+1}) \arrow[r, "\chi"] \arrow[d] & \pi_m(S^m) \arrow[r] & \pi_m(SO(m+2), SO(m)) \arrow[r] & \pi_m(S^{m+1}) \\ \pi_m(SO(m+1)) \arrow[ur] & & & \end{tikzcd}$$

Exercises 3.24 and 3.25, applied to the triangle in the diagram, show that the map denoted by χ is multiplication by the Euler number of TS^{m+1}, which is 2 if m is odd and 0 if m is even. From the exact sequence it follows that $\pi_m(SO(m+2), SO(m))$ is isomorphic to $Q_{(-1)^m}(\mathbb{Z})$. $\square$

We interpret this proposition in the following way: a stably framed m-dimensional oriented vector bundle $(V, \mathfrak{f})$ over S^m gives rise a class $\mathfrak{d}(V, \mathfrak{f}) \in Q_{(-1)^m}(\mathbb{Z})$, such that V admits a genuine framing compatible with its given stable framing if and only if the destabilization obstruction vanishes.

3.27. Definition. The class $\mathfrak{d}(V, \mathfrak{f})$ defined above is called the *destabilization obstruction* of the stably framed vector bundle $(V, \mathfrak{f})$.

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[2]A special case of the general *quadratic groups* that will be defined in Chapter 8.

[3]In terms of Lie groups this is just

$$SO(m+1)/SO(m) \to SO(m+2)/SO(m) \to SO(m+2)/SO(m+1).$$

If m is even we can identify the destabilization obstruction in terms of the Euler number.

3.28. Proposition. *For m even, the destabilization obstruction $\mathfrak{d}(V,\mathfrak{f})$ in $Q_+(\mathbb{Z}) = \mathbb{Z}$ of a stably framed oriented m-dimensional vector bundle V over S^m is half its Euler number (which is even by Proposition 3.22).*

PROOF. The proof of Proposition 3.26 above shows that, when m is even, the natural map

$$\mathbb{Z} \cong \pi_m(S^m) \to \pi_m(SO(m+2), SO(m)) = Q_{(-1)^m}(\mathbb{Z})$$

is an isomorphism. Exercise 3.25 now shows that the image of the generator, in the group $\pi_{m-1}(SO(m))$ of oriented m-dimensional vector bundles over S^m, is the tangent bundle TS^m, which has Euler number 2. $\square$

PROOF OF PROPOSITION 2.29. This follows from Proposition 3.22, Proposition 3.28, and the group structure of $\pi_m(SO(m+2), SO(m))$. $\square$

Notice that for m even the destabilization obstruction is independent of the choice of stable framing $\mathfrak{f}$ — it depends only on the bundle V. This is definitely *not* the case for m odd.

3.29. Example. Consider the simplest odd-dimensional example, $m = 1$. Let V be an oriented (therefore trivial) line bundle over the circle S^1, and suppose that $V \oplus \varepsilon^2$ is provided with a framing $\mathfrak{f}$. Then the 'difference' between $\mathfrak{f}$ and the framing coming from a trivialization of V gives a loop $S^1 \to SO(3)$. The destabilization obstruction $\mathfrak{d}(V,\mathfrak{f})$ is zero if this loop lifts to a loop in $\mathrm{Spin}(3)$, and it is one otherwise. (This follows easily from the exact sequence appearing in the proof of Proposition 3.26.)

Note in particular that if we take V to be the tangent bundle to S^1, and we give it the stable framing $\mathfrak{f}$ coming from the standard embedding $S^1 \to \mathbb{R}^2$, then $\mathfrak{d}(V,\mathfrak{f}) = 1$. However, since V is itself a trivial bundle, it can also be given a stable framing $\mathfrak{f}'$ for which $\mathfrak{d}(V,\mathfrak{f}') = 0$.

3.30. Exercise. Show that if $V = TS^m$ is given the stable framing $\mathfrak{f}$ arising from embedding $S^m \to \mathbb{R}^{m+1}$, then $\mathfrak{d}(V,\mathfrak{f}) = 1$ for all m.

We shall later need to work out the relationship between destabilization obstructions and self-intersections in algebraic detail. When we do this, a useful intermediate step will be supplied by a generalization of the Hopf invariant (Remark 1.10) due to Boardman and Steer [**?**].

Consider the diagonal map $S^1 \to S^1 \wedge S^1 = S^2$. This map is nullhomotopic. (For future reference, let us fix once and for all an explicit nullhomotopy by identifying S^1 with the unit interval modulo its boundary; then S^2 is identified with the unit square modulo its boundary and the diagonal map $t \mapsto (t,t)$ is nullhomotopic via the maps $t \mapsto (st,t)$, $0 \leqslant s \leqslant 1$.) Since the suspension of a (pointed) space X is just the smash product $S^1 \wedge X$, we see that the diagonal map of any suspension,

$$\Sigma X \to \Sigma X \wedge \Sigma X,$$

is canonically nullhomotopic.

Suppose now that $f\colon \Sigma X \to \Sigma Y$ is a map between suspensions. Arising from this map one obtains a commutative diagram

$$\begin{tikzcd}
\Sigma X \arrow[r] \arrow[d] & \Sigma Y \arrow[d] \\
\Sigma X \wedge \Sigma X \arrow[r] & \Sigma Y \wedge \Sigma Y
\end{tikzcd}$$

where the horizontal arrows are induced by f and the vertical arrows are diagonal maps. The composite map

$$\Sigma X \to \Sigma Y \wedge \Sigma Y$$

is then nullhomotopic in two different ways, one arising from the nullhomotopy of the diagonal map of ΣX and one arising from the nullhomotopy of the diagonal map of ΣY. In other words, there are two extensions of the displayed map to the cone on ΣX. Gluing together these two extensions, we obtain a 'differnce' element

$$H(f)\colon \Sigma^2 X \to \Sigma Y \wedge \Sigma Y.$$

3.31. Definition. The homotopy class of the map $H(f)$ so defined is the *Boardman-Steer Hopf invariant* of the original map f.

Notice that the homotopy class of $H(f)$ depends only on the homotopy class of f. Moreover, if f is itself the suspension of a map $X \to Y$, then the two nullhomotopies of $\Sigma X \to \Sigma Y \wedge \Sigma Y$ will agree and therefore their difference $H(f)$ will be trivial. Thus $H(f)$ is an obstruction to 'destabilizing' or 'desuspending' the map f.

3.32. Example. Let $f\colon S^3 = \Sigma S^2 \to S^2 = \Sigma S^1$ be a map. The Boardman-Steer invariant is a map from $\Sigma^2 S^2 = S^4$ to $\Sigma S^1 \wedge \Sigma S^1 = S^4$. Thus $H(f) \in \pi_4(S^4) = \mathbb{Z}$. In fact, the Boardman-Steer invariant is simply the Hopf invariant in this case.

Suppose now that V is a m-dimensional vector bundle over S^m, together with a framing $\mathfrak{f}$ of $V \oplus \varepsilon$. The pair $(V, \mathfrak{f})$ then defines an element of $\pi_m(SO(m+1), SO(m)) = \mathbb{Z}$. We shall show that this integer is detected by the Boardman-Steer Hopf invariant.

To do this notice that an isomorphism of vector bundles (over any base) gives rise to a map — in fact a homeomorphism — between their Thom spaces. Thus the stable framing $\mathfrak{f}$ gives rise to a map from the Thom space of ε^{m+1} to the Thom space of $V \oplus \varepsilon$. If we make use of the simple fact that the Thom space of $V \oplus \varepsilon$ is the suspension of the Thom space of V, this gives us a map

$$S^{m+1} \vee S^{2m+1} = \Sigma(S^m \vee S^{2m}) \to \Sigma \operatorname{Th} V.$$

Applying the Boardman-Steer construction gives a map

$$\Sigma^2(S^m \vee S^{2m}) \to \Sigma \operatorname{Th} V \wedge \Sigma \operatorname{Th} V.$$

Now $\Sigma \operatorname{Th} V$ is a CW complex with cells in dimensions 0, $m+1$, and $2m+1$ (see Exercise 3.17), so the right hand side of the above display is obtained by attaching higher-dimensional cells to S^{2m+2}. It follows that maps $\Sigma^2(S^m \vee S^{2m}) \to \Sigma \operatorname{Th} V \wedge \Sigma \operatorname{Th} V$ are classified up to homotopy by a single degree in $\pi_{2m+2}(S^{2m+2}) = \mathbb{Z}$. Let us denote the Boardman-Steer Hopf invariant obtained from $(V, \mathfrak{f})$ in this way by $H(V, \mathfrak{f})$.

3.33. Proposition. *The invariant $H(V, \mathfrak{f})$ is equal to the class defined by $(V, \mathfrak{f})$ in the group $\pi_m(SO(m+1), SO(m)) = \mathbb{Z}$.*

PROOF. It is convenient to consider the situation a little differently. Let U be the bundle $V \oplus \varepsilon \cong \varepsilon^{m+1}$. Then U is an oriented $(m+1)$-dimensional bundle over S^m provided with two distinct nonvanishing sections s_1, s_2: one being one of the coordinate sections of ε^{m+1} and the other coming from the trivial summand in $V \oplus \varepsilon$. Our problem is to detect the degree of 'mismatch' between these two sections.

If we make s_1, s_2 of unit length then they give us m-dimensional submanifolds of the sphere bundle of U, which is itself a $2m$-dimensional manifold. Generically therefore they intersect in a finite number of points, say k, and it is not hard to check that k is the element in $\pi_m(SO(m+1), SO(m))$ that describes the bundle V together with its stable trivialization. (As an aside, let us define a section of V by first taking the coordinate section s_1 of U and then projecting this into V using the direct sum $V \oplus \varepsilon \cong U$. We get a section of V which vanishes when $s_1 = s_2$ and when $s_1 = -s_2$. The number of such points (appropriately counted) is $2k$ and thus we find again that the Euler number of a stably trivial bundle over a sphere is even.)

Now let us calculate the Boardman-Steer Hopf invariant. Suppose that U is a bundle provided with a section S. Notice that $\mathrm{Th}\, U \wedge \mathrm{Th}\, U = \mathrm{Th}(U \times U)$, where $U \times U$ is the external direct sum of two copies of U (a bundle over $S^m \times S^m$). We identify the Thom space of anyy vector bundle with the total space modulo the vectors of norm $\geqslant 1$. Then a nowhere vanishing section s of U defines a null-homotopy of the diagonal map of Thom spaces via

$$(u, t) \mapsto (u + ts, u), \quad 0 \leqslant t \leqslant 2.$$

Attaching together the two null-homotopies arising form the sections s_1 and s_2 gives the Boardman-Steer Hopf invariant.

$$(u, t) \mapsto \begin{cases} (u + ts_1, u) & (t \geqslant 0) \\ (u - ts_2, u) & (t \leqslant 0) \end{cases}$$

We shall calculate the degree of this map using transversality theory. (see the next chapter.)

□

3.34. Exercise. Let V be an m-dimensional oriented vector bundle over S^m.

(i) Observe that $D(V)$ is homotopy equivalent to S^m. Prove that $D(V)$ is also homotopy equivalent to the space $S(V) \cup D^m \cup D^{2m}$ obtained from $S(V)$ by first attaching an m-cell along the map $S^{m-1} \to S(V)$ and then attaching a $2m$-cell along a map $S^{2m-1} \to (S(V) \cup D^m)$. Show that the homotopy class of the composite $S^{2m-1} \to (S(V) \cup D^m) \to S^m$ is the image of the element of $\pi_{m-1}(SO(m))$ that classifies V under the J-homomorphism $\pi_{m-1}(SO(m)) \to \pi_{2m-1}(S^m)$. (Compare Exercise 3.17).

(ii) Prove that $J(TS^m) = [\iota, \iota] \in \pi_{2m-1}(S^m)$, the Whitehead product of the generator $\iota \in \pi_m(S^m)$ with itself.

(iii) Prove that for any a stable framing $\mathfrak{f} : V \oplus \varepsilon^n \cong \varepsilon^{m+n}$ with $n > 1$ there exists a framing $\hat{\mathfrak{f}} : V \oplus \varepsilon \cong \varepsilon^{m+1}$ such that $\mathfrak{f}$ is equivalent to $\hat{\mathfrak{f}} \oplus 1_{\varepsilon^{n-1}}$. Use the EHP exact sequence (Example **??** below) to prove that for any such $\hat{\mathfrak{f}}$

$$J(V) \;=\; \hat{\mathfrak{d}}(V, \hat{\mathfrak{f}})[\iota, \iota] \in \ker(E : \pi_{2m-1}(S^m) \to \pi_{2m}(S^{m+1})) \;=\; \mathrm{im}(\mathbb{Z} \to \pi_{2m-1}(S^m))$$

for some integer $\hat{\mathfrak{d}}(V, \hat{\mathfrak{f}}) \in \mathbb{Z}$ with image the destabilization obstruction $\mathfrak{d}(V, \mathfrak{f}) \in Q_{(-1)^m}(\mathbb{Z})$.

3.4. Spherical fibrations

In this section we shall define the notion of a *spherical fibration* — a purely homotopy-theoretic counterpart to the definition of a vector bundle. We shall see that the Thom isomorphism theorem is still true for these much more general objects. Spherical fibrations give us a systematic way to keep track of the 'bundle data' that is important in surgery theory; we shall see in Chapter 11 that though a homotopy equivalence between manifolds may change the stable tangent (vector) bundle[4], it must preserve the underlying stable spherical fibration.

3.35. Definition. A *spherical fibration* is a Serre fibration whose fiber is homotopy equivalent to a sphere.

Recall that a
Serre fibration is a map that has the unrestricted homotopy lifting property: that is, $p\colon E \to B$ is a Serre fibration if any commutative diagram of the form

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

can be completed by filling in the dotted arrow. The 'fiber' of such a fibration is the inverse image of a point in B; its homotopy type is well defined (provided that B is path connected). There is a standard procedure ("Serre's construction") in homotopy theory whereby any map $f\colon X \to B$ can be 'made into' a Serre fibration, that is, X can be replaced by a space E with a canonical homotopy equivalence $X \to E$ and a Serre fibration $p\colon E \to B$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

commutes. (We define $E = \{(x, \varphi) \in X \times \mathrm{Maps}([0,1], B) : f(x) = \varphi(0)\}$.) The fiber of the new Serre fibration p is referred to as the *homotopy fiber* of f, and we will allow ourselves to speak of f as a spherical fibration if its homotopy fiber is a sphere, i.e. if p is a spherical fibration.

3.36. Example. Suppose $V \to B$ is a vector bundle of fiber dimension n. Then the sphere bundle $S(V)$ of V is an $(n-1)$-spherical fibration.

By analogy with this example, we shall usually denote a spherical fibration by a Greek letter such as ξ, and let $S(\xi)$ stand for its total space (the fibration is thus a map $\xi\colon S(\xi) \to B$).

The natural notion of equivalence for spherical fibrations (corresponding to isomorphism for vector bundles) is *fiber homotopy equivalence*:

3.37. Definition. Two spherical fibrations ξ and ξ' over the same base B are *fiber homotopy equivalent* if there is a commutative diagram

$$\begin{tikzcd} S(\xi) \arrow[r, "\simeq"] \arrow[d, "\xi"] & S(\xi') \arrow[d, "\xi'"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

in which the top row is a homotopy equivalence.

[4]We have seen an example of this in Section 1.6.

Many familiar operations on vector bundles have counterparts in the world of spherical fibrations. For example, corresponding to the addition of a trivial bundle to a vector bundle, there is an operation of 'fiberwise suspension' of a spherical fibration (replace $S(\xi)$ by $B \cup_p [0,1] \times S(\xi) \cup_p B$, with the obvious map to B) which replaces an n-spherical fibration by an $(n+1)$-spherical fibration. This means that it makes sense to speak of 'stable spherical fibrations', just as it does to talk of 'stable vector bundles'.

3.38. Lemma. *Any spherical fibration can be embedded (uniquely up to homotopy) as a subfibration of a fibration with contractible fiber.*

PROOF. $\square$

This contractible fibration is of course the counterpart of the disk bundle associated to a vector bundle; to maintain the analogy, we shall therefore denote it by $D(\xi)$.

We can now see that all the usual operations on vector bundles now have counterparts for spherical fibrations. Thus one can define pullbacks, external products, Whitney sum, and so on of spherical fibrations. For example, to define the Whitney sum of two spherical fibrations ξ_1 and ξ_2 of fiber dimensions $k_1 - 1$ and $k_2 - 1$ over B, one first forms the 'disk' fibrations $D(\xi_1)$ and $D(\xi_2)$. Their product is a fibration over $B \times B$, which can be restricted to the diagonal (i.e. pulled back over the diagonal map $B \to B \times B$) to yield a fibration over B. Then we define the Whitney sum of ξ_1 and ξ_2 to be the $(k_1 + k_2 - 1)$-spherical fibration over B with total space

$$S(\xi_1) \times D(\xi_2) \cup D(\xi_1) \times S(\xi_2) \subseteq D(\xi_1) \times D(\xi_2).$$

3.39. Exercise. Verify that this operation corresponds to the Whitney sum of vector bundles.

Similarly we can define the Thom space (compare Remark 3.16).

3.40. Definition. Let ξ be an $(n-1)$-spherical fibration over B. The *Thom space* $T(\xi)$ is the mapping cone of the projection $p\colon S(\xi) \to B$; in other words, it is $S(\xi) \times [0,1] \sqcup B$ modulo the equivalence relation which identifies all points $(x, 0)$, $x \in S(\xi)$, with each other and each point $(x, 1)$, $x \in S(\xi)$, with $p(x) \in B$.

It is easy to verify that the Thom space (in the sense of this definition) of the spherical fibration underlying a vector bundle is homotopy equivalent to the Thom space (in the old sense) of the vector bundle itself. Notice that the pair $(T(\xi), \infty)$ (where ∞ denotes the cone point) is equivalent under excision to $(D(\xi), S(\xi))$.

Since spherical fibrations are homotopically similar to vector bundles it is perhaps not surprising that the Thom isomorphism theorem can be proved for them as for vector bundles. The proof can be thought of as another Mayer-Vietoris argument; but this time we shall take advantage of the powerful machinery of spectral sequences to present it in a condensed form.

3.41. Definition. An $(n-1)$-spherical fibration ξ (with connected base B) is *orientable* if the action of $\pi_1(B)$ on $\pi_{n-1}(\text{fiber}) = \mathbb{Z}$ is trivial. In this case an *orientation* is a choice of generator for $\pi_{n-1}(\text{fiber}) = \mathbb{Z}$.

3.42. Proposition. *Let $\xi\colon S(\xi) \to B$ be an oriented $(n-1)$-spherical fibration. Then there is defined a* Thom class $\alpha \in H^n(T(\xi), \infty; \mathbb{Z})$ *such that cup and cap products with α define isomorphisms*

$$H_{n+k}(T(\xi), \infty) \to H_k(B), \qquad H^k(B) \to H^{n+k}(T(\xi), \infty).$$

These isomorphisms hold with arbitrary coefficients.

PROOF. Since the associated 'disk bundle' $D(\xi)$ has contractible fiber, the projection $S(\xi) \to B$ is homotopy equivalent to the inclusion $S(\xi) \to D(\xi)$. Thus the homotopy groups $\pi_{i+1}(D(\xi), S(\xi))$ of the projection are just the homotopy groups $\pi_i(S^{n-1})$ of the fiber, so that the pair $(D(\xi), S(\xi))$ is $(n-1)$-connected and there is a canonical isomorphism $\pi_n(D(\xi), S(\xi)) \to \mathbb{Z}$. Orientability tells us that $\pi_1(B)$ acts trivially on this group, so by the (relative) Hurewicz theorem we get

$$H_r(D(\xi), S(\xi); \mathbb{Z}) = \begin{cases} 0 & (r < n) \\ \mathbb{Z} & (r = n) \end{cases}$$

and hence by the Universal Coefficient Theorem $H^n(D(\xi), S(\xi); \mathbb{Z}) = \mathbb{Z}$. Let α be the (positive) generator of this group; this is the Thom class.

Now we appeal to the Serre spectral sequence of a fibration. (Because of our assumption of orientability we can use untwisted coefficients.) Specifically, we observe that cup product with α induces an isomorphism on the E_2 terms from the spectral sequence of the trivial fibration $B \to B$ to the spectral sequence of $(D(\xi), S(\xi)) \to B$. However, both spectral sequences collapse at E_2, the first one for trivial reasons, and the second one because the E_2^{pq} vanish for $q \neq n$. Thus cup product with α also induces an isomorphism on the E_∞ terms, that is, an isomorphism from $H^k(B)$ to $H^{n+k}(D(\xi), S(\xi))$. A similar argument works for homology. □

3.43. Remark. We can define a Thom isomorphism for *non-oriented* spherical fibrations (and in particular for non-oriented vector bundles) if we take coefficients in $\mathbb{Z}_2$. Then we have an isomorphism

$$H^k(B; \mathbb{Z}_2) \to H^{n+k}(T(\xi), \infty; \mathbb{Z}_2)$$

for any spherical fibration ξ.

3.44. Exercise. From the Thom isomorphism derive the *Gysin sequence* for an oriented $(n-1)$-spherical fibration ξ over B:

$$\ldots \to H^k(B) \to H^{k+n}(B) \to H^{k+n}(S(\xi)) \to H^{k+1}(B) \to \ldots.$$

What is the map $H^k(B) \to H^{k+n}(B)$ appearing here?

3.5. Stable bundles and the classifying space BG

3.45. Definition. $G(n)$ denotes the topological monoid of homotopy equivalences $S^{n-1} \to S^{n-1}$. G denotes the direct limit $\lim G(n)$ under suspension.

3.46. Theorem. (STASHEFF) *There is a classifying space BG, with $\Omega BG \simeq G$, such that the fiber homotopy equivalence classes of stable spherical fibrations over a finite complex X are in natural $1:1$ correspondence with homotopy classes of maps $X \to BG$.*

Thus one can think of spherical fibrations loosely as 'fiber bundles' with 'group' G. There are also classifying spaces $BG(n)$, as well as corresponding spaces BSG, etc, for *oriented* spherical fibrations ($SG(n)$ consists of *orientation-preserving* homotopy equivalences $S^{n-1} \to S^{n-1}$).

Every oriented vector-bundle gives rise to an oriented spherical fibration, so there is a map of classifying spaces $BSO \to BSG$. This map is closely related to the J-homomorphism which we studied in the previous chapter. Let us see why this is so.

Pick a base point $* \in S^n$. Then the action of $SG(n+1)$ on $*$ gives a map $SG(n+1) \to S^n$ and we have

3.47. Lemma. *This map is a Serre fibration, with fiber the monoid $SF(n)$ of orientation preserving homotopy equivalences $S^n \to S^n$ that preserve the basepoint.*

The inclusion $SG(n) \to SG(n+1)$ has image in $SF(n)$, so that the limit SG might equally be called SF; some authors use this notation, especially if they want the letter G for other purposes.

3.48. Proposition. *For $i \geqslant 1$, $\pi_i(SF(n)) \equiv \pi_{i+n}(S^n)$. Hence, $\pi_i(SF) = \pi_i(\mathbb{S})$.*

PROOF. By standard adjunction formulae,

$$\mathrm{Maps}_\bullet(S^n, S^n) = \Omega^n S^n;$$

the base-point-preserving maps $S^n \to S^n$ are just the n-fold loop space of S^n. This space is divided into connected components parameterized by the degree. A map $S^n \to S^n$ is an orientation-preserving homotopy equivalence if and only if it has degree 1. Hence, $SF(n)$ is a connected component (that corresponding to degree 1) of $\Omega^n S^n$; the result follows. $\square$

We now see that the map $SO \to SF$ which associates to a (stable) orthogonal transformation the corresponding (stable) self-homotopy-equivalence of a sphere induces $\pi_i(SO) \to \pi_i(SF) = \pi_i(\mathbb{S})$, and it is plain that this is another description of the J-homomorphism.

CHAPTER 4

General Position

In this chapter we will develop the ‘general position’ techniques, such as transversality, that will allow us to construct manifolds and embeddings. We have already seen how transversality is used at key points in the study of exotic spheres (compare Theorem 2.57 and the discussion of intersections in Proposition 2.50). The key to all these general position results is found in Sard’s theorem, which we study first.

4.1. Sard's theorem

4.1. Definition. Let $f\colon M^m \to N^n$ be a smooth map of manifolds. The *critical set* $C_f \subseteq M$ of M is the set of $m \in M$ such that the tangent map $df_m\colon T_mM \to T_{f(m)}N$ fails to be surjective. The image $f(C_f) \subseteq N$ is called the set of *critical values* of f.

4.2. Definition. With notation as above, the complement $N \setminus f(C_f)$ of the set of critical values of f is called the set of *regular values*.

Notice that, by definition, a point that is not in the image of f at all is a regular value of f.

4.3. Proposition. *Let $f\colon M^m \to N^n$ be a smooth map of manifolds. If $x \in N$ is a regular value of f, then the inverse image $f^{-1}\{x\}$ is a (possibly empty) smooth submanifold of M, of dimension $m-n$, and having trivial normal bundle.*

PROOF. This follows from the Inverse Function Theorem. □

4.4. Theorem (Sard's theorem)**.** *If f is a smooth map as above, then $f(C_f)$ has measure zero in N.*

4.5. Exercise. Prove Sard's theorem for maps $\mathbb{R} \to \mathbb{R}$. (Hint: By Taylor's theorem, f contracts an interval of length ε containing a critical point to an interval of length $O(\varepsilon^2)$ containing the corresponding critical value.) Note that this requires only C^1 differentiability.

The notion 'has measure zero' makes sense on any smooth manifold, even though there is no canonical choice of smooth measure. There is a disconcerting example which shows that high differentiability is in general necessary in Sard's theorem; this is an example of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}$, such that there is a (topological) arc γ in $\mathbb{R}^2$ for which $df = 0$ at all points of the arc, but nevertheless f is not constant along γ. The image of the critical set thus contains an open subset of $\mathbb{R}$.

4.6. Corollary. *If $f\colon M^m \to N^n$ is smooth, and $m < n$, then the image of f has measure zero.*

For this corollary, only C^1 differentiability is in fact necessary; the proof of Exercise 4.5 will work.

OUTLINE PROOF OF THEOREM 4.4. Clearly, it is enough to consider the case $M = \mathbb{R}^m$, $N = \mathbb{R}^n$. The proof involves a double induction on m and n.

We split the critical set C_f into two pieces: C' on which df is zero, and C'' on which df is non-zero (but nonetheless not surjective).

We further decompose C' into pieces C'_i on which f is critical to order i, in other words, all the partial derivatives of f up to order i vanish. Thus $C' = C'_1 \supseteq C'_2 \supseteq \cdots$. If i is big enough (roughly $m/n+1$) the argument of Exercise 4.5 generalizes to show that $f(C'_i)$ has measure 0. It is enough therefore to show that $f(C'_i \setminus C'_{i+1})$ has measure 0 for all i. Near a point of $C'_i \setminus C'_{i+1}$ there is, by definition, a ith order partial derivative g of f whose derivative dg does not vanish. Then $g^{-1}\{0\}$ is a lower-dimensional submanifold M' of M and $C'_i \setminus C'_{i+1}$ is contained in the critical set of f restricted to this submanifold. The result follows from induction on m.

On the other hand, near a point of C'', we can assume that there is a coordinate projection $p\colon \mathbb{R}^n \to \mathbb{R}$ for which $d(p \circ f)$ does not vanish. Then the inverse images $M_t =$

$(p \circ f)^{-1}\{t\}$ foliate M into smooth lower-dimensional submanifolds, and inductively we can assume that

$$f(C'' \cap M_t)$$

is a measure-zero subset of $\mathbb{R}^{n-1} \times \{t\}$. The result now follows from Fubini's theorem. $\square$

4.2. Embedding and immersion theorems

4.7. Definition. Let $f\colon M \to N$ be a smooth map between manifolds. Then f is called an *immersion* if the tangent map $df_x\colon T_xM \to T_{f(x)}N$ is injective for all $x \in M$. It is an *embedding* if it is an immersion and, in addition, is a homeomorphism of M onto the image $f(M)$ (equipped with the topology it inherits as a subspace of N)

We will mostly be interested in compact manifolds; in this case any injective immersion is an embedding. For it is a standard result of elementary topology that a continuous bijection from a compact space to a Hausdorff space is in fact a homeomorphism.

The smooth map $\mathbb{R} \to T^2$ which wraps $\mathbb{R}$ densely around the 2-torus, using an irrational slope, is an example of an injective immersion of a non-compact manifold which is not an embedding.

We are going to construct embeddings and immersions of compact manifolds into Euclidean space, and eventually into other manifolds. A first step is provided by

4.8. Proposition. *Let M^n be a smooth manifold, $K \subseteq M$ a compact subset. Then there is a smooth map $f\colon M \to \mathbb{R}^k$, for some large k, which is an embedding on a neighborhood of K. In particular, any compact manifold can be embedded in a Euclidean space.*

PROOF. Cover K by finitely many coordinate charts $U_1, \ldots, U_m$, with embeddings $h_i\colon U_i \to \mathbb{R}^n$. Let φ_i be a system of bump functions subordinated to U_i — by this I mean that φ_i is supported within U_i and that for each $x \in K$ there is some index i such that $\varphi_i(x) = 1$. Then the map $M \to \mathbb{R}^{n(m+1)}$ defined by

$$x \mapsto (\varphi_1(x)h_1(x), \ldots, \varphi_m(x)h_m(x), \varphi_1(x), \ldots, \varphi_m(x))$$

is easily seen to be an embedding. □

The partition of unity construction yields a very large k. We now seek to reduce k. This we can do by general position arguments.

4.9. Lemma. *Let $f\colon M \to \mathbb{R}^k$ be an embedding of a compact manifold (or of a compact piece of a manifold, as above), and suppose $k > 2n + 1$. Then for almost all unit vectors $v \in \mathbb{R}^k$ the projection $P_v\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ orthogonal to v has the property that $P_v f$ is an embedding. If $k = 2n + 1$, then for almost all unit vectors v, $P_v f$ is an immersion.*

PROOF. Think of M as a submanifold of $\mathbb{R}^k$. What can go wrong? $P_v f$ may fail to be one-to-one, or it may fail to be immersive. To say that $P_v f$ is not one-to-one is to say that v belongs to the image of the map

$$M \times M \setminus \Delta \to S^{k-1}, \qquad (x, y) \mapsto \frac{x - y}{\|x - y\|}.$$

However, by 4.6, the image of this map has measure zero, since $2n < k - 1$. To say that $P_v f$ is not immersive is to say that v belongs to the image of the unit tangent bundle T_1M under the map $T_1M \to S^{k-1}$ that sends each unit tangent vector to 'itself'. But again, by 4.6, the image of this map has measure zero; this is true even when $k = 2n + 1$. This proves both results. □

4.10. Theorem. *Let M^n be a compact n-manifold. Any map $M^n \to \mathbb{R}^{2n+1}$ may be arbitrarily well approximated by an embedding, and any map $M^n \to \mathbb{R}^{2n}$ may be arbitrarily well approximated by an immersion.*

PROOF. We do it for embeddings. Let $f\colon M \to \mathbb{R}^{2n+1}$ be the given map, and let $g\colon M \to \mathbb{R}^k$ be an embedding of M in some high-dimensional Euclidean space $\mathbb{R}^k$. Then $h = (f, g)\colon M \to \mathbb{R}^{2n+1+k}$ is an embedding, and $f = \Pi h$ where $\Pi\colon \mathbb{R}^{2n+1+k} \to \mathbb{R}^{2n+1}$

is the obvious projection. But by k applications of the previous lemma, we see that Π can be altered by an arbitrarily small amount so as to make Πh an embedding; and it is clear that this new Πh may be as close as we wish to f. □

We would like to prove a similar result for maps of one manifold to another. It will be useful to note the obvious fact that the set of embeddings of M in N, and the set of immersions of M in N, are both open subsets of $C^\infty(M; N)$. (We assume M is compact here.)

4.11. Theorem (Whitney)**.** *Let M^m, N^n be manifolds, M compact.*

(a) *If $2m + 1 \leqslant n$ then any smooth map $M \to N$ can be arbitrarily well approximated by an embedding. In fact, if $f\colon M \to N$ is a smooth map which is already an embedding on some closed subset C of M, then f can be arbitrarily well approximated by an embedding which agrees with f on C.*

(b) *If $2m \leqslant n$ then any smooth map $M \to N$ can be arbitrarily well approximated by an immersion. In fact, if $f\colon M \to N$ is a smooth map which is already an immersion on some closed subset C of M, then f can be arbitrarily well approximated by an immersion which agrees with f on C.*

PROOF. The relative version easily follows from the absolute version and the openness of the set of embeddings. Cover N by finitely many coordinate charts $V_1, \ldots, V_\ell$, let $U_i = f^{-1}(V_i)$, and let $K_i \subseteq U_i$ be closed subsets such that $\bigcup K_i = M$. Assume that f is an embedding on $K_1 \cup \cdots \cup K_{r-1}$. Using the previous theorem, we can make a small perturbation of f on U_r so that f becomes an embedding on K_r. Because the embeddings form an open set, if we choose the perturbation small enough it will not destroy the property that f is an embedding on $K_1 \cup \cdots \cup K_{r-1}$. An induction on r completes the proof. □

4.12. Remark. It is easy to see that sufficiently close maps $M \to N$ are homotopic (for instance, give N a complete Riemannian metric, and join nearby points by a minimal geodesic). Thus, any smooth map $M \to N$ is in particular *homotopic* to an embedding (if $2m + 1 \leqslant n$) or an immersion (if $2m \leqslant n$).

For some purposes it is important to have a more refined notion of equivalence of immersions (or embeddings).

4.13. Definition. An *isotopy* of embeddings $M \to N$ is a smooth map $h\colon M \times [0,1] \to N$ such that each $h_t\colon M \times \{t\} \to N$ is an embedding. An *ambient isotopy* is an isotopy which arises by composing a fixed embedding $M \to N$ with a one-parameter family of diffeomorphisms of N.

There is an analogous definition for immersions.

4.14. Definition. A *regular homotopy* of immersions $M \to N$ is a smooth map $h\colon M \times [0,1] \to N$ such that each $h_t\colon M \times \{t\} \to N$ is an immersion.

The notion of regular homotopy (of immersions) is more restrictive than that of homotopy. For example, consider immersions $S^1 \to \mathbb{R}^2$. They are all homotopic (after all, $\mathbb{R}^2$ is contractible), but they are not all regular homotopic: the *rotation number* of such an immersion γ, which is the homotopy class of the tangent map $\gamma'\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$, is a regular homotopy invariant.

4.15. Exercise. Prove the *Whitney-Graustein theorem*: two immersions $S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if they have the same rotation number. See [**34**]. (Beware that, contrary to the impression one might gain from the classical literature, it is not enough to require a regular homotopy merely to be continuous and smooth for each fixed t.)

We will discuss the regular homotopy classification of higher-dimensional immersions in Chapter 10.

4.3. Transversality

Let $f\colon M \to N$ be a smooth map, and let X be a submanifold of N. Then for each $p \in f^{-1}(X)$, $d_p f$ induces a map from the tangent bundle of M to the normal bundle of X,

$$T_pM \xrightarrow{df} T_{f(p)}N \longrightarrow T_{f(p)}N/T_{f(p)}X$$

We say that f is *transverse to* X *at* p if this composite map is surjective, and that f is *transverse to* X if it is transverse at all $p \in f^{-1}(V)$. Thus, if $X = \{x\}$ consists of a single point, f is transverse to X if and only if x is a regular value, that is, $x \notin f(C_f)$. By Sard's theorem, this is a generic condition.

A generalization will be of value. In the definition of transversality, it is clear that only the part of f that maps M to some tubular neighborhood of X is significant. Therefore we may consider the situation where M maps into the total space of some vector-bundle over X. In this situation, the condition that f be transverse does not involve the smooth structure on X at all. There is thus no reason to suppose X to be a manifold. We therefore end up with the following definition:

4.16. Definition. Let $\pi\colon V \to X$ be a vector-bundle over a space X and let $f\colon M \to \operatorname{Th} V$ be a vertically smooth map from M to the Thom space of V (see Remark 3.16). We say that f is *transverse at the zero section* X *of* $\operatorname{Th} V$ if for all $p \in f^{-1}(X)$, the vertical tangent map

$$d_{p,v}f\colon T_pM \to (\pi^*V)_{f(p)}$$

is surjective.

4.17. Exercise. Check that the notions of 'vertically smooth' and the 'vertical tangent map' make sense. Since we are interested in transversality only at the zero-section, it makes no difference whether we consider maps to V itself or to its Thom space; the latter has the advantage of being compact (provided that X itself is compact).

If f is a transverse map to the Thom space of the bundle $V \to X$, then by the inverse function theorem $f^{-1}(X)$ is a smooth manifold of dimension equal to the dimension of M minus the fiber dimension of V, and its normal bundle in M is identified with the pull-back f^*V. In other categories, and with an appropriate notion of 'bundle', this conclusion of the inverse function theorem may be taken as the *definition* of transversality.

We say that two *submanifolds* of M are transverse if the inclusion of one is transverse to the other. (This is a symmetric condition, equivalent to the statement that the tangent spaces of the submanifolds together span the tangent space of M at each point of intersection.) Using the inverse function theorem, one sees

4.18. Proposition. *Two submanifolds N_1 and N_2 of M (dimensions n_1, n_2, m) are transverse*

(a) *in case $n_1 + n_2 < m$, if and only if they don't intersect;*
(b) *in case $n_1 + n_2 \geqslant m$, if and only if near any point $p \in N_1 \cap N_2$, one can find local coordinates which identify a neighborhood U of p with $\mathbb{R}^m$ in such a way that $N_1 \cap U$ is identified with $\mathbb{R}^{n_1} \times \{0\}$ and $U \cap N_2$ is identified with $\{0\} \times \mathbb{R}^{n_2}$.*

□

This proposition reconciles our present definition of transversality with that used in Chapter 3 in the special case $n_1 + n_2 = m$.

The *transversality theorem* of Thom is

4.19. Theorem. *Any map from a compact manifold to the Thom space of a vector bundle can be arbitrarily well approximated by a map which is transverse at the zero section.*

PROOF. This is like the proof of 4.11. Let $f\colon M \to V$ be the given map. We start the proof by choosing an open cover $\{U_i\}$ of M, such that each $f(U_i)$ lies in a trivial part $\mathbb{R}^p \times V_i$ of the bundle, and such that there is a compact cover K_i contained in U_i. On each U_i the map f can be represented as (g, h), where $g\colon U_i \to \mathbb{R}^p$ and $h\colon U_i \to V_i$, and transversality just says that zero is a regular value of g_i. Thus, by Sard's theorem, it is possible to make an arbitrarily small perturbation of f to make it transverse on K_i. (Just perturb by a small constant.)

Now we remark that transversality is an open condition (in a suitable topology). Thus we can carry out inductively a sequence of smaller and smaller perturbations over the sets K_r, $r = 1, 2, \ldots$, in order to make f transverse as required. (This 'local-to-global' argument was already used in the proof of Theorem 4.11.) $\square$

It is convenient to make explicit a few points about transversality in the context of manifolds with boundary:

4.20. Definition. Let $(M, \partial M)$ be a manifold with boundary. A submanifold $N \subseteq M$ is called *neat* if $\partial N = N \cap \partial M$ and N meets ∂M transversely.

A map from $(M, \partial M)$ to the Thom space of a vector-bundle V is called *transverse at the zero-section* if it is transverse on the interior and its restriction to the boundary is transverse as a map from ∂M. If this is so, then the inverse image of the zero-section is a neat submanifold. The transversality theorem still applies: any map can be perturbed by an arbitrarily small amount so as to make it transverse. Moreover, there is a relative version: if the map is already transverse on ∂M, then the perturbation may be taken to be the identity on ∂M.

4.21. Example. As an illustration of the power of transversality, here is Hirsch's proof of the Brouwer fixed point theorem. You will observe that no algebraic topology is required.

As is well known, to prove Brouwer's theorem it is enough to show that there is no smooth retraction of D^n onto its boundary S^{n-1}. Suppose r is such a retraction. There is some point $p \in S^{n-1}$ such that r is transverse at p. Then $r^{-1}(p)$ is a 1-dimensional neat submanifold of D^n, so it is a finite union of circles and arcs with endpoints on the boundary. One of these arcs must run from p to some other point $q \in S^{n-1}$. Therefore, $r(q) = p$. But since r is a retraction, $q = p$, a contradiction.

4.22. Exercise. Show that the complement of a smooth (compact) manifold M^m embedded in $\mathbb{R}^{m+k}$ is $(k-2)$-connected. (Use transversality to show that any map of S^{k-2} into $\mathbb{R}^{m+k} \setminus M$ can be extended to a map of D^{k-1}.) In particular, the complement of a smoothly embedded circle in $\mathbb{R}^4$ is simply connected. (Smoothness is essential here.)

4.23. Exercise. Use transversality to show that if M^n is a closed orientable manifold, then any homology class in $H_{n-1}(M)$ or $H_{n-2}(M)$ is represented by the fundamental class of a closed oriented submanifold. (Use the fact from homotopy theory that the kth cohomology group of M is the collection of homotopy classes of maps from M to an Eilenberg-MacLane space of type $K(\mathbb{Z}, k)$; together with the identifications $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$.)

One important application of transversality is to make a map transverse to *itself*.

4.24. Definition. Let $f\colon M^m \to N^{2m}$ be an immersion. It is *self-transverse* if at most two points of M map to any point of N, and that if $x_1 \neq x_2$ with $f(x_1) = f(x_2)$, then there are neighborhoods U_i of x_i in M such that $f_{|U_1}$ and $f_{|U_2}$ are transverse embeddings of U_1 and U_2 into M.

Note that this condition implies that the double points of f are finite in number.

4.25. Proposition. *Any smooth map M^m to N^{2m} can be arbitrarily well approximated by (and in particular is homotopic to) a self-transverse immersion. In fact, if $f\colon M \to N$ is a smooth map which is already a self-transverse immersion on some closed subset C of M, then f can be arbitrarily well approximated by a self-transverse immersion which agrees with f on C.*

PROOF. $\square$

4.4. The Whitney lemma

Let N_1 and N_2 be smooth oriented submanifolds of the oriented manifold M having complementary dimensions and intersecting transversely. Then $N_1 \cap N_2$ consists of a finite number of isolated points each of which acquires a sign according to whether or not the orientations of N_1 and N_2 at that point combine to yield the orientation of M. (See Example 3.12.) Our objective in this section is to prove the *Whitney Lemma*, which states

4.26. Lemma. *Let M be an n-dimensional manifold. Suppose that*

(a) *$N_1^{k_1}$ and $N_2^{k_2}$ are transversely intersecting oriented submanifolds of M, $n = k_1 + k_2$, $k_1, k_2 \geqslant 3$,*
(b) *P and P' are intersection points of N_1 and N_2, having opposite signs, and*
(c) *there exist paths γ_1 and γ_2 from P to P', lying in N_1 and N_2 respectively, such that the loop $\gamma_1^{-1}\gamma_2$ is nullhomotopic in M.*

Then there is an ambient isotopy of N_1 to a submanifold N_1' transverse to M and such that $N_1' \cap N_2 = N_1 \cap N_2 \setminus \{P, P'\}$. The ambient isotopy is constant on a neighborhood of $N_1 \cap N_2 \setminus \{P, P'\}$, so in particular the signs of all the intersection points of $N_1' and N_2$ are the same as the signs of the corresponding intersection points of N_1 and N_2.

In other words, if two intersection points cancel ‘algebraically’, then they can be canceled ‘geometrically’. For example, the graph of $y = x^3 - x$ intersects the x-axis in three points -1,0,1; the signs alternate, so the algebraic intersection number is 1. By continuous deformation one can move the x-axis up to the line $y = 2$, which now intersects $y = x^3 - x$ only in one point — the number of intersections required by the algebra.

4.27. Remark. With some more careful hypotheses one can relax the dimension requirements somewhat[1]; this is important for the proof of the h-cobordism theorem. The proof of the Whitney lemma depends on the easy part of Whitney’s embedding theory for submanifolds (that is our Theorem 4.11), and the lemma itself is the key to the hard part of that theory (Chapter 10).

If M is oriented we can give each of the intersection points P and P' a sign in the usual way (see the discussion preceding Equation 3.13). In fact, though, we do not need to assume that M is oriented; to define the signs of the intersection points we can choose an arbitrary orientation of M at P and then transport the orientation to P' along either of the (homotopic) paths γ_1 and γ_2. While the actual signs associated to the two intersection points will of course depend on the choice of orientation of M at P, the notion ‘P and P' have opposite sign’ will not.

PROOF. The idea of the proof is illustrated in the figure below. Suppose that we want to cancel the two intersection points P and P' of N_1 and N_2 shown in the figure. Join p and q by embedded paths γ_1 and γ_2 in N_1 and N_2 respectively, so that the loop γ they form is nullhomotopic. We may assume without loss of generality that they do not meet any other intersection points. Now there is a homotopy class of maps $D^2 \to M$ with boundary γ realizing the nullhomotopy, and by Theorem 4.11 this homotopy class contains an embedding of a disk (because $n \geqslant 5 = 2 \cdot 2 + 1$); this disk may be assumed to be disjoint from N_1 and N_2 (this is an easy special case of transversality theory; remember that the codimension of both N_1 and N_2 is at least three). Such an embedded disk is called a *Whitney disk*. Now we use this as a guide for an isotopic ‘push’ of M parallel to the Whitney disk in a small neighborhood thereof; such a ‘push’ can be constructed to shove N_1 right through N_2, thereby getting rid of the intersection points, and to leave everything fixed outside a small neighborhood of the Whitney disc.

[1] To be precise, we can allow $n \geqslant 5$, $k_1 \geqslant 3$, provided that if $k_2 < 3$ we also assume that the induced map $\pi_1(M \setminus N_2) \to \pi_1(M)$ is an injection.

Size: 60 mm

FIGURE 1. The Whitney lemma

To be a bit more precise, let us define a 'standard Whitney model' to be the following configuration of two submanifolds intersecting transversely in $\mathbb{R}^n$. Write $\mathbb{R}^n = \mathbb{R}^{k_1-1} \times \mathbb{R}^2 \times \mathbb{R}^{k_2-1}$ and let γ_1 and γ_2 be the two transversely intersecting curves in the plane $\mathbb{R}^2$ given by the axis $y = 0$ and the parabola $y = x^2 - 1$. Let N_1 be the k_1-dimensional submanifold $\mathbb{R}^{k_1-1} \times \gamma_1 \times \{0\} \subseteq \mathbb{R}^n$, and let N_2 be the k_2-dimensional submanifold $\{0\} \times \gamma_2 \times \mathbb{R}^{k_2-1} \subseteq \mathbb{R}^n$. The proof of Whitney's lemma now has two parts:

(a) There is an ambient isotopy of the standard Whitney model which is equal to the identity off a compact set and which moves N_1 to a new submanifold N_1' of the model which does not intersect N_2;
(b) In the situation specified by the Whitney lemma, there is a neighborhood (in M) of the Whitney disk which is diffeomorphic to the standard model.

The proof of (a) is clear:

To prove (b) we use normal bundles and the tubular neighborhood theorem. We will find an obstruction. So, in our original set-up, let D^+ be an open disk slightly extending the closed disk D (and similarly for γ_i^+); let ν be the normal bundle to D^+ in M, and let ν_i, $i = 1, 2$, be the normal bundles to γ_i^+ in N_i. These bundles are all over contractible spaces, so they are all trivial. The bundles ν_i are sub-bundles of the restriction of ν to γ_i. By the tubular neighborhood theorem, there are tubular neighborhoods of D^+ and γ_i^+ which are diffeomorphic to the (trivial) total spaces of the bundles ν and ν_i. Moreover, with care we can arrange[2] that the inclusions of the tubular neighborhoods correspond to the inclusions of the sub-bundles ν_i in ν.

To embed our standard model, what we now need to do is to choose an orthonormal frame $\{v_1^1, \ldots, v_1^{k_1-1}, v_2^1, \ldots, v_2^{k_2-1}\}$ in the normal bundle ν in such a way that the vectors v_1 form an orthonormal frame for ν_1 along γ_1 and the vectors v_2 form an orthonormal frame for ν_2 along γ_2. The only question is whether ν_1 and ν_2 match up correctly at the two points of intersection. Notice that $\nu_1 = \nu_2^\perp$ at the intersection points, so we can define a vector-bundle ξ over the circle γ whose fiber is ν_1 over γ_1 and $\nu_2^\perp$ over γ_2. A bundle theory argument shows that we can find the frames we require if and only if the bundle ξ

[2]We use a Riemannian metric in which N_1 and N_2 are totally geodesic, and which is Euclidean near the intersection points.

is trivial. In general the bundle ξ defines a loop in $G(k_1 - 1, n - 2)$, the Grassmannian of $(k_1 - 1)$-planes in $(n - 2)$-space, and we need to know that this loop is null-homotopic.

Now our assumptions put us in the stable range for calculating the fundamental group of the Grassmannian, so $\pi_1 G(k_1 - 1, n - 1) = \pi_1(BO(k_1 - 1)) = \mathbb{Z}/2$. There is just a single element of $\{\pm 1\}$ to calculate, which can be detected by considering orientations, so that ξ is trivial if and only if it is orientable. Since γ_1 and γ_2 intersect with opposite orientations at P and P', ξ will be orientable if and only if the intersection indices $\varepsilon(P)$ and $\varepsilon(P')$ are opposite. Since this was our assumption, the bundle ξ is trivial, and we can embed the standard model and complete the proof. $\square$

4.28. Exercise. Verify the assertion made above about the fundamental group of the Grassmannian (you will need to use its description as $G(k, n) = O(n)/O(k) \times O(n - k)$, together with the homotopy exact sequence). Also, compute $\pi_1 G(1, 2)$ by the same method; hence find another reason why the Whitney trick fails in dimension four.

For a very careful account of the Whitney lemma and its consequences one should consult chapter 6 of Milnor's book [**23**].

CHAPTER 5

Products and the Symmetric Construction

The purpose of this chapter is to review some more-or-less standard material about the construction of products in cohomology theory, but to do so in the most functorial way possible. For instance, we don't just want to know that the cup product is (graded) commutative; we want to keep track of the chain homotopies that make it commutative; then we want to keep track of the chain homotopies between *them*, and so on for ever. It will turn out that the whole of this elaborate algebraic structure is important for surgery theory.

5.1. Diagonal approximations and the cup product

Cup products in de Rham theory are represented simply by the exterior product of differential forms. In fact, one can think of the exterior product of forms on a manifold M in the following way: if we identify the (suitably completed) tensor product $\Omega^*(M) \otimes \Omega^*(M)$ with the differential forms on the product manifold $M \times M$, then the wedge product is simply the map on forms

$$\Omega^*(M \times M) \to \Omega^*(M)$$

induced by the diagonal inclusion $M \to M \times M$.

When we use other homology and cohomology theories (such as singular or simplicial theory), there is no longer such a canonical choice of *diagonal approximation*. Instead, there are theorems which show that diagonal approximations exist and are unique up to an appropriate notion of chain homotopy.

Let $\mathcal{C}_\bullet$ denote

(a) either the singular chain functor, from topological spaces to chain complexes of abelian groups,
(b) or the simplicial chain functor, from ordered simplicial complexes[1] to chain complexes of abelian groups.

5.1. Definition. In either of the two cases above, a *diagonal approximation* is a chain map

$$\Delta = \Delta_X : \mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(X),$$

natural in X, with the property that $\Delta(x) = x \otimes x$ on each 0-simplex x of X.

5.2. Proposition. *Diagonal approximations exist, and any two diagonal approximations are naturally chain homotopic.*

PROOF. The proof is an application of a standard technique from algebraic topology, the *method of acyclic models*. It makes heavy use of the *naturality* of a diagonal approximation.

We are going to construct a diagonal approximation $\Delta = \{\Delta_n\}$ by induction on the degree n. In degree zero, the formula $\Delta_0(x) = x \otimes x$ is given us by the definition. Suppose inductively that natural maps

$$\Delta_j : \mathcal{C}_j(X) \to (\mathcal{C}(X) \otimes \mathcal{C}(X))_j$$

have been defined for $j < m$ and satisfy the chain map condition $\Delta_{j-1}\partial = \partial\Delta_j$.

Let X_m be the m-simplex, considered either as a topological space (if we are working with singular chains) or as a finite simplicial complex (if we are working with simplicial chains). Let $x_m \in \mathcal{C}_m(X_m)$ be the chain defined by the identity m-simplex. From the chain map condition

$$\partial\big(\Delta_{m-1}\partial x_m\big) = 0 \in (\mathcal{C}(X_m) \otimes \mathcal{C}(X_m))_{m-2}.$$

Thus $\Delta_{m-1}\partial x_m$ is a cycle, and therefore it is a boundary (the complex $\mathcal{C}(X_m) \otimes \mathcal{C}(X_m)$ is the tensor product of two acyclic complexes and is therefore acyclic itself). Choose some $y_m \in (\mathcal{C}(X_m) \otimes \mathcal{C}(X_m))_m$ such that

$$\Delta_{m-1}\partial x_m = \partial y_m$$

[1] That is, simplicial complexes with a specified ordering of the vertices

and define $\Delta_m(x_m) = y_m$. Because $\mathcal{C}_m(X)$ for a general space X is freely generated by the images of the class x_m under maps[2] $X_m \to X$, there is a unique natural extension of Δ_m to all such spaces. The induction is complete.

The uniqueness assertion is proved by a similar argument, which we omit. □

5.3. Example. The *Alexander-Whitney diagonal approximation* for an ordered simplicial complex is defined by the formula

$$\Delta[v_0, \dots, v_n] = \sum_{p=0}^{n} [v_0, \dots, v_p] \otimes [v_p, \dots, v_n]$$

where as usual we denote a simplex by its ordered set of vertices. There is an analogous Alexander-Whitney formula for the singular complex.

As the reader is no doubt aware, it is Proposition 5.2 which is 'responsible' for the well-definedness of cup and cap products in singular cohomology. For example a diagonal approximation gives rise to a map (well-defined on the homology level)

$$H_n(X; \mathbb{Z}) \to H_n(\mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(X)).$$

Combining this with the natural pairing

$$H_n(C \otimes D) \otimes H^m(D^*) \to H_{n-m}(C)$$

we obtain the *cap product*

$$H_n(X; \mathbb{Z}) \otimes H^m(X; \mathbb{Z}) \to H_{n-m}(X; \mathbb{Z}). \tag{5.4}$$

Similarly for the *cup product*

$$H^p(X; \mathbb{Z}) \otimes H^q(X; \mathbb{Z}) \to H^{p+q}(X; \mathbb{Z}) \tag{5.5}$$

which corresponds to the exterior product in de Rham cohomology.

We shall need to think more carefully about the properties of diagonal approximations. For instance, suppose that Δ is a diagonal approximation and let

$$T = T_X \colon \mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(X)$$

be the (natural) automorphism which switches the two factors of the tensor product. Then, clearly, $T \circ \Delta$ is another diagonal approximation. By the uniqueness part of Proposition 5.2, Δ and $T \circ \Delta$ are naturally chain homotopic. In particular, this gives us a proof that the cup-product is (graded) commutative on the level of cohomology.

However, there is no reason to stop here. Let us denote our original diagonal approximation Δ by φ_0 and let us now denote by φ_1 the natural chain homotopy that we have just constructed, so that

$$(1 - T)\varphi_0 = \partial\varphi_1 + \varphi_1\partial.$$

Since $(1 + T)(1 - T) = 0$, it follows from this equation that $(1 + T)\varphi_1$ is itself a natural chain map (raising degree by 1) from $\mathcal{C}_\bullet(X)$ to $\mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(X)$. Now we can extend Proposition 5.2 as follows.

5.6. Lemma. *Any natural chain map raising degree by $k > 0$*

$$\mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(X)$$

is naturally chain homotopic to zero. □

5.7. Exercise. Prove Lemma 5.6, once again using the method of acyclic models.

[2] Continuous maps if we are working with singular homology; simplicial maps if we are working with simplicial.

This lemma allows us to continue our construction inductively: we build a natural chain homotopy φ_2 such that

$$(1+T)\varphi_1 = \partial\varphi_2 - \varphi_2\partial.$$

Arguing inductively we can build a sequence of natural chain homotopies φ_n (raising degree by n) such that for all n,

$$(1+(-1)^{n+1}T)\varphi_n = \partial\varphi_{n+1} + (-1)^n\varphi_{n+1}\partial. \tag{5.8}$$

5.9. Definition. A collection $\{\varphi_n\}$ as above will be called a *refined diagonal approximation* for X.

We have proved

5.10. Proposition. *Refined diagonal approximations exist and are unique up to (natural) chain homotopy.*

A refined diagonal approximation gives a chain map from $\mathcal{C}_\bullet(X)$ to the *symmetric chain complex* associated to $\mathcal{C}_\bullet(X)$, which we shall now define. This *symmetric construction* (and the associated but more subtle *quadratic construction* which we shall meet later in this chapter) are closely related to the group (co)homology of the cyclic group $\mathbb{Z}_2$ with two elements. (The cyclic group in question is that generated by the transposition automorphism T.)

5.11. Lemma. *The complex*

$$\begin{array}{c} \cdots \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \\ \hfill \downarrow \\ \hfill \mathbb{Z} \end{array}$$

gives a resolution of the trivial $\mathbb{Z}[\mathbb{Z}_2]$-module $\mathbb{Z}$ by free $\mathbb{Z}[\mathbb{Z}_2]$-modules. □

We denote this complex by W, so that $W_n = \mathbb{Z}[\mathbb{Z}_2]$ if $n \geqslant 0$, and $d\colon W_n \to W_{n-1}$ equals $1 + (-1)^n T$.

5.12. Remark. For any $\mathbb{Z}[\mathbb{Z}_2]$-module K, we now have

$$H_n(\mathbb{Z}_2; K)) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} K), \quad H^n(\mathbb{Z}_2; K) = H^n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, K)),$$

by definition of group homology and cohomology.

Now let C be any finite-dimensional chain complex (of abelian groups). We can consider $C\otimes C$ to be a chain complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules by making use of the transposition involution T. Form the space of $\mathbb{Z}[\mathbb{Z}_2]$-module homomorphisms $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes C)$; this is now a double complex, which we make into a single complex by assigning total degree $p+q-r$ to $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_r, C_p \otimes C_q)$.

5.13. Definition. The chain complex $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes C)$ so defined is the *symmetric chain complex* of the chain complex C.

Let

$$Q^n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes C)) \tag{5.14}$$

be the homology groups of the symmetric chain complex of C. (Formally speaking these are the *hypercohomology* groups of $\mathbb{Z}_2$ with coefficients in $C \otimes C$. Compare the definition of ordinary group cohomology in Remark 5.12.)

5.15. Definition. The groups $Q^n(C)$ defined by Equation 5.14 are called the *symmetric groups* of the chain complex C.

The quadratic groups $Q_n(C)$ of C, to be defined and investigated shortly, are nothing but the corresponding hyper*homology* groups, i.e. the homology groups of the complex $W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C)$.

Writing out explicitly the definition of the boundary operator in the symmetric chain complex, we find that an n-cycle in that complex is given by a collection of chains $\varphi_s \in (C \otimes C)_{n+s}$ satisfying the relations

$$\partial \varphi_s \pm (\varphi_{s-1} + (-1)^s T \varphi_{s-1}) = 0, \quad s = 0, 1, 2 \dots$$

with (by convention) $\varphi_{-1} = 0$.

The quantities on the left of the equation give the expression for the boundary of a general chain.

Now if we compare this expression with Equation 5.8 we find that the maps φ_s constituting a refined diagonal approximation exactly give a chain map

$$\varphi \colon \mathcal{C}(X) \to \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathcal{C}(X) \otimes \mathcal{C}(X))$$

from $\mathcal{C}(X)$ to its symmetric chain complex, and this chain map is natural in X and is unique up to natural chain homotopy. In particular there is a natural map $\varphi \colon H_n(X; \mathbb{Z}) \to Q^n(\mathcal{C}(X))$. This process, which produces a map from the homology of any space X to the symmetric groups of its chain complex, is called the *symmetric construction*.

The symmetric groups $Q^\bullet(C)$ are obviously

functorial for chain maps of complexes: a chain map $\varphi \colon C \to D$ induces $\varphi_\% \colon Q^\bullet(C) \to Q^\bullet(D)$. In fact we have

5.16. Proposition. *The symmetric groups are chain homotopy invariant: chain homotopic chain maps $C \to D$ induce the same homomorphism on the symmetric groups. In fact, they introduce chain homotopic chain maps*

$$\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C) \to \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, D \otimes D)$$

on the symmetric chain complexes of C and D.

One should regard this as slightly surprising, since the symmetric groups depend ‘nonlinearly’ on C.

PROOF. A chain map $f \colon C \to D$ induces $f_\% = f \otimes f$ on the symmetric chain complexes. Let h be a chain homotopy between f and g, so that $f - g = h\partial + \partial h$. Define a map $h_\%$ on the symmetric chain complexes (raising degree by 1) by the formula

$$h_\%(\varphi)_{s+1} = (f \otimes h)\varphi_s \pm (h \otimes g)\varphi_s \pm (h \otimes h)T\varphi_{s-1}.$$

We write the total differential $d = \partial + b$, where $b = 1 \pm T$. Then calculation yields

$$\partial h_\% + h_\% \partial = (f \otimes f - g \otimes g) + ((f - g) \otimes h + h \otimes (f - g)T)$$

and

$$b h_\% - h_\% b = ((f - g) \otimes h + h \otimes (f - g))T.$$

Combining these shows that $h_\%$ gives the desired chain homotopy. □

5.17. Remark. Note in particular that the symmetric groups $Q^\bullet(\mathcal{C}(X))$ associated to a space X do not depend on whether we use the singular or the simplicial model for homology. For it is well known that the corresponding chain complexes are chain homotopy equivalent.

For our purposes it will turn out to be important to understand the behavior of the symmetric construction under *suspensions*. Let us use the notation $\widetilde{\mathcal{C}}_\bullet(X)$ for the *reduced* chain complex (singular or simplicial) of a space X with a base-point. The familiar suspension isomorphism between $\tilde{H}_r(X)$ and $\tilde{H}_{r+1}(\Sigma X)$ in fact comes from a natural chain equivalence

$$S\widetilde{\mathcal{C}}_\bullet(X) \to \widetilde{\mathcal{C}}_\bullet(\Sigma X),$$

where the 'algebraic suspension' S of a chain complex is defined by $(SC)_{r+1} = C_r$. Once again, this chain equivalence can be constructed by the method of acyclic models.

Let $\varphi_X : \widetilde{\mathcal{C}}(X) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \widetilde{\mathcal{C}}(X) \otimes \widetilde{\mathcal{C}}(X))$ denote the (reduced) symmetric construction for X. We can 'shift dimensions' algebraically to define

$$S\varphi_X : S(\widetilde{\mathcal{C}}(X)) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, S(\widetilde{\mathcal{C}}(X)) \otimes S(\widetilde{\mathcal{C}}(X)))$$

by

$$(S\varphi_X)_s = \begin{cases} 0 & \text{if } s = 0 \\ (\varphi_X)_{s-1} & \text{if } s > 0 \end{cases}$$

We can also consider the symmetric construction for the geometric suspension ΣX,

$$\varphi_{\Sigma X} : \widetilde{\mathcal{C}}(\Sigma X) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \widetilde{\mathcal{C}}(\Sigma X) \otimes \widetilde{\mathcal{C}}(\Sigma X)).$$

We thus obtain a diagram of chain complexes and maps

(5.18)
$$\begin{tikzcd} S(\widetilde{\mathcal{C}}(X)) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, S(\widetilde{\mathcal{C}}(X)) \otimes S(\widetilde{\mathcal{C}}(X))) \arrow[d] \\ \widetilde{\mathcal{C}}(\Sigma X) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \widetilde{\mathcal{C}}(\Sigma X) \otimes \widetilde{\mathcal{C}}(\Sigma X)) \end{tikzcd}$$

in which the vertical maps are chain equivalences. *This diagram need not commute.* It does, however, commute up to a natural chain homotopy. The non-triviality of this chain homotopy will give rise to 'destabilization obstructions' which we shall ultimately identify as a generalization of the homotopy-theoretic destabilization obstructions discussed in Section 3.3.

5.19. Exercise. Prove that the diagram 5.18 commutes up to natural chain homotopy. (Acyclic models again.)

5.20. Exercise. Use the commutativity of the diagram to explain why it is that cup-products vanish in the (reduced) cohomology of a suspension ΣX. Suppose that you know that Y is the suspension of some other space X; how can you use the symmetric construction for Y to recover the cup-product structure of X?

5.2. Steenrod squares

The symmetric construction exactly encodes the algebra needed to define the classical *Steenrod squares*. They are *cohomology operations* — natural transformations from the cohomology of a space to itself.

Let X be a space. The symmetric construction for X gives a natural chain map

$$\Delta\colon \mathcal{C}(X) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathcal{C}(X) \otimes \mathcal{C}(X))$$

which is unique up to chain homotopy.

Let $\alpha \in \mathrm{Hom}(\mathcal{C}(X), D)$ be a chain map from $\mathcal{C}(X)$ to a complex D with mod 2 coefficients. Then $\alpha \otimes \alpha$ defines a map $\mathcal{C}(X) \otimes \mathcal{C}(X) \to D \otimes D$, which vanishes on the image of the b-differential $b = 1 \pm T$ in the symmetric (double) complex of $\mathcal{C}(X)$. It follows that $\alpha \otimes \alpha$ defines a chain map

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathcal{C}(X) \otimes \mathcal{C}(X)) \to D \otimes D.$$

Composing with the symmetric construction we obtain a family of chain maps

$$\mathcal{C}_\bullet(X) \to (D \otimes D)_{\bullet - s}, \quad s = 0, 1, 2, \ldots.$$

In particular we can consider the case when α is a cocycle for $H^r(X; \mathbb{Z}_2)$, which we consider as a chain map from $\mathcal{C}(X)$ to the complex D having one copy of $\mathbb{Z}_2$ in degree r and zero elsewhere. The symmetric construction then gives us cocycles

$$\mathcal{C}_{r+s} \to \mathbb{Z}_2$$

which correspond (by definition) to the *Steenrod squares* $\mathrm{Sq}^s(\alpha) \in H^{r+s}(X; \mathbb{Z}_2)$.

The standard reference for Steenrod squares and their properties is the book of Steenrod and Epstein [**30**].

5.21. Exercise. The following are Steenrod and Epstein's axioms for the Steenrod squares:

(a) $\mathrm{Sq}^0 = 1$.
(b) If $x \in H^m(X; \mathbb{Z}_2)$, then $\mathrm{Sq}^m(x) = x \smile x$.
(c) If $x \in H^m(X; \mathbb{Z}_2)$, then $\mathrm{Sq}^n(x) = 0$ for $n > m$.
(d) The *total squaring operation* $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ is a ring homomorphism from $H^*(X; \mathbb{Z}_2)$ to itself. (The multiplicative part of this statement, $\mathrm{Sq}(x \smile y) = \mathrm{Sq}(x) \smile \mathrm{Sq}(y)$, is called the *Cartan product formula*.)

Verify, from our definition, as many of these as you have the energy for. (a) and (b) should be no problem, but (c) and (especially) (d) are trickier.

5.22. Exercise. Use our discussion of the relationship of the symmetric construction to suspensions (specifically the fact that the diagram 5.18 commutes up to natural chain homotopy) to show that the Steenrod squares commute with suspensions. (This can also be deduced from Steenrod and Epstein's axioms. Remember how we used the fact that Steenrod squares commute with suspensions in the proof of Proposition 3.22.)

5.23. Exercise. Show that Sq^1 is the Bockstein homomorphism $H^k(X; \mathbb{Z}_2) \to H^{k+1}(X; \mathbb{Z}_2)$ associated to the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

of coefficient groups.

5.24. Exercise. Verify that for s odd, the Steenrod squares Sq^s are defined on *integral* cohomology (but their images are still 2-torsion elements).

5.25. Example. Let $f\colon S^{2n-1} \to S^n$ be any map. Let X be the CW-complex $S^n \cup_f D^{2n}$. This is a CW-complex with three cells: 0-dimensional, n-dimensional (corresponding to a generator x of $H^n(X;\mathbb{Z})$), and $2n$-dimensional (corresponding to a generator y of $H^{2n}(X;\mathbb{Z})$). There is an integer m such that

$$x \smile x = my \in H^{2n}(X;\mathbb{Z}).$$

This integer m is called the *Hopf invariant* of the map f. In the classical examples of the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$, and $S^{15} \to S^8$, X is a manifold (the complex, quaternionic or Cayley projective space) and thus $m = 1$ by the unimodularity of Poincaré duality.

The Steenrod squares satisfy certain relations (the Adem relations, see [**30**]). Using these relations it can be shown that if $\mathrm{Sq}^n(x) \neq 0$ then $\mathrm{Sq}^m(x) \neq 0$ for some $m \leqslant n$ which is a power of 2. It follows easily that if $f\colon S^{2n-1} \to S^n$ has odd Hopf invariant, then n is a power of 2.

It is a theorem of Adams that, in fact, an odd Hopf invariant is possible only for $n \in \{1, 2, 4, 8\}$. The easiest proof of this uses operations on K-theory; see [**2**].

One can use the Steenrod squares to define the *Stiefel-Whitney* characteristic classes of a real vector bundle. Let V be an n-dimensional vector bundle over X. We will make use of the Thom isomorphism for V (whether or not V is orientable, this exists by remark 3.43). Let

$$\Phi\colon H^p(X;\mathbb{Z}_2) \to H^{p+n}(D(V), S(V);\mathbb{Z}_2)$$

denote the Thom isomorphism and let $\alpha = \Phi(1) \in H^n(D(V), S(V);\mathbb{Z}_2)$ be the Thom class.

5.26. Definition. With the above notation, the Stiefel-Whitney classes of V are the cohomology classes

$$w_q(V) = \Phi^{-1}(\mathrm{Sq}^q(\alpha)) \in H^q(X;\mathbb{Z}_2).$$

It is plain from this definition that Stiefel-Whitney classes may be defined for a spherical fibration — the 'linear' structure of a vector bundle plays no rôle.

5.27. Remark. We can also define the *total Stiefel-Whitney class* $w = 1 + w_1 + w_2 + \ldots$, by analogy with the total Steenrod square. Cartan's product formula for Steenrod squares then becomes the *Whitney sum formula*

$$w(V \oplus W) = w(V) \oplus w(W)$$

for Stiefel-Whitney classes.

5.28. Exercise. Show that if V is an oriented n-dimensional vector bundle, then $w_n(V)$ is the mod 2 reduction of the Euler class $e(V)$. (This is immediate from the definition 3.18 of the Euler class, and the fact that $\mathrm{Sq}^n(x) = x \smile x$ for $x \in H^n(X;\mathbb{Z}_2)$.)

Basic geometrical properties of the two lowest Stiefel-Whitney classes are given in:

5.29. Proposition. *Let V be an n-dimensional vector bundle.*

(a) *The first Stiefel-Whitney class w_1 vanishes if and only if V is orientable, i.e. if and only if the structural group of V can be reduced from $O(n)$ to $SO(n)$ (via the inclusion $SO(n) \to O(n)$).*
(b) *Supposing that $w_1 = 0$, the second Stiefel-Whitney class w_2 vanishes if and only if V is spinable, i.e. if and only if the structural group of V can be reduced from $SO(n)$ to $\mathrm{Spin}(n)$ (via the double cover $\mathrm{Spin}(n) \to SO(n)$).*

Note that (b) is a vital component of the analytic proof of Rochlin's theorem, Remark 2.23. However, it will not be used elsewhere in the book.

PROOF. (a) The exact sequence $SO \to O \to \mathbb{Z}_2$ of groups gives rise to a fibration of classifying spaces

$$BSO \to BO \to B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1)$$

where we recall that $B\mathbb{Z}_2$ can be taken as the real projective space $\mathbb{RP}^\infty$. It follows that a map $X \to BO$ can be lifted to BSO (i.e., the corresponding bundle can be oriented) if and only if the composite map $X \to K(\mathbb{Z}_2, 1)$ is nullhomotopic. But homotopy classes of maps to an Eilenberg-MacLane space correspond to cohomology classes, so we conclude that there exists a characteristic class in $H^1(BO; \mathbb{Z}_2)$ which is the obstruction to orientability. We now appeal to the calculation $H^1(BO; \mathbb{Z}_2) = \mathbb{Z}_2$ (see [**26**]) which shows that there is precisely one possibility for such a (nontrivial) characteristic class, namely w_1.

(b) This is similar: the exact sequence $\mathbb{Z}_2 \to \mathrm{Spin} \to SO$ gives a fibration

$$K(\mathbb{Z}_2, 1) \to B\,\mathrm{Spin} \to BSO.$$

Using $K(\mathbb{Z}_2, 1) = \Omega K(\mathbb{Z}_2, 2)$ we can rearrange this into a fibration

$$B\,\mathrm{Spin} \to BSO \to K(\mathbb{Z}_2, 2),$$

and arguing as before we see that the obstruction to 'spinability' of an oriented bundle is a characteristic class in $H^2(BSO; \mathbb{Z}_2)$. Again, calculation shows that this group has rank 1, generated by w_2; so w_2 is the desired obstruction. $\square$

5.30. Exercise. Give an alternative proof of (a) above by making use of the relationship between Sq^1 and the Bockstein, Exercise 5.23.

5.31. Exercise. For a connected manifold X define a homomorphism $w\colon \pi_1(X) \to \mathbb{Z}_2$ by sending each loop in X to ± 1 according to whether it preserves or reverses the orientation at the basepoint of X. Since $\mathbb{Z}_2$ is abelian, w defines an element of $\mathrm{Hom}(H_1(X); \mathbb{Z}_2) = H^1(X; \mathbb{Z}_2)$. Show that this element is, in fact, $w_1(X)$.

5.32. Remark. The reader may wonder if this series of results continues: can the vanishing of w_1, w_2 and w_3 be interpreted in terms of some still more refined geometric structure on the tangent bundle? The answer is no, at least if we understand 'geometric structure' in classical terms of finite Lie structural groups. However, it is possible to interpret invariants related to H^3 in terms of bundles with *infinite*-dimensional structure groups, such as the projective unitary group of a Hilbert space. This is one side of the theory of *gerbes* — see [**?**].

Let us now consider the particular case of the tangent bundle of a manifold M^n. (In this case we refer for short to the *Stiefel-Whitney classes of* M.) In this case there is another recipe for 'characteristic' cohomology classes due to Wu. Consider the linear map $H^{n-s}(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ defined by

$$x \mapsto \langle \mathrm{Sq}^s(x), [M] \rangle.$$

Because Poincaré duality is nondegenerate, there is a unique class $v_s = v_s(M) \in H^s(M; \mathbb{Z}_2)$ which represents this map, so that

$$\langle \mathrm{Sq}^s(x), [M] \rangle = \langle v_s \smile x, [M] \rangle.$$

The classes $v_s(M)$ (which are determined by the homotopy type and Poincaré duality structure of M only) are called the *Wu classes* of M. The reader should check that this definition does indeed generalize the definition of the Wu class appearing in the proof of Proposition 2.14.

Now we shall show that the Wu classes can be expressed in terms of the Stiefel-Whitney classes. (This fact was already used in the proof of Proposition 2.14.)

5.33. Theorem. *The total Wu class $v = 1 + v_1 + \dots$ of a manifold M is related to the total Stiefel-Whitney class w of its tangent bundle by $w = \mathrm{Sq}(v)$. In detail we have*

$$w_q = \sum_{s=0}^{q} \mathrm{Sq}^{q-s}(v_s).$$

In particular, the Wu classes $v_1, \dots, v_s$ vanish if and only if the corresponding Stiefel-Whitney classes $w_1, \dots, w_s$ vanish.

PROOF. We shall need to know that the total Steenrod squaring operation is invertible. To be precise, there is an operation Sq^{-1} in the *Steenrod algebra* (the algebra generated by the Steenrod squares) such that $\mathrm{Sq}^{-1}\mathrm{Sq} = \mathrm{Sq}\,\mathrm{Sq}^{-1} =$ identity. This follows from the Adem relations, see [**30**, reference].

Embed the manifold M in a Euclidean space $\mathbb{R}^N$ (Proposition 4.8) and let ν be the normal bundle of M in $\mathbb{R}^N$. Because $\nu \oplus TM$ is a trivial bundle, the total Stiefel-Whitney class w of TM and the total Stiefel-Whitney class $\bar{w}$ of ν are related by the equation

$$w \smile \bar{w} = 1.$$

This equation allows each of $w, \bar{w}$ to be calculated in terms of the other (see [**26**, Prop ?]).

We are going to make some calculations using the Thom isomorphism for the normal bundle ν. This normal bundle has the special property that its top homology class $[\nu] \in H_N(\mathrm{Th}\,\nu, \infty; \mathbb{Z}_2)$ is *spherical* ; that is, $[\nu]$ belongs to the image of the Hurewicz homomorphism. Indeed, the Pontrjagin-Thom construction gives a map $S^N \to \mathrm{Th}\,\nu$ which generates the top homology class. From the naturality of cohomology operations it follows that

$$\langle \gamma(x), [\nu] \rangle = 0$$

for any $x \in H^*(\mathrm{Th}\,\nu, \infty; \mathbb{Z}_2)$ and any γ in the Steenrod algebra of positive degree. In particular,

$$\langle \mathrm{Sq}^{-1}(x), [\nu] \rangle = \langle x, [\nu] \rangle \tag{5.34}$$

for all x.

Now let α denote the Thom class for the normal bundle, and let $y \in H^*(M; \mathbb{Z}_2)$. We have

$$\langle \mathrm{Sq}(x), [M] \rangle = \langle \mathrm{Sq}(x) \smile \alpha, [\nu] \rangle = \langle \mathrm{Sq}^{-1}(\mathrm{Sq}(x) \smile \alpha), [\nu] \rangle = \langle x \smile \mathrm{Sq}^{-1}(\alpha), [\nu] \rangle$$

using the properties of the Thom isomorphism and the result of equation 5.34. On the other hand, from the definition of the Wu class v we have

$$\langle \mathrm{Sq}(x), [M] \rangle = \langle x \smile v, [M] \rangle = \langle x \smile v \smile \alpha, [\nu] \rangle.$$

Comparing the two displayed equations gives us

$$v \smile \alpha = \mathrm{Sq}^{-1}(\alpha).$$

But by definition of the Stiefel-Whitney classes

$$\bar{w} \smile \alpha = \mathrm{Sq}(\alpha).$$

Putting these together

$$\alpha = \mathrm{Sq}(v) \smile \mathrm{Sq}(\alpha) = \mathrm{Sq}(v) \smile \bar{w} \smile \alpha.$$

Therefore, $\mathrm{Sq}(v) \smile \bar{w} = 1$, which means that $\mathrm{Sq}(v) = w$ as asserted. $\square$

5.35. Exercise. Use Stiefel-Whitney classes to show that when n is a power of 2, there is no immersion $\mathbb{RP}^n \to \mathbb{R}^{2n-2}$. See [**26**]. (We shall prove, in Chapter 10, that any closed n-manifold can be immersed in $\mathbb{R}^{2n-1}$. The example of $\mathbb{RP}^n$ shows that this immersion theorem is sharp.)

5.3. The quadratic construction

The material in this section will not be required until Chapter 14. The first-time reader might well wish to postpone the study of this section at least until after reading Chapter 8. The discussion there about the relationship between symmetric and quadratic forms may be regarded as a special case of the ideas of this section, applied to chain complexes concentrated in a single degree.

Let X and Y be spaces with base-points.

5.36. Definition. A *stable map* from X to Y is a map from $\Sigma^p X$ to $\Sigma^p Y$, $p \geqslant 0$; two stable maps are *stably homotopic* if they become homotopic after some further suspensions.

Compare our discussion of stable vector bundles, in

. We are interested in deciding whether a stable map is stably homotopic to a genuine map $X \to Y$. If this is the case, we shall say that the given stable map can be *destabilized*.

5.37. Example. Let V be a stably framed k-vector bundle over a base X. A stable framing of V gives a stable map

$$\mathrm{Th}(V) \to \mathrm{Th}(\varepsilon^k) = \Sigma^k (X \sqcup \bullet).$$

If the stable framing can be destabilized to a genuine framing (in the sense of Section 3.3), then the stable map of Thom spaces can be destabilized to a genuine map.

5.38. Example. (A 'symmetric' destabilization obstruction) A stable map from X to Y induces maps of reduced cohomology $\tilde{H}^*(Y) \to \tilde{H}^*(X)$. Using the cup product we obtain a commutative diagram

$$\begin{tikzcd}
\tilde{H}^m(Y) \otimes \tilde{H}^n(Y) \arrow[r] \arrow[d] & \tilde{H}^{m+n}(Y) \arrow[d] \\
\tilde{H}^m(X) \otimes \tilde{H}^n(X) \arrow[r] & \tilde{H}^{m+n}(X)
\end{tikzcd}$$

which will commute if our stable map can be destabilized. Thus the difference between the two ways around the diagram gives a homomorphism

$$\tilde{H}^k(Y \wedge Y) \to \tilde{H}^k(X)$$

which is an obstruction to destabilization.

5.39. Exercise. Show that in the case of m-dimensional vector bundles over S^m, this obstruction is the Euler number. Also, relate the obstruction to the Hopf invariant.

We are going to implement the idea of the preceding example on the level of chain complexes and chain maps. Thus, let $f\colon \Sigma^p X \to \Sigma^p Y$ be a stable map. Then f induces a natural chain homotopy class of chain maps

$$\widetilde{\mathcal{C}}(X) \to \widetilde{\mathcal{C}}(Y)$$

and therefore there is a diagram of symmetric constructions

(5.40)
$$\begin{tikzcd}
\tilde{H}_n(X) \arrow[r, "\varphi_X"] \arrow[d, "f_*"] & Q^n(\widetilde{\mathcal{C}}(X)) \arrow[d, "f^{\%}"] \\
\tilde{H}_n(Y) \arrow[r, "\varphi_Y"] & Q^n(\widetilde{\mathcal{C}}(X))
\end{tikzcd}$$

This diagram need not commute in general; its non-commutativity is an obstruction to destabilizing f. However, the fact that diagram 5.18 commutes up to natural chain homotopy tells us that the difference e between the two paths around the diagram 5.40 will vanish after p applications of the algebraic shift map

$$S\colon Q^*(\widetilde{\mathcal{C}}(X)) \to Q^{*+1}(S(\widetilde{\mathcal{C}}(X)))$$

defined by

$$(S\varphi)_s = \begin{cases} 0 & \text{if } s = 0 \\ \varphi_{s-1} & \text{if } s > 0 \end{cases}$$

More is true, in fact. The natural chain homotopy expressing the commutativity of 5.18 gives us an algebraic 'reason' for the vanishing of the p-fold suspension of e and that 'reason', together with the diagram of chain maps underlying 5.40, combine to give us a chain map

$$\psi\colon S(\widetilde{\mathcal{C}}(X)) \to C(S^p) \tag{5.41}$$

from $S(\widetilde{\mathcal{C}}(X))$ to the *algebraic mapping cone* $C(S^p)$ of

$$S^p\colon \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \widetilde{\mathcal{C}}(X) \otimes \widetilde{\mathcal{C}}(X)) \to S^{-p} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, S^p(\widetilde{\mathcal{C}}(X)) \otimes S^p(\widetilde{\mathcal{C}}(X)))$$

5.42. Remark. Recall that the *algebraic mapping cone* $C(\varphi)$ of a chain map $\varphi\colon C \to D$ is the chain complex with chain groups and differential

$$C(\varphi)_r = C_r \oplus D_{r+1}, \qquad \partial_{C(\varphi)} = \begin{pmatrix} \partial_C & 0 \\ \varphi & -\partial_D \end{pmatrix}.$$

There is a short exact sequence of chain complexes

$$0 \to S^{-1}(D) \to C(\varphi) \to C \to 0$$

whose associated long exact homology sequence has boundary map $f_*\colon H_*(C) \to H_*(D) = H_{*-1}(S^{-1}(D))$.

What is the algebraic mapping cone of the shift map S^p appearing above? Let $W[0, p-1]$ denote the truncation of the complex W (defined in Lemma 5.11) at the $(p-1)$st stage. Thus $W[0, p-1]_r$ is $\mathbb{Z}[\mathbb{Z}_2]$ if $0 \leqslant r \leqslant p-1$, and is 0 otherwise. The nonzero maps of the complex are the same as those appearing in W.

5.43. Proposition. *Let C be a finite-dimensional chain complex. Then the algebraic mapping cone of $S^p\colon \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C) \to S^{-p} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, S^p(C) \otimes S^p(C))$ is naturally chain equivalent to the complex $S(W[0, p-1] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C))$. Moreover, under this chain equivalence the natural chain map $C(S^p) \to C^(S^{p+1})$ corresponds to the map induced by $W[0, p-1] \to W[0, p]$.*

PROOF. The suspension map is induced by applying $\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\circ, C \otimes C)$ to the map of complexes

$$W[-p, \infty] \to W[0, \infty]$$

with the obvious notation. Thus its mapping cone is chain equivalent to

$$\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[-p, -1], C \otimes C) \cong W[1, p] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C),$$

which gives the result. □

5.44. Definition. The *quadratic groups* $Q_n(C)$ of the chain complex C are the homology groups of the complex $W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C)$.

Note that the quadratic groups $Q_n(C)$ are the direct limit of the homology groups of the direct system of complexes

$$\cdots \to W[0, p-1] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C) \to W[0, p] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes C) \to \cdots$$

that is implicit in Proposition 5.43.

5.45. Definition. The *quadratic construction* of the stable map $f\colon \Sigma^p X \to \Sigma^p Y$ is the homomorphism $\psi_f\colon \tilde{H}_n(X) \to Q_n(\widetilde{\mathcal{C}}(Y))$ coming from the chain map ψ of Equation 5.41 followed by the identification of Proposition 5.43 and, finally, the stabilization coming from the direct system of complexes displayed above. (If we omit the final stabilization step we may refer to the *unstable quadratic construction*.)

From our previous discussion we have

5.46. Proposition. *If a stable map $f\colon \Sigma^p X \to \Sigma^p Y$ can be destabilized (i.e., is stably homotopic to a genuine map $X \to Y$), then the quadratic construction ψ_f associated to f is zero.* □

5.47. Remark. Let us contrast the quadratic and symmetric groups. As we previously observed, a cycle for the symmetric group $Q^n(C)$ is represented by a collection of chains $\varphi_s \in (C \otimes C)_{n+s}$, $s = 0, 1, 2, \ldots$, satisfying

$$\partial(\varphi_s) + (-1)^{n+s-1}(1 + (-1)^s T)\varphi_{s-1} = 0.$$

By contrast a cycle for the quadratic group $Q_n(C)$ is represented by a collection of chains $\psi_s \in (C \otimes C)_{n-s}$, $s = 0, 1, 2 \ldots$, satisfying

$$\partial(\psi_s) + (-1)^{n-s-1}(1 + (-1)^{s+1} T)\psi_{s+1} = 0.$$

Given a cycle $\{\psi_s\}$ for the quadratic group, the collection of chains

$$\varphi_s = \begin{cases} (1+T)\psi_s & (s = 0) \\ 0 & (s > 0) \end{cases}$$

is a cycle for the symmetric group; this process defines the *symmetrization map* $(1+T)\colon Q_n(C) \to Q^n(C)$. The proof of proposition 5.43 shows that the image of the symmetrization map is equal to the kernel of the iterated suspension $Q^n(C) \to \lim_k Q^{n+k}(S^k(C))$.

The following result is an immediate consequence of our definitions.

5.48. Proposition. *Let $f\colon \Sigma^p X \to \Sigma^p Y$ be a stable map. Then the quadratic construction $\psi_f\colon \tilde{H}_n(X) \to Q_n(\widetilde{\mathcal{C}}(Y))$ is related to the non-commutativity in the square of symmetric constructions 5.40 by*

$$(1+T)\psi_f = \varphi_Y f_* - f^{\%}\psi_X. \quad \square$$

5.49. Example. We shall now directly relate the quadratic construction to the destabilization obstruction for vector bundles of Section 3.3. Let V be an oriented m-dimensional vector bundle over S^m, and let $\mathfrak{f}$ be a stable framing for V. Recall that the *destabilization obstruction* $\mathfrak{d}(V, \mathfrak{f})$ is the class defined by this data in $\pi_m(SO, SO(m)) = \pi_m(SO(m+2), SO(m))$.

The framing $\mathfrak{f}$ defines a stable map $\mathrm{Th}(V) \to \mathrm{Th}(\varepsilon^m) = S^{2m} \vee S^m$. Applying the quadratic construction to this map we obtain

$$\psi_{\mathfrak{f}}\colon \tilde{H}_n(\mathrm{Th}\, V) \to Q_n(\widetilde{\mathcal{C}}(S^{2m} \vee S^m)).$$

Take $n = 2m$. Then $\tilde{H}_{2m}(V)$ has a canonical generator (the image of the top homology class of S^m under the Thom isomorphism). Moreover, we can consider a cocycle representing the generator of $\tilde{H}^m(S^{2m} \vee S^m)$ as a chain map from $\widetilde{\mathcal{C}}(S^{2m} \vee S^m)$ to the chain complex $S^m(\mathbb{Z})$ having a single $\mathbb{Z}$ in dimension m. Composition

$$\mathbb{Z} \longrightarrow \tilde{H}_{2m}(\mathrm{Th}\, V) \longrightarrow Q_{2m}(\widetilde{\mathcal{C}}(S^{2m} \vee S^m)) \longrightarrow Q_{2m}(S^m(\mathbb{Z}))$$

produces an element of $Q_{2m}(S^m(\mathbb{Z}))$, which is clearly just the same as the group $Q_{(-)^m}(\mathbb{Z})$ that was defined in Section 3.3. Thus we have used the quadratic construction to associate an element of $Q_{(-)^m}(\mathbb{Z})$ to the stably framed vector bundle $(V, \mathfrak{f})$.

5.50. Proposition. *The element of $Q_{(-)^m}(\mathbb{Z})$ associated to the stably framed vector bundle $(V, \mathfrak{f})$ by the construction above is just the destabilization obstruction $\mathfrak{d}(V, \mathfrak{f})$ of Definition 3.27.*

PROOF. □

5.51. Exercise. Show that a 'symmetric' version of the above construction can also be given, which maps $\pi_{m-1}(SO(q))$ to $Q^{m+q}(S^q\mathbb{Z})$. Show that the homomorphism thus obtained factors through the J-homomorphism. In the case $m = q = 2k$, also relate it to the Hopf invariant.

CHAPTER 6

Poincaré duality and intersections

At the beginning of Chapter 1 we sketched a proof of the Poincaré duality theorem with real coefficients, using de Rham theory. In this chapter we are going to develop a more general approach to Poincaré duality. We begin by abstracting the main idea of the Mayer-Vietoris proof of duality sketched in Remark 1.3.

Given a finite open cover $\mathcal{U}$ of the closed manifold W, we can build a simplicial complex called the *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ as follows: the vertices of the nerve are the members of $\mathcal{U}$, and $U_1, \dots, U_k \in \mathcal{U}$ span a simplex if and only if their intersection $U_1 \cap \dots \cap U_k$ is a non-empty subset of W. Let F be a functor which attaches to each open subset U of X a chain complex (of real vector spaces) and which is covariant for inclusions; the examples we have in mind are $F_1(U) = \Omega_c^{n-*}(U)$ the compactly supported forms on U (with a shift of grading), and $F_2(U) = \Omega_*^c(U)$ the compactly supported currents on U. Then to each simplex of $N(\mathcal{U})$ is associated a chain complex (via the functor F) and to each face map is associated a morphism of chain complexes. These data allow us to define a double complex (as in [**7**]) combining the given differentials on the functor F and the simplicial differential on the nerve $N(\mathcal{U})$. The duality map D defines a natural transformation of functors $F_1 \to F_2$ and the key point in the proof of Poincaré duality is a 'local-to-global' principle stating that if such a natural transformation is an isomorphism 'locally' — over every simplex of N — then it is an isomorphism 'globally' — on the total complex of the double complex. We will develop these ideas in the next section.

6.1. Geometric modules and duality

In order to understand the structure of Poincaré duality, and for other purposes, it will be helpful to develop some 'geometric algebra' — algebra carried out on objects (such as modules) which are 'located' at some point of a 'control space'. In this section we shall develop one version of this idea, which is of central importance in modern topology.

Let K be a finite simplicial complex and let R be a ring.

6.1. Remark. For the purposes of this chapter it will suffice to take R to be a *commutative* ring, in fact we shall usually be working with $R = \mathbb{Z}$. However, our algebra does not depend strongly on the commutativity of R and the reader should note for future reference that all our statements remain valid for general rings. Over a general ring R, the term 'module' will refer to a *right* module.

6.2. Definition. A *geometric R-module M over K* (or *(R,K)-module* for short) is a list $\{M_\sigma\}$ of R-modules parameterized by the simplices of K. The *total module* of M is the direct sum $\bigoplus_\sigma M_\sigma$ (over all simplices of K). Usually we'll use the same notation M for the total module as we do for the geometric module itself. We will call M_σ the part of M *anchored* at σ. A geometric module M is *free* if each M_σ is free.

6.3. Definition. A *geometric morphism* or simply *morphism* $\varphi\colon M \to N$ of (R,K)-modules is a list $\{\varphi_{\sigma,\tau}\}$ of R-module morphisms $M_\sigma \to N_\tau$, such that $\varphi_{\sigma,\tau}$ is zero unless $\sigma \leqslant \tau$ (that is, unless σ is a face of τ). We also use the notation φ for the *total morphism* induced by φ, that is the direct sum $\bigoplus_{\sigma,\tau} \varphi_{\sigma,\tau}$ considered as a morphism on the total modules.

Geometric R-modules and morphisms form an (additive) category.

6.4. Example. Here is a key example. Let $\mathcal{C}^q(K)$ be the geometric module whose component over a simplex σ is R if σ is a q-simplex, and 0 otherwise. The total module of this geometric module may be identified with the space of simplicial q-cochains of K (with coefficients in R). Moreover, the simplicial cochain complex of K,

$$\mathcal{C}^0(K;R) \longrightarrow \mathcal{C}^1(K;R) \longrightarrow \mathcal{C}^2(K;R) \longrightarrow \cdots$$

now becomes a complex in the category of geometric modules. (This is because the coboundary of a simplex σ is a sum of simplices of which σ is a face.)

6.5. Example. Let X be a topological space, $\mathcal{U}$ a finite open cover, $K = N(\mathcal{U})$ the nerve of $\mathcal{U}$ (as in Remark **??**). Suppose that Γ is a sheaf of R-modules over X. Let $C^q(\mathcal{U};\Gamma)$ be the geometric (R,K)-module which sends each q-simplex $\sigma = (U_1,\ldots,U_q)$ to the R-module $\Gamma(U_1 \cap \cdots \cap U_q)$, and is zero on simplices of other dimensions. The total module of this geometric module may be identified with the space of Čech q-cochains of the cover $\mathfrak{U}$ with coefficients in Γ. Moreover, the Čech cochain complex of the cover

$$C^0(\mathcal{U};\Gamma) \to C^1(\mathcal{U};\Gamma) \to C^2(\mathcal{U};\Gamma) \to \cdots$$

now becomes a complex in the category of geometric modules.

6.6. Exercise. Let M and N be geometric (R,K)-modules. Show that the space $\mathrm{Hom}_{(R,K)}(M,N)$ of geometric morphisms from M to N is itself a geometric module, where we consider the component $\varphi_{\sigma,\tau}$ to be anchored at τ. Show that composition on the left with a geometric morphism $M' \to M$, or on the right with a geometric morphism $N \to N'$, themselves define geometric morphisms

$$\mathrm{Hom}_{(R,K)}(M,N) \to \mathrm{Hom}_{(R,K)}(M',N), \quad \mathrm{Hom}_{(R,K)}(M,N) \to \mathrm{Hom}_{(R,K)}(M,N'),$$

respectively.

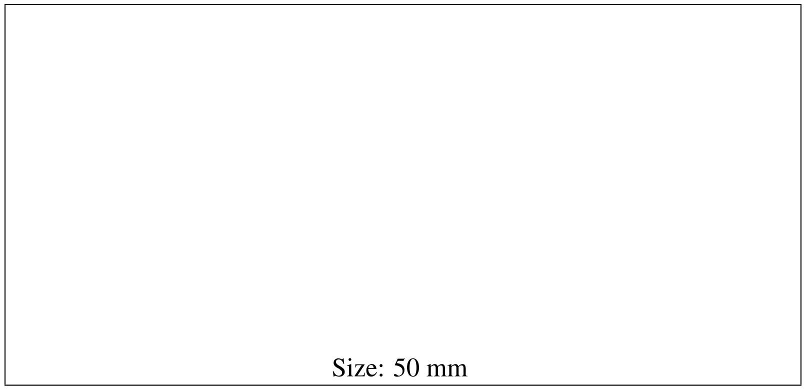

FIGURE 1. Barycentric subdivision, and a dual cell

Our definition of geometric morphism has a certain asymmetry, which is why it is easier to build cohomological examples than homological ones. However, homology can also be incorporated into the picture by the device of *dual cell decomposition*, which goes right back to Poincaré's proof of Poincaré duality.

Let K be a simplicial complex, as before. Remember that the *barycentric subdivision* K' of K may be defined (abstractly) as the simplicial complex whose vertices correspond to the simplices of K, with a simplex of K' being a *flag* of simplices of K. That is to say, the simplices of K' are spanned by vertices $\hat{\sigma}_0, \hat{\sigma_1}, \ldots, \hat{\sigma}_q$ corresponding to simplices $\sigma_0, \sigma_1, \ldots, \sigma_q$ of K having $\sigma_0 < \sigma_1 < \cdots < \sigma_q$. The figure shows the geometric picture of a barycentric subdivision.

As a matter of terminology, if $[\hat{\sigma}_0, \ldots, \hat{\sigma}_q]$ is a simplex of K', we shall refer to the simplex σ_0 of K as its *root* and the simplex σ_q as its *tip*. If σ is a simplex of K, its *dual cell* $D(\sigma, K)$ is the subcomplex of K' comprising those simplices whose root σ_0 satisfies $\sigma \leqslant \sigma_0$; the condition of strict inequality $\sigma < \sigma_0$ defines a subcomplex of the dual cell which is called its *boundary* $\partial D(\sigma, K)$. The dual cell is contractible; there is an obvious 'linear' contraction to the vertex represented by σ.

6.7. Example. Let K be a finite simplicial complex. Let $\mathcal{C}_q(K', R)$ be the geometric (K, R)-module which assigns to a simplex $\sigma \in K$ the free R-module generated by those q-simplices of K' whose root is σ. As an R-module, this is canonically isomorphic to the q'th relative simplicial chain module of the pair $(D(\sigma, K), \partial D(\sigma, K))$. The total module of the geometric module $C_q(K', R)$ is just the module of simplicial chains on K'. Moreover, the simplicial chain complex of K'

$$\mathcal{C}_0(K'; R) \longleftarrow \mathcal{C}_1(K'; R) \longleftarrow \mathcal{C}_2(K'; R) \longleftarrow \cdots$$

is now a complex in the category of geometric (R, K)-modules. This is because a face of a simplex of K' must have as vertices only simplices of K which have the root of the original simplex among their faces.

We have constructed various chain complexes in the category of geometric modules. We will need a 'local-global principle' for deciding when two such complexes are chain equivalent.

6.8. Definition. Let φ be a morphism of geometric (R, K)-modules. It is said to be *diagonal* if $\varphi_{\sigma,\tau} = 0$ unless $\sigma = \tau$. For a general morphism φ, its *diagonal part* is the diagonal morphism $\hat{\varphi}$ defined by

$$\hat{\varphi}_{\sigma,\tau} = \begin{cases} \varphi_{\sigma,\tau} & \text{if } \sigma = \tau \\ 0 & \text{otherwise} \end{cases}$$

6.9. Exercise. Check that this process of 'taking the diagonal part' is functorial (it preserves composition of morphisms). The reason is essentially that the map from upper triangular matrices to their diagonal part preserves matrix multiplication.

6.10. Exercise. Show that an (R, K)-module morphism is an isomorphism if and only if its diagonal part is an isomorphism. (Hint: If the diagonal part of φ is invertible, show that its inverse defines an (R, K)-module morphism ψ such that $\varphi\psi - 1$ and $\varphi\psi - 1$ are nilpotent. Remember that K is a *finite* complex.)

We can define a category (call it the category of 'diagonal modules') whose objects are geometric modules and whose morphisms are diagonal morphisms. The exercise shows that taking the diagonal part defines a functor from the category of geometric modules to the category of diagonal modules. In particular we can take the diagonal part of a chain complex of geometric modules, obtaining an associated chain complex of diagonal modules.

Notice that the total complex of a chain complex of diagonal modules splits into a direct sum of subcomplexes, one for each simplex σ. This means that properties of complexes of diagonal modules are 'local' — they can be verified one simplex at a time.

6.11. Example. The diagonal part of the cochain complex $C^\bullet(K; R)$, considered as a complex of geometric modules as in Example 6.4, assigns to each q-simplex σ the complex which has one free generator in dimension q and zero boundary maps. The diagonal part of the chain complex $C_\bullet(K'; R)$, considered as a complex of geometric modules as in Example 6.7, assigns to each q-simplex σ the relative chain complex of the pair $(D(\sigma, K), \partial D(\sigma, K))$ (the nontrivial statement here is that the diagonal part of the boundary map is exactly the relative boundary map of the pair).

We can now state the local-global principle

6.12. Proposition. *A finite chain complex of geometric (R, K)-modules is chain contractible (in the category of geometric modules) if and only if its diagonal part is chain contractible (in the category of diagonal modules). Similarly, a chain map between such complexes is a chain equivalence if and only if the induced map on the diagonal parts is a chain equivalence.*

PROOF. Let (C, d) be a finite chain complex of geometric modules. It is clear that if C is chain contractible then so is $\hat{C}$. Conversely, suppose that $\hat{C}$ is chain contractible and let $\hat{\Gamma} \colon \hat{C} \to \hat{C}$ be a chain contraction, defined by diagonal morphisms $\hat{\Gamma} : C_r \to C_{r+1}$ such that

$$\hat{d}\hat{\Gamma} + \hat{\Gamma}\hat{d} = 1.$$

The morphisms α defined by

$$\alpha = d\hat{\Gamma} + \hat{\Gamma}d$$

have diagonal parts $\hat{\alpha} = 1$, so that they are automorphisms by Exercise 6.10. Moreover, the calculation $d\alpha = d\hat{\Gamma}d = \alpha d$ shows that they are chain maps. Then the morphisms $\Gamma = \hat{\Gamma}\alpha^{-1}$ satisfy

$$d\Gamma + \Gamma d = 1,$$

and so they define a chain contraction of C.

The second part of the proposition follows from the first by considering mapping cylinders. □

6.13. Exercise. Show that the cochain complex $C^\bullet(K';R)$ of the barycentric subdivision of K becomes a chain complex of (R,K)-modules if we take each simplex $[\hat\sigma_0,\ldots,\hat\sigma_q]$ of K' to be anchored at its tip σ_q.

Show that the barycentric subdivision chain map [**8**, IV.17] defines a chain equivalence of complexes of (R,K)-modules between $C^\bullet(K;R)$ and $C^\bullet(K';R)$.

We are now going to discuss the cap product in the context of geometric modules. To do so we need a diagonal approximation (Definition 5.1) and we shall make use of the specific diagonal approximation given by Alexander and Whitney (Example 5.3) Recall that to define it, we must first order (arbitrarily) the vertices of the complex K, and decide to represent each simplex by a symbol $[v_0\cdots v_q]$ where the vertices appear in increasing order. The *Alexander-Whitney diagonal approximation* is the chain map

$$\mathcal{C}_\bullet(K)\to\mathcal{C}_\bullet(K)\otimes\mathcal{C}_\bullet(K)$$

defined by

$$[v_0\cdots v_q]\mapsto\sum_{i=0}^{q}[v_0\cdots v_i]\otimes[v_i\cdots v_q].$$

The tensor products are taken over R. Here is one point where the assumption that R is commutative does make our life easier; see Chapter 8 for the appropriate notions of tensor product over noncommutative rings with involution.

We are going to apply the Alexander-Whitney diagonal approximation not to the complex K itself but to its barycentric subdivision K'. In order to do this we must order the vertices of K'. Remembering that each vertex of K' corresponds to a simplex of K, we order these by increasing dimension:

$$\text{0-simplices of } K < \text{1-simplices of } K < \cdots;$$

and within each fixed dimension we order the simplices lexicographically. This choice of ordering gives us a chain level cap product map

$$\mathcal{C}_\bullet(K';R)\to\operatorname{Hom}_R(\mathcal{C}^\bullet(K';R),\mathcal{C}_\bullet(K';R)) \tag{6.14}$$

which is defined by

$$[\hat\sigma_0\cdots\hat\sigma_q]\mapsto\varphi([\hat\sigma_0\cdots\hat\sigma_p])[\hat\sigma_p\cdots\hat\sigma_q]$$

if φ is a p-cochain.

6.15. Proposition. *The pairing of Equation 6.14 in fact defines a (R,K)-module chain map*

$$\mathcal{C}_\bullet(K';R)\to\operatorname{Hom}_{(R,K)}(\mathcal{C}^\bullet(K';R),\mathcal{C}_\bullet(K';R))$$

where the chain and cochain complexes are made into geometric modules as in Examples 6.7 and 6.13.

PROOF. There are two statements to verify here,

(i) that for a fixed simplex $[\hat\sigma_0\cdots\hat\sigma_q]$ of K' the map $\mathcal{C}^\bullet(K';R)\to\mathcal{C}_\bullet(K';R)$ defined by $\varphi\mapsto\varphi([\hat\sigma_0\cdots\hat\sigma_p])[\hat\sigma_p\cdots\hat\sigma_q]$ is an (R,K)-module homomorphism,
(ii) and that the map assigning to $[\hat\sigma_0\cdots\hat\sigma_q]$ the (R,K)-module homomorphism defined in item (i) is itself an (R,K)-module homomorphism from $\mathcal{C}_\bullet(K';R)$ to $\operatorname{Hom}_{(R,K)}(\mathcal{C}^\bullet(K';R),\mathcal{C}_\bullet(K';R))$.

It is easy to check these facts: remember that a simplex of the chain complex of K' is anchored at its root, whereas a simplex of the cochain complex is anchored at its tip. $\square$

6.16. Exercise. Show that the map

$$\mathcal{C}_\bullet(K';R) \to \operatorname{Hom}_{(R,K)}(\mathcal{C}^\bullet(K';R), \mathcal{C}_\bullet(K';R))$$

defined in the proposition is in fact an (R,K)-module chain equivalence (use Proposition 6.12).

6.2. Geometric Poincaré Duality

Let K be a finite complex. For a vertex v of K, let $K \ominus v$ denote the subcomplex of K comprising all those simplices which do not have v as a vertex (this is the complement of the 'open star' of v in K).

6.17. Definition. Let R be a commutative ring (usually $\mathbb{Z}$). The complex K is a (combinatorial) *homology n-manifold* (with coefficients R) if

$$H_k(K', K' \ominus \hat{\sigma}; R) = \begin{cases} R & \text{when } k = n \\ 0 & \text{otherwise} \end{cases}$$

for every vertex $\hat{\sigma}$ of the barycentric subdivision K'.

6.18. Exercise. A compact Hausdorff space X is called a homology n-manifold if, for each point $x \in X$, one has

$$H_k(X, X \setminus \{x\}; R) = \begin{cases} R & \text{when } k = n \\ 0 & \text{otherwise} \end{cases}$$

using singular homology. Show that the complex K is a homology manifold by our definition in 6.17 if and only if its geometric realization $|K|$ is a homology manifold in the topological sense above.

6.19. Example. Every compact smooth manifold can be triangulated (that is, it is homeomorphic to the geometric realization of a finite simplicial complex). This result, which is far from trivial, is due to Cairns and Whitehead [**33**], and it can also be deduced from the handlebody decomposition of smooth manifolds which will be sketched in the appendix. Using excision and local coordinate charts, it is easy to check that a smooth manifold is a homology manifold, in the topological sense of the previous exercise. Therefore, by that exercise, any triangulation of a smooth manifold is a combinatorial homology manifold.

6.20. Definition. Let K be a homology n-manifold (with coefficients R). An *orientation* for K is a homology class $[K] \in H_n(K'; R)$ (called a *fundamental class* for the orientation) which restricts to a generator of $H_n(K', K' \ominus \hat{\sigma}; R) \cong R$ for each vertex $\hat{\sigma}$ of K'.

Suppose that K is an oriented homology n-manifold, and pick a specific cycle representing the fundamental class $[K]$. By Proposition 6.15, cap-product with $[K]$ defines an (R, K)-module chain map from $C^{n-\bullet}(K'; R)$ to $C_\bullet(K'; R)$.

6.21. Theorem (Geometric Poincaré Duality)**.** *For an oriented homology n-manifold K as above, the (R, K)-module chain map defined by cap product with the fundamental class*

$$\mathcal{C}^{n-\bullet}(K'; R) \to \mathcal{C}_\bullet(K'; R)$$

is a chain equivalence (in the category of (R, K)-modules).

6.22. Remark. In particular, cap-product with $[K]$ defines a chain equivalence in the category of R-modules, and therefore an isomorphism of homology and cohomology groups $H^{n-*}(K; R) \to H_*(K; R)$, which is the classical statement of Poincaré duality. But the local form of duality given by this theorem is more precise.

PROOF. According to Proposition 6.12 above, it will be enough to show that cap-product with $[K]$ gives a chain equivalence on the level of the diagonal parts of the (R, K)-module chain complexes $C^{n-\bullet}(K'; R)$ and $C_\bullet(K'; R)$.

The diagonal part of $C_\bullet(K'; R)$ anchored over a k-simplex σ is the simplicial chain complex of the dual cell $D(\sigma, K)$ relative to its boundary (see Example 6.11). Let us note

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FIGURE 2. Suspension of the dual cell gives a star

that the k-fold suspension of the pair $(D(\sigma,K),\partial D(\sigma,K))$ is the pair consisting of the closed star of $\hat\sigma$ relative to its boundary, or equivalently (by excision) the pair $(K', K'\ominus\hat\sigma)$. See Figure 2. In particular, $H_\bullet(D(\sigma,K),\partial D(\sigma,K))$ is R in dimension $n-k$, 0 elsewhere.

Similarly, the diagonal part of $C^\bullet(K';R)$ anchored over σ is spanned by all those simplices of K' which have σ as their tip. Let $(R)^k$ denote the cochain complex that has a single copy of R in dimension k, and zero elsewhere. There is a chain map $(R)^k \to C^\bullet(K';R)_\sigma$ given by sending the generator to the sum of all the k-simplices of K' whose tip is σ; by Exercise 6.13, this chain map is a chain equivalence. Thus the cohomology of the diagonal part of $C^\bullet(K';R)_\sigma$ is R in dimension k and 0 elsewhere.

One sees geometrically that the cap-product with the cohomology generator described above is just the suspension isomorphism

$$H_r(M, M\ominus\hat\sigma;R)\to H_{r-k}(D(\sigma,K),\partial D(\sigma,K)).$$

Taking $r=n$ this tells us that cap-product with the fundamental homology class maps the cohomology of the diagonal cochain complex anchored at σ isomorphically to the homology of the diagonal chain complex anchored at σ. Finally, we recall that a chain map between free chain complexes which induces a homology isomorphism is necessarily a chain equivalence. $\square$

6.23. Remark. By elaborating these techniques slightly we can also prove the *Alexander duality theorem*: Let K be an oriented combinatorial homology n-manifold, and let L be a subcomplex of K'. Then cap-product with the fundamental class induces an isomorphism of (R,K)-module chain complexes

$$\mathcal{C}^{n-\bullet}(L;R)\to\mathcal{C}_\bullet(K', K'\ominus L;R).$$

Notice that when L consists of a single vertex, this is just the definition of orientation.

Although we have followed the classical approach to duality using triangulations and dual cells, Poincaré duality does not depend on the existence of such a combinatorial structure. Using Mayer-Vietoris arguments similar to those we employed for de Rham cohomology, one can for instance prove an Alexander duality theorem for topological homology manifolds:

6.24. Theorem. *Let M be an oriented topological homology n-manifold (compact or not), and let $C\subseteq M$ be a compact subset. Then the cap-product with the orientation class defines duality isomorphisms*

$$D\colon \check{H}^r(C;R)\to H_{n-r}(M, M\setminus C;R)$$

where $\check{H}$ denotes Čech cohomology.

SKETCH OF PROOF. One verifies the theorem first when C is either empty (obvious) or is a *small cell* in M, that is a closed ball in some coordinate chart. In the latter case K is homotopy equivalent to a point and $M \setminus C$ is homotopy equivalent to the complement of that point, so the result follows from the definition of orientation. Now by the usual Mayer-Vietoris 'assembly' argument we can handle the case where C is a finite 'good' union of small cells. Given any closed set C and any open neighborhood U one can find C', $C \subseteq C' \subseteq U$, which is such a union of small cells; using the continuity property of Čech cohomology we can therefore complete the proof. For more details see Dold [**?**]. □

Some standard consequences are the separation theorems of Brouwer, generalizing the Jordan curve theorem.

6.25. Exercise. Let C be a closed subset of a compact connected n-manifold. Show that the number of connected components of $M \setminus C$ is equal to $1 + \dim \operatorname{Coker}(H^{n-1}(M;\mathbb{Z}/2) \to \check{H}^{n-1}(C;\mathbb{Z}/2))$. (Use duality and exact sequences.)

6.26. Exercise. Prove the *Jordan-Brouwer separation theorem:* Any homeomorphic image K of a compact connected $(n-1)$-manifold (in particular, of S^{n-1}) in S^n separates S^n into two connected components, of which it is the common boundary. (Use the previous exercise.)

6.27. Exercise. Prove the theorem of *invariance of domain*: Let $U \subseteq \mathbb{R}^n$ be open, $f\colon U \to \mathbb{R}^n$ be continuous and injective; then $f(U)$ is open in $\mathbb{R}^n$. (Let $p \in U$ and surround p by a small sphere S^{n-1} in U; argue that $f(p)$ must belong to the unique bounded component of the complement of $f(S^{n-1})$, which must be the image under f of the interior disc to S^{n-1} in U; hence $f(p)$ belongs to the interior of the image.)

Suppose now that $(W, \partial W)$ is a compact smooth $(n+1)$-manifold with boundary. An *orientation* in this case is by definition a class $[W] \in H_n(W, \partial W; R)$ that restricts to a generator of $H_n(W, W \setminus \{x\}; R)$ for each $x \in W^\circ$, the interior of W. It is easy to check that $\partial[W] \in H_{n-1}(\partial W; R)$ is then an orientation for ∂W. Cap-product with the relevant orientation classes gives a diagram of duality maps

$$\begin{tikzcd}
{} \arrow[r] & H^{n-r+1}(W) \arrow[r] \arrow[d] & H^{n-r+1}(\partial W) \arrow[r] \arrow[d] & H^{n-r}(W,\partial W) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_r(W,\partial W) \arrow[r] & H_{r+1}(\partial W) \arrow[r] & H_{r+1}(W) \arrow[r] & {}
\end{tikzcd}$$

which commutes up to sign.

6.28. Proposition (Lefschetz duality)**.** *All the duality maps in the diagram above are isomorphisms.*

PROOF. We already know that the absolute duality map for ∂W is an isomorphism, so by the five lemma it suffices to prove that one of the relative duality maps is an isomorphism too, say the map $H^{n-r}(W) \to H_r(W, \partial W)$. One can regard this as an Alexander duality map for W considered as a closed subset of its 'double', obtained by joining two copies of W, with opposite orientations, along their common boundary. □

A corollary whose importance that we have already seen is

6.29. Proposition (Cobordism invariance of the signature)**.** *Let W^{4j+1} be an oriented manifold with boundary ∂W. Then the signature* $\operatorname{Sign}(\partial W) = 0$.

PROOF. Let $M = \partial W$, let $i\colon M \to W$, and consider the subspace V which is the image of $i^*\colon H^{2j}(W) \to H^{2j}(M)$ in the middle-dimensional cohomology of M (we take coefficients in $\mathbb{R}$ throughout this proof). Then I claim that V is exactly equal to its own annihilator with respect to the intersection form $(x, y) \mapsto \langle x, D(y)\rangle$. For the proof, consider the diagram of duality maps, and write

$$x \in V \Leftrightarrow i_* D(x) = 0 \Leftrightarrow \langle H^{2j}(W), i_* D(x)\rangle = \{0\} \Leftrightarrow \langle V, D(x)\rangle = \{0\}.$$

But elementary linear algebra shows that if a symmetric bilinear form over $\mathbb{R}$ admits a subspace which is equal to its own annihilator (such a subspace is called *Lagrangian*) then it has signature zero. $\square$

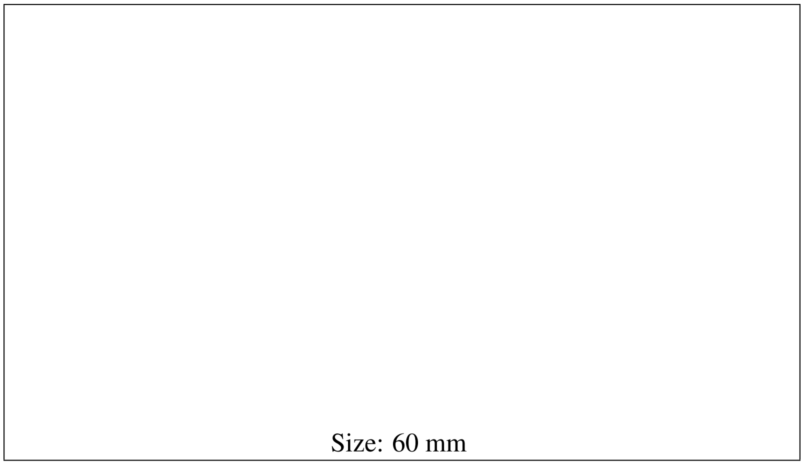

FIGURE 3. Geometric versus algebraic intersections

6.3. Geometric versus algebraic intersections I

Let M be a closed, oriented n-manifold. The *intersection form* of M is the bilinear pairing

$$\lambda\colon H_r(M) \otimes H_{n-r}(M) \to \mathbb{Z}$$

that is defined by Poincaré duality. When our homology classes are given by embedded, transversely intersecting submanifolds N_1 and N_2, the intersection form $\lambda([N_1],[N_2])$ counts (with sign) the intersection points of N_1 and N_2; see Example 3.12.

6.30. Example. Figure 3 depicts two closed 1-dimensional submanifolds N_1 and N_2 in a surface of genus 2. There are two intersection points of opposite signs, so $\lambda([N_1],[N_2]) = 0$. Nevertheless, it is intuitively clear that the two intersection points cannot be 'deformed away': there is no isotopy of N_1 to a new position in which it does not intersect N_2 (or *vice versa*).

As this example makes clear, the vanishing of the intersection form is in general not a sufficient condition for N_1 and N_2 to be disjoint after isotopy. The situation can be analyzed further by means of the Whitney lemma.

6.31. Theorem. *Let M be an n-dimensional oriented* simply-connected *manifold. Suppose that $N_1^{k_1}$ and $N_2^{k_2}$ are transversely intersecting oriented submanifolds of M, $n = k_1 + k_2$, $k_1, k_2 \geqslant 3$. Then there exists an ambient isotopy of N_1 to a submanifold N_1' which intersects N_2 in precisely $|\lambda([N_1],[N_2])|$ points. In particular, if $\lambda([N_1],[N_2]) = 0$, then N_1 and N_2 can be made disjoint by an ambient isotopy.*

PROOF. Repeatedly apply Lemma 4.26 to cancel pairs of intersection points of opposite sign. The crucial hypothesis (c) of that lemma, that certain loops are nullhomotopic, is assured by our assumption that M is simply connected. $\square$

In Chapter 9 we shall see what can be salvaged from this argument when the manifold M is no longer simply connected.

6.4. Linking numbers

CHAPTER 7

Cobordism and the signature theorem

In this chapter we shall systematically develop the properties of the Pontrjagin-Thom construction, which makes a link between the geometric problem of classifying manifolds up to cobordism and the topological problem of computing stable homotopy groups. Pontrjagin's original idea was to use geometry to give information about homotopy theory; later, after the development of new methods in homotopy theory, Thom reversed the argument and used homotopy theory to yield geometric information. We have already seen the Pontrjagin-Thom construction at work in Chapter 2 (see Equation 2.20).

The Pontrjagin-Thom construction can be applied in many slightly different examples. We shall develop the classical application to *framed* cobordism in detail: other applications, to oriented cobordism in this chapter and to normal cobordism in Chapter 12, will merely be sketched.

7.1. Cobordism and surgery

7.1. Definition (Thom)**.** Two closed n-dimensional manifolds M and M' are said to be *cobordant* if their disjoint union $M \sqcup M'$ is the boundary of a compact $(n+1)$-dimensional manifold W.

There are many variations on this basic definition. For instance, we shall need to consider *oriented cobordism* (everything is oriented, and the cobordism condition is $\partial W = M \sqcup (-M')$), *framed cobordism* (everything is equipped with a framing of its stable normal bundle), and so on.

It is clear that cobordism is an equivalence relation. It is a rather weak one: for instance, every oriented 2-manifold is cobordant to zero.

There is a close connection between cobordism and surgery. Proposition 2.39 tells us that if M' is obtained from M by performing a surgery, then there is a cobordism W between M and M', obtained by attaching a handle to $M \times [0, 1]$. (Such a cobordism is called an *elementary cobordism*.) The following observation is then immediate.

7.2. Proposition. *If the closed manifold M' is obtained from the closed manifold M by performing surgeries, then M' is cobordant to M.* □

Morse theory provides a means of generating elementary cobordisms.

7.3. Definition. Let $f\colon M \to \mathbb{R}$ be a smooth function on a manifold M. It is called a *Morse function* if the differential $df\colon M \to T^*M$ is transverse to the zero-section of T^*M.

7.4. Proposition. *Morse functions are dense. More precisely, suppose that M is embedded in some $\mathbb{R}^k$. Then any smooth function f on M can be perturbed by an (arbitrarily) small linear function on $\mathbb{R}^k$, so as to make it a Morse function.*

PROOF. Look first on a coordinate patch, where T^*M is trivial. By Sard's theorem, we may perturb df by an arbitrarily small constant in order to make it transverse to the zero-section on this patch. Since a constant is the differential of a linear map, this means that we can perturb f by a small linear map to make df transverse on the given patch. Now, noting that the Morse condition is an open one, we may apply the same local-to-global argument as in the proof of Theorem 4.11 to get the result. □

An equivalent definition of a Morse function is this: at each critical point of f (that is, point with $df = 0$), the *Hessian* — the symmetric matrix (in local coordinates) of second derivatives of f — should be nonsingular. By definition, the *index* of the critical point is the number of negative eigenvalues of the Hessian there. One can easily check that this does not depend on the choice of local coordinates.

7.5. Lemma (Morse Lemma)**.** *Let $f\colon M \to \mathbb{R}$ be a Morse function having a critical point at p. Then one can choose local coordinates $x_1, \ldots, x_n$ near p (with p corresponding to the origin) such that, relative to these coordinates, f takes the form*

$$f = -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_n^2$$

where r is the index of the critical point.

PROOF. □

The basic result of Morse theory is contained in the next proposition.

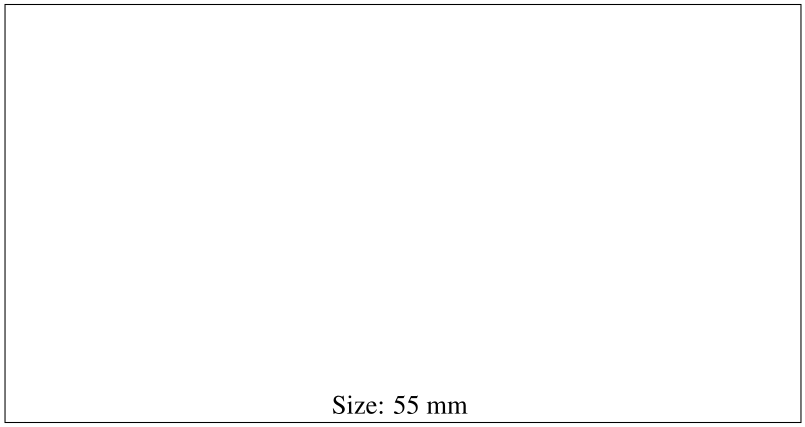

FIGURE 1. Neighborhood of a critical point

7.6. Proposition. *Let W be a cobordism, with boundary $\partial W = \partial_- W \sqcup \partial_+ W$. Suppose that W admits a Morse function f with no critical values on the boundary. Then*

(i) *If f has no critical values on the interior of W, then W is a product;*
(ii) *If f has exactly one critical value, of index r say, then W is an elementary cobordism, obtained by attaching an r-handle to $\partial_- W \times [0, 1]$.*

SKETCH PROOF. First consider the case in which there are no critical points. Equip W with a Riemannian metric, which allows us to define the *gradient vector field* ∇f of f as the dual to df. The flow lines of this vector field foliate W, and they always run in the direction of decreasing f; so they give W a product structure.

Now consider the case of just one critical point. Using the Morse lemma we can choose local coordinates so that the Morse function f is just a quadratic form

$$f(x_1, \ldots, x_n) = \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2$$

where the first r signs (r being the index) are negative and the last $n - r$ are positive. Using the first result we can localize matters to a neighborhood of the critical point; we then just need to observe that if f is the quadratic form given above, the region $\{x \in \mathbb{R}^n : -1 \leqslant f(x) \leqslant 1\}$ is naturally diffeomorphic to $D^r \times D^{n-r}$ minus the corner set. See Figure 1.

For more details of this argument, consult [**22**]. □

If M is a manifold, $f \colon M \to \mathbb{R}$ a Morse function, then it is quite easy to adjust f so that all the different critical points of f have different critical values (Just make an appropriate small linear perturbation.) Consequently, we may pick a sequence $a_0 < a_1 < \ldots$ of regular values for f such that there is exactly one critical value of f between a_i and a_{i+1} for each i. By the above result, $f^{-1}([a_i, a_{i+1}])$ is an elementary cobordism. Consequently, we obtain

7.7. Proposition (Milnor)**.** *Cobordism (of closed manifolds) is exactly the equivalence relation generated by surgeries.* □

Moreover

7.8. Proposition. *Any (compact) manifold can be built up by successively attaching handles to the empty set. Any cobordism between manifolds can similarly be built up by* *successive attachment of handles. The number of r-handles equals the number of critical points of index r of the Morse function used to construct the handle decomposition.* □

7.9. Corollary. *A closed manifold has the homotopy type of a finite CW-complex.*

PROOF. Attaching a $(q+1)$-handle to a cobordism W has the same effect on homotopy type as attaching a $(q+1)$-cell D^{q+1} via the attaching map, since up to homotopy equivalence we may contract the complementary disk D^p to a point. But a CW-complex just is a space built up by attaching cells to the empty set. □

7.10. Exercise. Let M be a connected manifold of dimension m, having non-empty boundary. Show that M has the homotopy type of an $(m-1)$-dimensional CW-complex.

7.11. Exercise.

7.2. Framed cobordism

7.12. Definition. Let M be a closed submanifold of the closed manifold N. A *framing* for M in N is a framing of the normal bundle ν_M of the embedding $M \to N$. We say that M is a *framed submanifold* if it is provided with a framing.

We have already made use of this definition in special cases; compare Definition 2.38.

7.13. Definition. A *framed cobordism* between framed submanifolds M, M' of N is a neat framed submanifold W of $N \times [0,1]$ whose boundary is $M \times \{0\} \cup M' \times \{1\} \subseteq N \times [0,1]$ (as a framed manifold).

7.14. Lemma. *The collection $\Omega_m^{fr}(N)$ of framed cobordism classes of m-dimensional submanifolds of N^n is an abelian group provided that $2m+1 < n$.*

PROOF. We need to know that given two framed m-dimensional submanifolds, we can adjust one of them by a cobordism so that it is disjoint from the other one. This is assured by transversality: just push one of the manifolds slightly in a normal direction. Now we can define the addition operation on $\Omega_m^{fr}(N)$ to be disjoint union. The empty m-manifold is the identity element, and the inverse of a framed manifold M is M with a 'mirror image' framing (one vector in the framing is replaced by its negative). $\square$

7.15. Exercise. Show that if N is a sphere then the condition $2m+1 < n$ can be replaced by $m < n$ in the proposition above. (Show that there are enough diffeomorphisms $S^n \to S^n$ to allow us to move one submanifold into the northern hemisphere and the other into the southern hemisphere.)

7.16. Exercise. Say that two framings are *equivalent* if one can be obtained from the other by a (fixed) element of $SO(m)$. Show that equivalent framings of a submanifold M define the same element of $\Omega_m^{fr}(N)$. Deduce that when we define the mirror image of a framing, it doesn't matter *which* vector we replace by its negative.

We shall now apply the Pontrjagin-Thom construction as in Equation 2.20. Given a framed submanifold M^m of N^n, this constructs a map $N \to S^{n-m}$, as

$$N \to U^+ = \Sigma^{n-m}(M \sqcup \bullet) \to \Sigma^{n-m}(S^0) = S^{n-m} \tag{7.17}$$

where U is a tubular neighborhood of M in N, identified with $N \times \mathbb{R}^{n-m}$ by the given framing and the tubular neighborhood theorem.

7.18. Theorem. *The construction of equation 7.17 gives a well-defined isomorphism from $\Omega_m^{fr}(N)$ to the* cohomotopy group $\pi^{n-m}(N) := [N, S^{n-m}]$. *In particular when $N = S^n$ we obtain an isomorphism*

$$\Omega_m^{fr}(S^n) \to \pi_n(S^{n-m}) = \pi^{n-m}(S^n).$$

PROOF. It is important to observe first that the construction depends on the choice of tubular neighborhood. Thus we need first of all to appeal to the uniqueness theorem[1] for tubular neighborhoods, which states that given two tubular neighborhoods U and V of M in N, there exists an ambient isotopy of U onto V. That is, there exists a 1-parameter family of diffeomorphisms of N, all of which fix M, beginning with the identity and ending with a diffeomorphism which maps U onto V. This ambient isotopy gives rise to a homotopy between the Pontrjagin-Thom maps constructed using the tubular

[1] Many textbooks prove the existence but not the uniqueness of tubular neighborhoods, e.g. [**26**]. The uniqueness is given a careful treatment in [**32**].

neighborhoods U and V. We conclude that a framed submanifold of N does give rise to a well-defined element of $\pi^{n-m}(N)$.

If M and M' are framed cobordant, we can apply the Pontrjagin-Thom construction to a framed cobordism W between them. This produces a map $N \times [0,1] \to S^{m-n}$ which implements a homotopy between the Pontrjagin-Thom maps constructed from M and M'. We conclude that the Pontrjagin-Thom construction gives a well-defined map $\Omega_m^{fr}(N) \to \pi^{n-m}(N)$. The proof that this map is a group homomorphism is left to the reader[2].

To show that this homomorphism is an isomorphism, we shall use transversality to construct an inverse. Suppose that $f\colon N \to S^{n-m}$ is a map. Pick a point $p \in S^{n-m}$ and apply transversality (Theorem 4.19: think of the sphere as the Thom space of a trivial bundle over p) to perturb f slightly so as to be transverse at p. The perturbation does not change the homotopy class of f, so we may assume without loss of generality that the original f was transverse at p. Then $M = f^{-1}\{p\}$ is a framed submanifold of N, and it has a tubular neighborhood U such that the restriction of f to U can naturally be identified with the projection $M \times D \to D$, D being a disk around p in S^{n-m}. The Pontrjagin-Thom map associated to this framed submanifold is the map $g\colon N \to S^n$ which is obtained by composing f with the map $S^{n-m} \to S^{n-m}$ which maps the complement of D to the point at infinity. But this latter map is homotopic to 1, hence g is homotopic to f. We have therefore shown that $\Omega_m^{fr}(N) \to \pi^{n-m}(N)$ is surjective.

The proof of injectivity is similar (apply relative transversality to a homotopy $N \times [0,1] \to S^{n-m}$), and will be omitted. □

7.19. Example. Let us use Pontrjagin-Thom theory to calculate the groups $\pi_n(S^n)$. We need to study the cobordism classes of framed 0-manifolds in S^n. Now a framed 0-manifold is just a point with a sign $\pm$ depending on the orientation of the chosen frame; the inverse of a point with sign $+$ is a point wit sign $-$. The only possible non-trivial cobordisms (1-manifolds) are cancellations of pairs of points of opposite sign. Thus we recover the familiar result

$$\pi_n(S^n) = \Omega_0^{fr}(S^n) = \mathbb{Z},$$

together with the identification of the degree of a map $S^n \to S^n$ as the number of inverses (counted with sign) of a generic point in the range.

7.20. Example. We can proceed similarly to study $\pi_{n+1}(S^n)$ (details).

The difficulties of this method of computing the homotopy groups of spheres obviously increase rapidly. See Section 8.6 for the connection between $\pi_{n+2}(S^n) = \mathbb{Z}_2$ and the Arf invariant. See Section 17.2 for the connection between $\pi_{n+3}(S^n) = \mathbb{Z}_{24}$ and PL manifolds, Rochlin's theorem and the Hauptvermutung.

7.21. Remark. Consider the 'equatorial sphere' S^m in S^n. It has a standard framing (which makes it the framed boundary of a disk). Any other framing is obtained from this one by a map $S^m \to O(n-m)$, or to $SO(n-m)$ if we insist that orientation is preserved. Homotopic maps $S^m \to O(n-m)$ give rise to cobordant framings, so in this way we obtain a homomorphism

$$\pi_m(O(n-m)) \to \Omega_m^{fr}(S^n) = \pi_n(S^m).$$

It is easy to see that this is simply the J-homomorphism of Remark 1.43.

[2]If you are not familiar with the group structure on $\pi^{n-m}(N)$, just restrict attention to $N = S^n$ and think about the familiar homotopy group structure there. This is the most interesting case anyway.

7.22. Exercise. Use the Pontrjagin-Thom construction to prove the Freudenthal suspension theorem: the suspension map $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism for $n > k+1$ and an epimorphism for $n = k+1$.

7.23. Remark. As this last exercise indicates, the Pontrjagin-Thom construction gives a nice way to think of the stable homotopy groups of spheres, π_k^s; they are the bordism groups of k-manifolds M equipped with a (stable) framing for the stable normal bundle. Note that the stable normal bundle to M can be defined without appealing to any embedding into a sphere: it is just a vector bundle ν_M such that $TM \oplus \nu_M$ is trivialized.

7.3. Computations with exotic spheres

A manifold is called *stably parallelizable* if its tangent bundle has a stable framing. From Proposition 2.28, we see that a manifold M is stably parallelizable if and only if $TM \oplus \varepsilon^1$ is a trivial bundle.

In this section we are follow Milnor and Kervaire in using framed cobordism to prove that every homotopy sphere is stably parallelizable. We shall see that this implies that the quotient group Θ_n/bP_{n+1} in the exact sequence 2.12 is a finite group. In fact, it is naturally identified with a subgroup of Coker $J\colon \pi_n(SO) \to \pi_n^s$.

It is useful to introduce a somewhat weaker notion than stable parallelizability.

7.24. Definition. A manifold M is called *almost parallelizable* if $M \setminus \{p\}$ is parallelizable for $p \in M$.

Clearly, a homotopy sphere is almost parallelizable, since the result of removing a point from it is contractible. By Exercise 7.10, the result of removing a point from a connected, compact manifold of dimension n has the homotopy type of a CW-complex of dimension $(n-1)$. Therefore, by 2.28, a compact connected stably parallelizable manifold is almost parallelizable. We ask: When is the converse true?

7.25. Theorem. *Let M be a compact, connected, oriented, almost parallelizable n-manifold. Then*

(a) *If n is not a multiple of 4, M is always stably parallelizable;*
(b) *If n is a multiple of 4, M is stably parallelizable if and only if its signature is zero.*

PROOF. Let M be an almost parallelizable manifold, and consider a disk D^n around $p \in M$. Then TM is trivial over D and trivial over $M \setminus D$, so it is completely described by the map $S^{n-1} \to SO(n)$ relating the two trivializations. (Notice that this argument shows that TM is the pull-back of some bundle over S^n. In particular, all its lower Pontrjagin classes vanish.) The bundle $TM \oplus \varepsilon^1$ will be trivial if and only if the composite

$$\gamma\colon S^{n-1} \to SO(n) \to SO(n+1)$$

is nullhomotopic, i.e. if and only if a certain element of the group $\pi_{n-1}(SO(n+1)) = \pi_{n-1}(SO)$ vanishes.

We claim that $\gamma \in \operatorname{Ker} J\colon \pi_{n-1}(SO) \to \pi_{n-1}(S)$. Indeed, the n-manifold $M \setminus D$ provides a framed cobordism from the sphere S^{n-1}, with the framing described by γ, to zero. By our discussion in Remark 7.21, the element $J(\gamma) \in \pi_{n-1}^s = \Omega_{n-1}^{fr}$ is equal to zero.

According to Bott periodicity, the groups $\pi_{n-1}(SO)$ are determined by the congruence class of n modulo 8, according to the following table

n modulo 8	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

which is reproduced from 2.18. Moreover, a theorem of Adams [**1**] states that when n is congruent to 1 or 2 modulo 8, the stable J-homomorphism $\pi_{n-1}(SO) \to \pi_{n-1}^s$ is injective. Thus the group $\operatorname{Ker} J$, in which γ lies, is zero unless n is a multiple of 4. To complete the proof, we need only show that the obstruction γ in this case is proportional to the signature of M.

This is the same calculation we have already made in the proof of Proposition 2.15. The top Pontrjagin class of M is a multiple of γ ; the signature theorem shows that the signature of M is a multiple of the top Pontrjagin class (since the lower Pontrjagin classes all vanish). (The specific constants involved are not relevant to the argument here, except insofar as they are nonzero; see Equation 2.19 and Proposition 7.40). $\square$

7.26. Corollary (Kervaire-Milnor)**.** *Homotopy spheres are stably parallelizable.*

PROOF. The signature of a homotopy sphere is certainly zero. $\square$

We will now use this calculation to investigate the quotient group Θ_n/bP_{n+1}. Let Σ^n be a homotopy sphere; since, as we have just proved, it is stably parallelizable, its normal bundle will be trivial when it is embedded in a sphere S^{n+k} for sufficiently large k. Choosing a framing for the normal bundle we obtain an element of $\Omega_n^{fr} = \pi_n^s$. The element we obtain depends on the choice of framing, of course; but two different choices of framing will give rise to elements of π_n^s which differ by an element of $\operatorname{Im} J$. Therefore we have defined a homomorphism

$$\varphi\colon \Theta_n \to \operatorname{Coker} J = \pi_n^s / \operatorname{Im} J.$$

7.27. Lemma. *The kernel of φ is exactly the group bP_{n+1} of homotopy spheres that bound parallelizable manifolds.*

PROOF. If a homotopy sphere Σ belongs to $\operatorname{Ker}\varphi$, then by suitable choice of framing it can be embedded in S^{n+k} as the framed boundary of a framed submanifold W of D^{n+k+1}. Since W is a framed submanifold of a parallelizable manifold, it is stably parallelizable. But a stably parallelizable manifold W with non-empty boundary is parallelizable (by Proposition 2.28 and Exercise 7.10). Conversely if Σ is the boundary of a parallelizable manifold, then there is a framing for which it defines the zero element of Ω_n^{fr}. $\square$

Thus we have shown that Θ^n/bP_{n+1} is isomorphic to a subgroup of $\operatorname{Coker} J_n$. In particular, since $\operatorname{Coker} J_n \subseteq \pi_n^s$ is a finite group (by Serre's theorem, Proposition 1.19), we have

7.28. Proposition. *Θ^n/bP_{n+1} is a finite group.* $\square$

7.4. Thom spaces and oriented cobordism

The set Ω_n of oriented cobordism classes of n-dimensional closed oriented manifolds is an abelian group, using the empty set as the identity element, disjoint union as addition, and $-M$ as the inverse of M. More is true: the operation of cartesian product of manifolds passes to cobordism classes, giving $\Omega_* = \bigoplus_n \Omega_n$ the structure of a graded ring, the *oriented cobordism ring*. In this section we shall follow Thom's computation of the torsion-free part $\Omega_* \otimes \mathbb{Q}$. (The full structure of Ω_* was later obtained by Wall. We shall not need this.)

Recall that oriented k-dimensional vector bundles are classified by maps to the space $BSO(k)$, which one can think of as the Grassmannian of oriented k-planes in 'infinite dimensional Euclidean space'. (More formally, $BSO(k)$ is defined as a direct limit of finite-dimensional Grassmannians.) This means that given such a vector bundle over a space X, there is a unique homotopy class of maps $X \to BSO(k)$ that pulls back the universal bundle over $BSO(k)$ to the given vector bundle over X.

7.29. Definition. $MSO(k)$ denotes the Thom space of the universal bundle over $BSO(k)$.

Thom proved

7.30. Proposition. *There is a canonical isomorphism*

$$\lim_{k\to\infty} \pi_{n+k}(MSO(k)) \to \Omega_n.$$

PROOF. Suppose that f is a map from S^{n+k} to $MSO(k)$. By Theorem 4.19, we can make f transverse at the zero-section; and $f^{-1}(BSO(k))$ then becomes a manifold M of dimension n. This defines maps $\pi_{n+k}(MSO(k)) \to \Omega_n$ which are compatible with suspension.

Let M^n be a closed oriented manifold. By Whitney's embedding theorem (4.11), M can be embedded in S^{2n+1}. Let ν be the normal bundle to such an embedding. By collapsing all of S^{2n+1} outside a tubular neighborhood of M we get a map $S^{2n+1} \to T(\nu)$, and then by composing with the classifying map for ν we get a map $S^{2n+1} \to MSO(n+1)$, which is already transverse at the zero-section and such that the inverse image of the zero-section is M. This shows that $\pi_{2n+1}(MSO(n)) \to \Omega_n$ is surjective. A refinement of this argument (embedding a cobordism rel boundary) proves injectivity if k is a little larger. The details of the proof are similar to those in the case of framed cobordism (Theorem 7.18); we omit them. $\square$

Now we use this result to compute the cobordism ring modulo torsion. This needs the theory of Pontrjagin numbers. Let $k = (k_1, \ldots, k_r)$ be a *partition* of n (that is, a list of nonnegative integers adding up to N). For an oriented $4n$-manifold M, the *Pontrjagin number* $p_k[M]$ corresponding to the partition **k** is the number

$$\langle p_{k_1}(TM) \ldots p_{k_r}(TM), [M]\rangle$$

where on the left we have the Pontrjagin classes of the tangent bundle of M.

7.31. Lemma. *Pontrjagin numbers are cobordism invariants.*

PROOF. Suppose $M = \partial W$. The Pontrjagin classes of TM are the same as the Pontrjagin classes of TW restricted to M; indeed, these two bundles differ only by a trivial 1-dimensional bundle. Let $i\colon M \to W$ be the inclusion. Then we have (denoting by p the relevant product of Pontrjagin classes)

$$\langle p(TM), [M]\rangle = \langle i^* p(TW), [M]\rangle = \langle p(TW), i_*[M]\rangle.$$

But $i_*[M] = 0$, since $[M]$ is the boundary of the orientation class $[W] \in H_{4n+1}(W, \partial W)$, so the Pontrjagin numbers are zero. □

The Pontrjagin numbers therefore give homomorphisms $\Omega_{4n} \to \mathbb{Z}$. In particular they give $\mathbb{Q}$-linear maps $\Omega_{4n} \otimes \mathbb{Q} \to \mathbb{Q}$.

Given a partition **k** of n, let $\mathbb{P}^k$ denote the product $\mathbb{CP}^{2k_1} \times \cdots \times \mathbb{CP}^{2k_r}$ of complex projective spaces, which is a $4n$-dimensional manifold. Let $\varphi(n)$ denote the number of partitions of n. Then we have

7.32. Lemma. *For any n, the $\varphi(n) \times \varphi(n)$ matrix whose entries are the Pontrjagin numbers $p_j[\mathbb{P}^k]$ has nonzero determinant.*

PROOF. A computation with symmetric functions. See [**26**, Chapter 16]. □

As a corollary, the manifolds $\mathbb{P}^k$ are linearly independent elements of $\Omega_{4n} \otimes \mathbb{Q}$, and the dimension of this vector space is therefore at least $\varphi(n)$. In fact we have

7.33. Theorem. *(Thom) The rational cobordism algebra $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra on the complex projective spaces $\mathbb{CP}^2, \mathbb{CP}^4, \ldots$. In particular, the dimension of $\Omega_{4n} \otimes \mathbb{Q}$ is exactly $\varphi(n)$.*

7.34. Exercise. It is a consequence of the theorem that if M^r is a manifold and r is not a multiple of 4, then some finite disjoint union of copies of M is a boundary. Try to see this directly in some examples. For instance, what happened in the case of $\mathbb{CP}^m$, m odd? Hint: You should be able to represent $\mathbb{CP}^m$ in this case as the total space of a circle bundle over a quaternionic projective space.

PROOF. Given the linear independence lemma above, it is plain that all we need to do is to find an upper bound for the dimension $\dim_{\mathbb{Q}} \Omega_n \otimes \mathbb{Q}$; the upper bound should be $\varphi(m)$ if $n = 4m$ is a multiple of 4, and 0 otherwise.

We start by noting that by Proposition 7.30 we can identify

$$\Omega_n \otimes \mathbb{Q} \cong \pi_{n+k}(MSO(k)) \otimes \mathbb{Q}$$

for k large. The space $MSO(k)$ is highly connected (in fact it is $(k-1)$-connected), and so we can apply the Hurewicz theorem tensored with[3] $\mathbb{Q}$; this theorem says that if X is a $(k-1)$-connected space then the Hurewicz map $\pi_r(X) \otimes \mathbb{Q} \to H_r(X; \mathbb{Q})$ is an isomorphism for $r < 2k-1$. Now by the Thom isomorphism, $\tilde{H}_r(MSO(k); \mathbb{Q}) \cong H_{r-k}(BSO(k); \mathbb{Q})$. Thus we find an isomorphism

$$\Omega_n \otimes \mathbb{Q} \to H_n(BSO(k); \mathbb{Q})$$

for k large.

The rational cohomology of the classifying space $BSO = \lim_k BSO(k)$ is well known; it is a polynomial algebra generated by the Pontrjagin classes. It follows that $\dim_{\mathbb{Q}} H_n(BSO(k); \mathbb{Q})$ is equal to $\varphi(n/4)$ if n is a multiple of 4, and zero otherwise. The proof is completed. □

[3]This is a theorem of Serre. The main ingredient in the proof is the computation of the homotopy groups of spheres modulo torsion, which may be found for instance in Spanier, Chapter 9 Section 7. The computation is that $\pi_r(S^n)$ is a finite group for $r \neq n, 2n-1$, and this verifies that the theorem is true for a sphere. One then extends to prove the theorem for a bouquet of spheres, and then for an arbitrary finite complex X by considering a map $S^{r_1} \vee \cdots \vee S^{r_p} \to X$ obtained by combining the generators of the torsion-free parts of all the homotopy groups up to dimension $2k-1$. See Milnor and Stasheff, theorem 18.3.

7.5. The Hirzebruch signature theorem

We stated the signature theorem somewhat loosely in Chapter 1. Now we will give a more precise statement, and an outline of the proof.

We need the notion of a *multiplicative sequence* of polynomials, due to Hirzebruch. This is a sequence of polynomials $K_0 = 1, K_1(p_1), K_2(p_1, p_2), \ldots$ and so on in the universal Pontrjagin classes, with $K_n \in H^{4n}(\cdot; \mathbb{Q})$, such that the *total K-genus* $K(V) = 1 + K_1(V) + K_2(V) + \cdots$ of a vector bundle V is multiplicative: $K(V_1 \oplus V_2) = K(V_1)K(V_2)$. For example, the sequence of polynomials $K_n = p_n$ is multiplicative (this is just the Whitney sum formula.)

Recall the *splitting principle* from the theory of characteristic classes (Proposition 1.22). The real form of the splitting principle tells us that given any reasonable space X and (real) vector-bundle V over X, we can find a map $f\colon Y \to X$ such that the induced map f^* on cohomology is injective and the pulled-back bundle f^*V splits as a direct sum of 2-plane bundles, together (possibly) with a line bundle. It follows that any multiplicative sequence K_n is determined uniquely by its value on 2-plane bundles, which is a formal power series $f(t)$ in the first Pontrjagin class. (The coefficients of $f(t)$ are just the coefficients of p_1^n in $K_n(p_1, \ldots, p_n)$.) Conversely, Hirzebruch showed that every formal power series with leading coefficient 1 determines uniquely a multiplicative sequence of polynomials, called the multiplicative sequence *belonging* to the given formal power series. The proof is a computation with symmetric functions: write formally

$$1 + p_1 t + p_2 t^2 + \cdots = (1 + u_1 t)(1 + u_2 t) \cdots ,$$

where the formal variables u_i may be identified with the first Pontrjagin classes of the splitting 2-plane bundles. Then we must have

$$1 + K_1 t + K_2 t^2 + \cdots = f(u_1 t) f(u_2 t) \cdots ,$$

so to find K_1, K_2 and so on we just expand the right-hand side as a power series in t whose coefficients are symmetric functions in the u_i, and then write these coefficients in terms of the elementary symmetric functions p_i.

7.35. Example. The multiplicative sequence of polynomials belonging to the formal power series

$$f(t) = (1 + t)^{-1} = 1 - t + t^2 - \cdots$$

expresses the Pontrjagin classes of the stable inverse of a vector bundle V (a bundle V' such that $V \oplus V'$ is trivial) in terms of the Pontrjagin classes of V.

7.36. Exercise. Consider the multiplicative sequence of polynomials K_n belonging to the formal power series $f(t)$. Show that the coefficient of p_n in K_n is the same as the coefficient of t^n in the formal power series

$$1 - t\frac{d}{dt}\big(\log f(t)\big) = f(t)\frac{d}{dt}\left(\frac{t}{f(t)}\right).$$

(This is originally due to Cauchy. Hint: Take logarithms of the generating identity to write

$$\sum \log f(u_i t) = \log(1 + K_1 t + K_2 t^2 + \cdots).$$

Use this to find the coefficient in K_n of the power sum $u_1^n + \cdots + u_n^n$. Now use Newton's identities (see **[17]**) relating elementary symmetric functions and power sums.)

7.37. Theorem (Hirzebruch Signature Theorem)**.** *Let L_n be the multiplicative sequence of polynomials in the Pontrjagin classes belonging to the formal power series*

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots .$$

Then for any compact oriented $4n$-manifold M we have

$$\operatorname{Sign} M = \langle L_n(p_1, \ldots, p_n), [M]\rangle$$

where the p_i are the Pontrjagin classes of the tangent bundle of M.

PROOF. (See Hirzebruch [**15**].) Both sides of the equation define ring homomorphisms $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$. For the right side this is obvious from the definition of a multiplicative sequence and the cobordism invariance of the Pontrjagin numbers. For the left side, we proved the cobordism invariance of the signature in 6.29 as a consequence of Poincaré duality for manifolds with boundary; the multiplicative property can similarly be proved using the Künneth theorem.

Since both sides of the Hirzebruch signature formula define ring homomorphisms from $\Omega_* \otimes \mathbb{Q}$, it suffices to check the theorem on a set of generators for this ring. By Thom's theorem 7.33, such a set of generators is provided by the even-dimensional complex projective spaces $\mathbb{CP}^{2k}$. These all have signature $+1$. On the other hand, the total Pontrjagin class of $\mathbb{CP}^{2k}$ is equal to $(1+a^2)^{2k+1}$ (Exercise 1.27), where $a \in H^2(\mathbb{CP}^{2k}; \mathbb{Q})$ is the canonical generator (the hyperplane class). Thus

$$L(p) = (a/\tanh a)^{2k+1}$$

and a direct calculation shows that the coefficient of a^{2k} in this power series is equal to 1. Thus the theorem is verified for the generators, hence it is true. $\square$

7.38. Exercise. Verify that, as asserted above, the coefficient of z^{2k} in the power series expansion of $(z/\tanh z)^{2k+1}$ is equal to 1. (Use contour integration.)

The *Bernoulli numbers B_k* may be defined by

$$\frac{z}{\tanh z} = \sum_{k=0}^{\infty} \frac{2^{2k}B_k}{(2k)!} z^{2k}. \tag{7.39}$$

We may now deduce

7.40. Proposition (Milnor-Kervaire [**25**])**.** *Let M^{4k} be an oriented manifold all of whose Pontrjagin classes except for p_k vanish (for example, M could be an almost parallelizable manifold, see 7.24). Then*

$$\operatorname{Sign}(M) = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!} p_k(TM).$$

This result was used in Chapter 2, in the proof of Proposition 2.15.

PROOF. By Exercise 7.36, we need to calculate the coefficient of t^k in the power series expansion of

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} \frac{d}{dt}\left(\sqrt{t}\tanh\sqrt{t}\right) = \frac{1}{2}\left(1 + \frac{2\sqrt{t}}{\sinh(2\sqrt{t})}\right).$$

Using the identity

$$\frac{2}{\sinh 2u} = \frac{1}{\tanh 2u} - \frac{1}{\tanh u},$$

together with the definition of the Bernoulli numbers in Equation 7.39, we obtain the stated result. □

CHAPTER 8

Quadratic Algebra

This chapter has two purposes. First, it develops the basic machinery of 'quadratic algebra' over noncommutative rings with involution. In Chapter 2 we used the algebra of integral quadratic forms to describe the intersection theory of middle-dimensional homology classes in a $(2k-1)$-connected, parallelizable $4k$-manifold. The corresponding theory for general manifolds needs to be elaborated in several different ways. The most basic point is this: to deal with manifolds that are not simply connected, we shall need to study 'intersection forms' which are quadratic forms on free modules over the group ring $\mathbb{Z}[\pi]$ of the fundamental group. This is connected with the Whitney Lemma (4.26) via the requirement that certain loops in the manifold span 2-disks.

The second part of this chapter gives the basic theory of the L-groups $L_{2n}(R)$, due to Wall. These groups give a 'stable' classification of quadratic forms over R, in just the right sense to be useful for surgery theory. It will turn out that, after preliminary surgery below the middle dimension to make matters highly connected, the middle-dimensional intersection form of a $2n$-dimensional 'surgery problem' defines an element of $L_{2n}(\mathbb{Z}[\pi])$, which vanishes precisely when surgery is possible. (This is the main result of Chapter 15; compare Proposition 2.37 and its proof.)

If we do not wish to assume that the geometric situation has been simplified by preliminary surgery below the middle dimension, we shall in fact need to study not quadratic *forms*, but their homological counterparts, quadratic *complexes*. Quadratic complexes of length one, otherwise known as 'formations', are a necessary ingredient in the surgical study of odd-dimensional manifolds also. We shall not develop the theory of quadratic complexes in detail in this chapter; in Section 14.4 we shall organize them into an 'algebraic bordism' group which gives a generalized definition of L-theory.

The material of this chapter is almost purely algebraic, and provides a necessary foundation for the geometric developments later in the book. The geometrically-minded reader might skim the whole chapter on first reading, and then refer back as necessary.

8.1. Linear algebra over rings with involution

We are going to think about linear and multilinear algebra over a possibly noncommutative ring R. The basic objects of linear algebra are modules, tensor-products, and Hom-sets. In the noncommutative context one must draw a distinction between

(a) *left modules* V over R (equipped with a multiplication $R \times V \to V$, satisfying the associativity law $(rs)v = r(sv)$),
(b) *right modules*, equipped with a multiplication $V \times R \to V$ satisfying the associativity law $v(rs) = (vr)s$, and
(c) *bimodules*, equipped with both a left and a right module structure and satisfying the compatibility law $(rv)s = r(vs)$.

The distinction corresponds to that between left, right, and two-sided ideals in a noncommutative ring.

8.1. Remark. One needs to exercise care in forming tensor products and Hom-sets. For instance, if V is a right R-module and W a left R-module, then $V \otimes_R W$ may be defined: it is the quotient of the tensor product in the category of additive groups by the subgroup generated by expressions

$$vr \otimes w - v \otimes rw, \quad v \in V, r \in R, w \in W.$$

Notice that this tensor product has no module structure — it is simply an abelian group. However, if V is a bimodule then the tensor product inherits a left R-module structure from V; if W is a bimodule it inherits a right R-module structure from W; and of course if both V and W are bimodules, then $V \otimes_R W$ is a bimodule as well. Similar remarks apply to Hom-sets $\mathrm{Hom}_R(V, W)$ (now V and W need to be modules of the same handedness, both left or both right, in order that $\mathrm{Hom}(V, W)$ be defined.)

8.2. Definition. By convention, we will use the terminology 'module over R' to refer to a *right* module.

The rings of interest to us will come equipped with an extra piece of structure which allows us to relate 'left' and 'right'.

8.3. Definition. An *involution* on a ring R is a map $R \to R$, denoted $x \mapsto x^*$, which is a homomorphism of abelian groups, preserves the unit, and has $(xy)^* = y^*x^*$ and $x^{**} = x$ for all $x, y \in R$.

A ring with involution will be called a $*$-*ring*.

8.4. Example. Conjugation on $\mathbb{C}$ or on $\mathbb{H}$ is an involution. The conjugate transpose on a ring of matrices over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is an involution. The adjoint on the ring of bounded operators on a Hilbert space (or on any C^*-subalgebra, such as the ring of compact operators) is an involution.

More relevant to our purposes is the following.

8.5. Proposition. *The map $g \mapsto g^{-1}$ extends (by linearity) to an involution (the* standard involution*) on the group ring $\mathbb{Z}[\pi]$. More generally, the map $g \mapsto w(g)g^{-1}$ where $w\colon \pi \to \{\pm 1\}$ is any group homomorphism, extends to an involution (the w-*twisted involution*) on $\mathbb{Z}[\pi]$.* □

Let R denote a ring with involution. Given a right R-module V, the *opposite* left R-module, V^o, is defined to be V with the left action of R given by

$$(r, v) \mapsto vr^*.$$

Similarly we can define the opposite of a left R-module, and even the opposite of an R-bimodule (take the opposite of each structure).

8.6. Exercise. Let R be a ring with involution. Verify that R^o is isomorphic to R as an R-bimodule.

8.7. Exercise. Suppose that the group ring $\mathbb{Z}[\pi]$ is provided with the involution associated to $w\colon \pi \to \{\pm 1\}$. Give $\mathbb{Z}$ the trivial right $\mathbb{Z}[\pi]$-module structure in which each group element acts as 1. What is the structure of the left $\mathbb{Z}[\pi]$-module $\mathbb{Z}^o$? Show that the left and right actions commute so that $\mathbb{Z}$ becomes a $\mathbb{Z}[\pi]$-bimodule. (We denote it by $\mathbb{Z}^w$ when it is considered as a bimodule in this way.)

We isolate here a useful algebraic calculation.

8.8. Proposition. *Let U and V be right $\mathbb{Z}[\pi]$-modules. Equip $\mathbb{Z}[\pi]$ with the w-twisted involution, and let $\mathbb{Z}^w$ denote the integers considered as a left $\mathbb{Z}[\pi]$-module as in Exercise 8.7. The tensor product $U \otimes_{\mathbb{Z}} V$ in the category of abelian groups is made into a right $\mathbb{Z}[\pi]$-module by the diagonal action $(u \otimes v)g = ug \otimes vg$. Then there is a natural isomorphism*

$$(U \otimes_{\mathbb{Z}} V) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w \cong U \otimes_{\mathbb{Z}[\pi]} V^o$$

in the category of abelian groups.

PROOF. Send an element $x = u \otimes v \in U \otimes_{\mathbb{Z}[\pi]} V^o$ to the element $u \otimes v \otimes 1 \in (U \otimes_{\mathbb{Z}} V) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w$. The map is well-defined because if we represent x also by $(ug) \otimes (g^{-1}v)$, the image is

$$(ug) \otimes (g^{-1}v) \otimes 1 = w(g)(ug) \otimes (vg) \otimes 1 = u \otimes v \otimes 1 \in (U \otimes_{\mathbb{Z}} V) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w$$

using the definitions of the opposite module and of the involution in $\mathbb{Z}[\pi]$. The reader may verify similarly that the map is, in fact, an isomorphism. $\square$

Let V be a (right) R-module. Then $\mathrm{Hom}_R(V, R)$ is a left R-module.

8.9. Definition. We define the *dual module* of V to be the right R-module $V^* = \mathrm{Hom}_R(V, R)^o$.

8.2. Symmetric and quadratic forms

8.10. Definition. Let V be an R-module, where R is a ring with involution. A *sesquilinear form* on V is a R-module homomorphism $\lambda\colon V \to V^*$. It is *nonsingular* if it is an isomorphism of R-modules. The abelian group of all sesquilinear forms on V is denoted by $\mathrm{Ses}(V)$.

We may identify λ with the function $V \times V \to R$ given by $(x, y) \mapsto (\lambda(x))(y)$. By a slight abuse of notation we denote this function by λ also. The sesquilinearity condition then states that

$$\lambda(xa, yb) = a^*\lambda(x, y)b$$

for $a, b \in R$ and $x, y \in V$. We now study symmetry conditions on these forms.

8.11. Definition. Let $\varepsilon = \pm 1$. The *ε-symmetrization map* $T_\varepsilon\colon \mathrm{Ses}(V) \to \mathrm{Ses}(V)$ is defined by

$$T_\varepsilon\lambda(x, y) = \varepsilon\lambda(y, x)^*.$$

8.12. Definition. A sesquilinear form λ on an R-module V is called *ε-symmetric* if $\lambda = T_\varepsilon\lambda$. The space of ε-symmetric forms on V is denoted $Q^\varepsilon(V)$.

We can also say 'symmetric' or 'skew-symmetric' for 'ε-symmetric', according as $\varepsilon = 1$ or $\varepsilon = -1$, but the more uniform terminology saves some writing).

8.13. Remark. In Chapter **??** we defined what we called the *symmetric groups* associated to a chain complex C (of $\mathbb{Z}$-modules), and we denoted these by $Q^n(C)$. Those symmetric groups are related to the groups of symmetric forms defined above in the following way. Let V be a $\mathbb{Z}$-module (abelian group) and let C be the chain complex having a single copy of V^* in degree n and 0 elsewhere. Then the chain complex symmetric group $Q^n(C)$ is the same as the group $Q^\varepsilon(V)$ of ε-symmetric forms on V for $\varepsilon = (-1)^n$. Later, we shall be motivated by this example to generalize the symmetric and quadratic constructions of Chapter 5 to chain complexes of modules over a noncommutative ring R with involution.

8.14. Example. Let V be any R-module. Then $W = V \oplus V^*$ is also an R-module. We can define an ε-symmetric form on W by making use of the natural pairing between V and V^*:

$$\lambda\big((x_1, \varphi_1), (x_2, \varphi_2)\big) = \varphi_1(x_2) + \varepsilon\varphi_2(x_1)^*.$$

One checks easily that this is indeed a sesquilinear form and is ε-symmetric. It is called the *hyperbolic* ε-symmetric form associated to V.

Hyperbolic forms arise as the intersection forms of pairs of n-spheres in a $2n$-manifold which are embedded and meet transversely in a single point. This observation, made suitably precise, is at the core of surgery theory.

We have noticed several times that the information given in the usual intersection form does not fully account for all the geometry of *self*-intersections of middle-dimensional spheres in a manifold. The simplest example of this is the observation that the intersection matrix of a parallelizable $4k$-manifold has to be *even* (see the proof of Proposition 2.14); that is, the self-intersection numbers (diagonal entries) must be multiples of 2. We now seek a more refined algebra which includes this extra information.

8.15. Definition. Let V be an R-module and let $T_\varepsilon\colon \mathrm{Ses}(V) \to \mathrm{Ses}(V)$ be the ε-transposition map, as before. Define $Q_\varepsilon(V) = \mathrm{Coker}(1 - T_\varepsilon)\colon \mathrm{Ses}(V) \to \mathrm{Ses}(V)$. An element of $Q_\varepsilon(V)$ is called a *quadratic form* on V.

Compare this with Definition 8.12.

8.16. Example. Consider the case $V = R$. One easily checks that $\mathrm{Ses}(R)$ is identified with (the additive group of) R itself: the element $a \in R$ corresponds to the sesquilinear form $\lambda(x, y) = x^*ay$ on R. Thus if, for example, R is a commutative ring with the trivial involution $x^* = x$, $Q^{+1}(R) = R$ and $Q_{-1}(R) = R/\langle 2\rangle$. (Here $\langle 2\rangle$ denotes the principal ideal generated by $2 = 1 + 1$.)

Since $T_\varepsilon^2 = 1$, $\mathrm{Im}(1 - T_\varepsilon) \subseteq \mathrm{Ker}(1 + T_\varepsilon)$ and $\mathrm{Im}(1 + T_\varepsilon) \subseteq \mathrm{Ker}(1 - T_\varepsilon)$, so that $1 + T_\varepsilon$ gives a well-defined map $Q_\varepsilon(V) \to Q^\varepsilon(V)$. This map from quadratic to symmetric forms is called the *symmetrization map*. We can therefore regard a quadratic form as a symmetric form 'with extra structure'.

8.17. Definition. A quadratic form is said to be *nonsingular* if its symmetrization is nonsingular.

8.18. Lemma. *If 2 is invertible in R, then symmetrization gives an isomorphism between $Q_\varepsilon(V)$ and $Q^\varepsilon(V)$.*

PROOF. An easy consequence of the identity $\frac{1}{2}(1 + T_\varepsilon)(1 - T_\varepsilon) = 1$. □

8.19. Definition. A ε-symmetric form λ on a module V over a ring R is *even* if $\lambda(x, x) \in (1 + T_\varepsilon)R$ for all $x \in V$.

This extends the notion of even form over $\mathbb{Z}$ (Remark 1.8).

8.20. Proposition. *Let R be a ring with involution and let V be a free R-module. Then the image of the symmetrization map consists precisely of the even forms.*

PROOF. Clearly everything in the image of the symmetrization map is even. Suppose that the symmetric form λ is even and let $v_1, \ldots, v_n$ be a basis[1] for V. Then λ is completely determined by the matrix $\lambda_{ij} = \lambda(v_i, v_j)$, which satisfies $\lambda_{ij} = \varepsilon\lambda_{ji}^*$. Choose $a_{ij} \in R$ such that $a_{ij} = \lambda_{ij}$ if $i < j$, $a_{ij} = 0$ if $i > j$, and $(1 + \varepsilon)a_{ii} = \lambda_{ii}$. The matrix a then defines a sesquilinear form which symmetrizes to λ. □

8.21. Exercise. If R has trivial involution and the 'multiplication by 2' map $R \to R$ is injective, prove that every even $+1$-symmetric form on a free R-module is the symmetrization of a *unique* element of $Q_{+1}(V)$. Thus, in this case, quadratic forms correspond $1 : 1$ with even symmetric forms.

We can take this ideas further to give a precise description of the 'extra structure' in a quadratic form, at least over a free module.

8.22. Definition. Let V be an R-module and let λ be an ε-symmetric bilinear form on V. A *quadratic refinement* for λ consists of a function $\mu\colon V \to Q_\varepsilon(R)$, such that

(i) The identity $\mu(x + y) - \mu(x) - \mu(y) = [\lambda(x, y)]$ holds in $Q_\varepsilon(R)$ (here $[\lambda(x, y)]$ denotes the equivalence class of $\lambda(x, y) \in R$ under the quotient map $R \to Q_\varepsilon(R)$;

(ii) The identity $\mu(x) + \varepsilon\mu(x)^* = \lambda(x, x)$ holds in R. (Notice, here, that $\mu(x) + \varepsilon\mu(x)^* = (1 + T_\varepsilon)\mu(x)$ is a well-defined element of $Q^\varepsilon(R) \subseteq R$, since $1 + T_\varepsilon$ maps Q_ε to Q^ε.)

(iii) The identity $\mu(ax) = a^*\mu(x)a$ holds in $Q_\varepsilon(R)$. (Here we need to remark that even though $Q_\varepsilon(R)$ is *not* an R-module, the 'quadratic operation' $\mu \mapsto a^*\mu a$ is well-defined on it.)

[1] Everything works in the infinitely generated case but we don't burden the proof with the notation for that.

8.23. Proposition. *Over a free module V, there is a $1:1$ correspondence between quadratic forms, on the one hand, and symmetric forms equipped with quadratic refinements, on the other.*

PROOF. If λ is obtained by symmetrizing a sesquilinear form ψ, then $\mu(x) = \psi(x,x) \in Q_\varepsilon(R)$ is a quadratic refinement of λ.

Conversely, if V is free with basis $\{v_1, \dots, v_n\}$ and we are given a symmetric form λ on V with a quadratic refinement μ, then set

$$\lambda_{ij} = \lambda(v_i, v_j), \qquad \mu_i = \mu(v_i).$$

The matrix

$$b_{ij} = \begin{cases} \lambda_{ij} & \text{if } i < j \\ \mu_i & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

then specifies a well-defined element of $Q_\varepsilon(V)$ which symmetrizes to λ and has μ as associated quadratic refinement. ☐

8.24. Exercise. Extend the above argument to *projective* R-modules V (by embedding into a free module).

8.25. Example. Consider the hyperbolic ε-symmetric form on $W = V \oplus V^*$, defined in Example 8.14 above. An underlying ε-quadratic structure is provided by the function $\mu \colon W \to Q_\varepsilon R$ with $\mu(x, \varphi) = [\varphi(x)]$. As a matter of notation, the space W equipped with this *hyperbolic* ε-quadratic form will be denoted by $\mathcal{H}_\varepsilon(V)$. Note that the hyperbolic form can be defined directly as an element of $Q_\varepsilon(V)$; it corresponds to the sesquilinear ψ on $V \oplus V^*$ given by

$$\psi\big((x_1, \varphi), (x_2, \varphi_2)\big) = \varphi_1(x_2).$$

We are going to classify the nonsingular ε-quadratic forms on finite-dimensional vector spaces over certain fields.

8.26. Example. We begin with the case $R = \mathbb{R}$. Since $\frac{1}{2} \in \mathbb{R}$, there is no difference between quadratic and symmetric forms. By Sylvester's law of inertia, nonsingular symmetric bilinear forms over $\mathbb{R}$ are classified completely by their rank and signature. On the other hand, *skew*-symmetric bilinear forms over $\mathbb{R}$ are necessarily hyperbolic, with no invariant other than their rank (which must be even).

8.27. Example. The case $R = \mathbb{C}$ (with complex conjugation as the involution) is similar as regards symmetric (now usually known as *hermitian*) forms, which are classified by their rank and signature. Now, however, since the ring contains an element i such that $i^2 = -1$, there is no difference between symmetric and skew-symmetric forms; so skew-symmetric forms are also classified by their rank and signature.

8.28. Example. Now we consider the fundamental 2-torsion example, $R = \mathbb{F}_2$ the field of 2 elements. Thus we have a finite-dimensional vector space V over $\mathbb{F}_2$, and on this we have a bilinear $\lambda \colon V \times V \to \mathbb{F}_2$ and a function $\mu \colon V \to \mathbb{F}_2$ such that

$$\mu(x+y) = \mu(x) + \mu(y) + \lambda(x,y). \tag{8.29}$$

Notice that since $+1 = -1$ the distinction between symmetric and skew-symmetric forms has disappeared. By (8.29) applied to $x + x$ we have that $\lambda(x,x) = 0$ for all x. I claim that the symmetric part λ of the given quadratic form is hyperbolic, in fact it is a direct sum of elementary hyperbolic forms each of which has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on a two-dimensional subspace. We prove this inductively, so let $x \in V$ be any element, and let $y \in V$ be an element such that $\lambda(x, y) = 1$, $\lambda(v, y) = 0$ for all $v \in V \setminus \{x\}$. (There is such a y since the map $x \mapsto 1$, $V \setminus \{x\} \mapsto 0$ is linear over $\mathbb{F}_2$.) Then the subspace H spanned by x and y has $H \cap H^\perp = 0$, so that $V = H \oplus H^\perp$, and we have split off an elementary hyperbolic subspace. The assertion follows by induction.

Even though the *symmetric* structure of our form is now revealed to be hyperbolic, its *quadratic* structure need not be so. In fact, suppose now that x and y span an elementary hyperbolic subspace for λ. The identity (8.29) easily shows that, of the three numbers $\mu(x)$, $\mu(y)$, $\mu(x+y)$, either all three are 1 (call this case V_1) or two are 0 and the third is 1 (call this case V_0). We have proved therefore that our given quadratic form is isomorphic to a direct sum of copies of V_0 and V_1.

There is however a relation to be taken into account, namely $V_0 \oplus V_0 \cong V_1 \oplus V_1$. This can be proved by writing down an explicit isomorphism. In fact, if $\{x_1, y_1, x_2, y_2\}$ (with the obvious notation) is a basis for $V_0 \oplus V_0$, then the basis $\{x_1 + y_1 + x_2, x_1 + y_1 + y_2, x_1 + x_2 + y_2, y_1 + x_2 + y_2\}$ has $\mu = 1$ on each element, so exhibits an isomorphism with $V_1 \oplus V_1$. We conclude that our form is in fact isomorphic to the direct sum of a number of copies of V_0 together with at most one copy of $\mathcal{H}_1$.

This is as far as we can go: $\bigoplus^n V_0$ is *not* isomorphic to $\bigoplus^{n-1} V_0 \oplus V_1$, because one can count the number of elements of the vector space on which μ is nonzero, and this number is greater in the second case (see Exercise 8.30 below). We have therefore obtained a complete classification of quadratic forms (on finite-dimensional vector spaces) over $\mathbb{Z}_2$. If an V_1 factor appears we say that the form has *Arf invariant* 1; otherwise it has *Arf invariant* 0. Notice that the Arf invariant (considered as a member of $\mathbb{Z}_2$) is additive on direct sums.

8.30. Exercise. For a finite-dimensional $\mathbb{F}_2$-vector space V equipped with a quadratic form (as above), set $p(V) =$ number of elements of V on which $\mu = 1$, and $n(V) =$ number of elements of V on which $\mu = 0$. Show that

$$p(V \oplus V_0) = 3p(V) + n(V), \quad n(V \oplus V_0) = p(V) + 3n(V).$$

Deduce that $\bigoplus^n V_0$ is not isomorphic to $\bigoplus^{n-1} V_0 \oplus V_1$, as asserted above. [**9**, Lemma III.1.10]

8.31. Exercise. Show that a quadratic form over $\mathbb{F}_2$ with Arf invariant 0 is hyperbolic (in the sense of 8.25). Show that a quadratic form with Arf invariant 1 is *not* hyperbolic.

8.32. Exercise. Show that the Arf invariant of a quadratic form (λ, μ) on V may be defined as $\sum \mu(x_i)\mu(y_i)$, where $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is any *symplectic* basis for V. (A symplectic basis satisfies $\lambda(x_i, y_j) = \delta_{ij}$, $\lambda(x_i, x_j) = \lambda(y_i, y_j) = 0$.)

8.3. Lagrangians and hyperbolic forms

Suppose that V is an R-module equipped with a nonsingular bilinear form λ, which we assume is either symmetric or skew-symmetric. Then, for any submodule $U \leq V$ we can define the *orthogonal* $U^\perp$ in the natural way, namely

$$U^\perp = \{y \in V : \lambda(x,y) = 0 \forall x \in U\}.$$

In fact, we have already made use of this notion for vector spaces in our discussion of the Arf invariant. Notice that, in contrast to the familiar situation of orthogonal complements relative to a positive-definite inner product, it is possible for U and $U^\perp$ to intersect non-trivially.

The submodule U is said to be *complemented* in V if there is another submodule W such that $U \oplus W = V$.

8.33. Exercise. Show that if U is complemented then so is $U^\perp$.

8.34. Definition. Let V be an R-module equipped with a nonsingular ε-quadratic form (λ, μ). Let U be a submodule of V.

(i) U is called a *sublagrangian* if $U \subseteq U^\perp$, U is complemented, and $\mu_{|U} = 0$.
(ii) U is called a *lagrangian* if it is a sublagrangian and, in addition, $U = U^\perp$.

If V is a finite-dimensional vector space over a field R of characteristic not 2, then quadratic and symmetric forms coincide, and moreover all submodules are complemented. A sublagrangian is then what is usually called an 'isotropic subspace', and a lagrangian is a 'maximal isotropic subspace'.

8.35. Example. Let V be any R-module, and consider the hyperbolic ε-quadratic form on $W = V \oplus V^*$. Then V and V^* are complementary lagrangians.

In fact, this is the only example which can occur.

8.36. Theorem (Witt)**.** *If a nonsingular ε-quadratic form (V, λ, μ) on a finitely generated free R-module V admits a lagrangian U, then it is isomorphic to the hyperbolic form $\mathcal{H}_\varepsilon(U)$ generated by U.*

PROOF. We note first that if we can find another lagrangian W such that $V = U \oplus W$, then λ will identify W with the dual U^* of U and μ will be determined by the fact that it is zero on U and on W, so that it will follow that V is hyperbolic. We therefore aim to find a complementary lagrangian to U. The idea is to choose any complementary subspace and then modify it, à la Gram-Schmidt, so that it becomes lagrangian.

Let $i\colon U \to V$ be the inclusion. Dualizing and composing with the isomorphism $\lambda\colon V \to V^*$ we get a map $j\colon V \to U^*$. By definition, $\operatorname{Ker} j = U^\perp$, which equals U since U is lagrangian. Since a lagrangian is assumed to be complemented, j is a split surjection, and there is a $1:1$ correspondence between splittings $\theta\colon U^* \to V$ of j and complementary submodules to U in V. Fix one such splitting θ; then any other splitting is of the form $\theta + \varphi$, where φ is a homomorphism from U^* to $\operatorname{Ker} j = U$.

Now (using the fact that V is free) choose a sesquilinear form ψ on V such that $[\psi] \in Q_\varepsilon(V)$ corresponds to the quadratic form (λ, μ). We would like to choose our splitting $\theta + \varphi$ so that it corresponds to a complementary lagrangian to U, which is to say that $(\theta + \varphi)^*\psi(\theta + \varphi) = 0$. However we may compute

$$(\theta + \varphi)^*\psi(\theta + \varphi) = \theta^*\psi\theta + \varphi \in \operatorname{Hom}(U^*, U).$$

Thus we can achieve what we want by choosing $\varphi = -\theta^*\psi\theta$. $\square$

8.37. Remark. By an extension of this argument we may also prove that if U is a *sub*lagrangian in V, then there is an isomorphism of quadratic forms $V \cong \mathcal{H}_\varepsilon(U) \oplus U^\perp/U$.

8.38. Corollary. *Let (V, λ, μ) be a quadratic form on a finitely generated free R-module; then $(V, \lambda, \mu) \oplus (V, -\lambda, -\mu)$ is isomorphic to a hyperbolic form.*

PROOF. The diagonally embedded copy of V is lagrangian. $\square$

8.4. The even-dimensional L-groups

We may now define the even-dimensional L-groups. Let $\varepsilon = (-1)^n$. We define a group $L_{2n}(R)$ as follows: consider a semigroup with one generator for each isomorphism class of ε-quadratic forms on finitely generated free R-modules, with addition by direct sum, and with the imposed relation that every hyperbolic form represents zero (in other words, we take the free semigroup as described above, and divide by the subsemigroup generated by hyperbolic forms). By the corollary above, the quotient semigroup so defined is in fact a group; the inverse of (V, λ, μ) being $(V, -\lambda, -\mu)$.

8.39. Definition. This quotient group is denoted $L_{2n}(R)$.

We observe that $L_{2n}(R)$ is a covariant functor of R (by "change of rings").

8.40. Example. Here are the L-groups for the three fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_2$ that we previously considered.

R	$L_0(R)$	$L_2(R)$
$\mathbb{R}$	$\mathbb{Z}$	0
$\mathbb{C}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

These follow from the classification of symmetric and skew-symmetric forms over these fields, which we previously discussed. The isomorphisms $L_0(\mathbb{R}) \to \mathbb{Z}$, $L_0(\mathbb{C}) \to \mathbb{Z}$, and $L_2(\mathbb{C}) \to \mathbb{Z}$ are given by the signature. The isomorphisms $L_0(\mathbb{Z}_2) \to \mathbb{Z}_2$ and $L_2(\mathbb{Z}_2) \to \mathbb{Z}_2$ are given by the Arf invariant

8.41. Example. An example of some interest to analysts occurs when the ring R is a complex C^*-algebra A. In this case there is an identification between $L_0(A) = L_2(A)$ and the topological K-theory of the algebra A. We will assume that the reader is familiar with K-theory for C^*-algebras.

We consider, then, symmetric forms on free A-modules; in fact, without essential loss of generality, it is enough to consider symmetric forms on A itself. Such a form is given by

$$\lambda(x, y) = yTx^*$$

for some self-adjoint $T \in A$. If the form is nondegenerate, then T is invertible, that is, the spectrum $\sigma(T)$ of T does not contain zero.

Choose a continuous and bounded function $f\colon \mathbb{R} \to \mathbb{R}$ which is equal to zero on $\mathbb{R}^-$ and equal to one on $\mathbb{R}^+ \cap \sigma(T)$. The operator $e_+(T) = f(T)$ defined by the functional calculus does not depend on the choice of function f; it is called the *positive spectral projection* of T. Similarly we may define the *negative spectral projection* $e_-(T)$ and we note that $e_-(T) + e_+(T) = 1$. Now define a new symmetric form λ' by

$$\lambda'(x, y) = y(-e_-(T) + e_+(T))x^*.$$

I claim that λ' and λ are isomorphic as forms; indeed, the spectral theorem provides an invertible operator $S = |T|^{1/2}$ such that $\lambda'(Sx, Sy) = \lambda(x, y)$. We conclude that, over a C^*-algebra A, any nondegenerate form is isomorphic to one arising as the 'difference' of two projections.

The *analytic signature* of λ is the class $[e_+] - [e_-]$ in $K_0(A)$. The analytic signature of a hyperbolic form is zero, so we get a map $L_0(A) \to K_0(A)$. The discussion above shows that this map is almost an isomorphism from L-theory to K-theory. The reason it isn't exactly an isomorphism is that the two projections are related by the requirement that their sum represent a free module. If we use the variant definition L^p of L-theory, made out of quadratic forms on f.g. *projective* modules, then the analytic signature gives an isomorphism $L_0^p(A) \to K_0(A)$ for any unital C^*-algebra A.

8.42. Exercise. Show that we can describe our original L-theory $L_0(A) = L_0^h(A)$ in terms of K-theory as well. In fact, show that the map which sends the form to the pair (e_-, e_+) gives an isomorphism between $L_0(A)$ and the group G which consists of pairs $(x, y) \in K_0(A) \times K_0(A)$ such that $x + y$ vanishes in reduced K-theory $\tilde{K}_0(A)$, modulo the subgroup which is the image of the diagonal embedding $\mathbb{C} \to A \oplus A$. Show that this group only differs from $K_0(A)$ by 2-primary torsion.

8.5. Computation of $L_{2k}(\mathbb{Z})$

The main point of L-theory is that it should be applicable to group rings, and the alert reader will have noticed that we have not computed the L-theory groups of a single group ring so far. In this section we will make some amends by calculating the simplest example, the L-theory of $\mathbb{Z}$ (which is of course the group ring of the trivial group). This object made an implicit appearance in Chapter 2. Even in this simple case, some substantial input from number theory is required.

Let $\alpha\colon \mathbb{Z} \to \mathbb{R}$ and $\beta\colon \mathbb{Z} \to \mathbb{Z}_2$ denote the obvious homomorphisms. We aim to prove the following two results.

8.43. Proposition. *$\alpha_*\colon L_0(\mathbb{Z}) \to L_0(\mathbb{R}) = \mathbb{Z}$ is injective, and its image is $8\mathbb{Z}$.*

8.44. Proposition. *$\beta_*\colon L_2(\mathbb{Z}) \to L_2(\mathbb{Z}_2) = \mathbb{Z}_2$ is an isomorphism.*

We begin with the symmetric case ($n = 0$). Here our task is to classify *even* nonsingular symmetric forms (which are the same as nonsingular quadratic forms by Exercise 8.21) on finitely generated free abelian groups. It turns out to be helpful to begin with a stable classification of *all* nonsingular symmetric forms.

8.45. Definition. We define $\mathcal{K}$ to be the Grothendieck group constructed from the semigroup of isomorphism classes of nonsingular symmetric forms on finitely generated free abelian groups.

Let I_+ and I_- denote the rank 1 forms $\lambda(x, y) = xy$ and $\lambda(x, y) = -xy$.

8.46. Proposition. *The group $\mathcal{K}$ is free abelian generated by $[I_+]$ and $[I_-]$.*

PROOF. It will suffice to prove that any odd indefinite form is isomorphic to a direct sum of copies of I_+ and I_-. For certainly any form is can be made odd and indefinite by adding $I_+ \oplus I_-$, so this will prove that $[I_+]$ and $[I_-]$ generate $\mathcal{K}$; on the other hand, the pair (rank,signature) gives a homomorphism $\mathcal{K} \to \mathbb{Z} \times \mathbb{Z}$ under which the images of $[I_+]$ and $[I_-]$ are linearly independent, so $\mathcal{K}$ must in fact be free on these classes.

We work by induction on the rank. Let λ be an odd indefinite form on a $\mathbb{Z}$-module V of rank n. By the number-theoretic Theorem 2.46 there exists $x \in V$ such that $\lambda(x, x) = 0$. We may assume that x is *indivisible* (i.e. that it cannot be written as a nontrivial integer multiple of any other vector) and from this and the unimodularity of the form it follows that there exists $y \in V$ such that $\lambda(x, y) = 1$. Because λ is odd, a simple argument shows that we may choose such a y with $\lambda(y, y) = 2m+1$ an odd number. Now let $x' = y - mx$, $y' = y - (m + 1)x$; then we have the following table of values for λ:

$$\begin{array}{ccc} & x' & y' \\ x' & 1 & 0 \\ y' & 0 & -1 \end{array}$$

and hence $V = I_+ \oplus I_- \oplus W$, where W is a module of rank $n - 2$. Now one of $I_+ \oplus W$, $I_- \oplus W$ is odd indefinite and of rank $n - 1$, so the induction may proceed. □

Now we can prove something we have already asserted and used, that the signature maps $L_0(\mathbb{Z})$ to $8\mathbb{Z}$.

8.47. Proposition (van der Blij)**.** *The signature of an even symmetric form (that is, a quadratic form) over $\mathbb{Z}$ is a multiple of 8.*

PROOF. We have observed that the signature gives a homomorphism $\mathcal{K} \to \mathbb{Z}$. We will now define a related map $\sigma\colon \mathcal{K} \to \mathbb{Z}/8$. Given a symmetric form λ on a free $\mathbb{Z}$-module V, let $\bar\lambda$ be the associated form on the vector space $\bar V = V \otimes \mathbb{F}_2 = V/2V$ over $\mathbb{F}_2$. On $\bar V$ the functional $\xi \mapsto \bar\lambda(\xi,\xi)$ is *linear* and hence, by duality, there is a canonical element $\zeta \in \bar V$ such that

$$\lambda(\zeta,\xi) = \lambda(\xi,\xi)$$

for all $\xi \in \bar V$. Let $z \in V$ be a lift of ζ; it is unique modulo $2V$. Then $\lambda(z,z) \in \mathbb{Z}$ is well-defined modulo 8, since

$$\lambda(z+2x, z+2x) = \lambda(z,z) + 4(\lambda(x,z) + \lambda(x,x))$$

and $\lambda(x,x)$ agrees modulo 2 with $\lambda(z,x)$. This residue class modulo 8 is the invariant σ of the form (V,λ).

I now claim that σ is exactly the reduction of the signature modulo 8. Since σ and the signature both give homomorphisms on $\mathcal{K}$, it suffices to check this assertion on the generators $[I_+]$ and $[I_-]$ of $\mathcal{K}$, and there it is easy. But now, if λ is an *even* form, then $\bar\lambda(\xi,\xi)$ (in the notation above) vanishes identically on $\bar V$, so $\zeta = 0$ and we may take $z = 0$, whence $\sigma = 0$. The conclusion follows. $\square$

8.48. Remark. Note that in the geometric situation of Proposition 2.14, the quantity ζ appearing in the above proof is exactly the Wu class.

8.49. Exercise.

Now we can do the computation of $L_0(\mathbb{Z})$. Let us first prove

8.50. Lemma. *Every (nonzero) class in $L_0(\mathbb{Z})$ can be represented by a definite form.*

PROOF. It suffices to show that a hyperbolic summand can be split off from any even indefinite form. Let (V,λ) be such a form, let $x \in V$ be indivisible with $\lambda(x,x) = 0$ and let $y \in V$ have $\lambda(x,y) = 1$. Then $\lambda(y,y) = 2m$ for some m, and $x' = x$ and $y' = y - mx$ span a hyperbolic summand in V. $\square$

PROOF OF PROPOSITION 8.43. Lemma 8.50 proves that the signature homomorphism $L_0(\mathbb{Z}) \to \mathbb{Z}$ is injective (since the signature of a definite form is equal to plus or minus its rank). On the other hand, van der Blij's lemma shows that the image of this homomorphism is contained in $8\mathbb{Z}$, and the existence of the even definite form E_8 of rank 8 shows that the image is actually equal to $8\mathbb{Z}$. $\square$

PROOF OF PROPOSITION 8.44. Suppose that λ is a skew-symmetric form on a free $\mathbb{Z}$-module V. An inductive argument (similar to but simpler than those we have already carried out) shows that V is an orthogonal direct sum of 2-dimensional subspaces on each of which the form λ is hyperbolic, having matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If we now consider a quadratic refinement $\mu\colon V \to Q_-(\mathbb{Z}) = \mathbb{Z}_2$, essentially the same discussion as in Example 8.28 applies to show that there are only two distinct possibilities for μ on each hyperbolic summand; moreover, denoting these possibilities V_0 and V_1 as before, we still have the isomorphism $V_0 \oplus V_0 \cong V_1 \oplus V_1$. At this point we know that there are at most two skew-quadratic forms over $\mathbb{Z}$ up to stable isomorphism; but they are certainly distinguished by the Arf invariants of their mod 2 reductions. This is 8.44. $\square$

8.6. The Arf invariant and topology

In this section we shall describe one natural topological situation in which the Arf invariant arises. This shows a relationship between the Arf invariant and the stable homotopy group π_2^s. At the same time, the techniques that we shall use to find a quadratic refinement of the intersection form — counting self-intersections and destabilization obstructions of immersed spheres — will be exactly the ones that are needed to define the surgery obstruction in the general case.

We are going to consider stably framed $(4k+2)$-manifolds M which are $2k$-connected. It follows that the Hurewicz homomorphism $\pi_{2k+1}(M) \to H_{2k+1}(M)$ is an isomorphism. The most important example for this section is the 2-torus $M = \mathbb{T}^2$.

It follows from Proposition 4.25 that each element of $H_{2k+1}(M)$ can be represented by a self-transverse immersion $f\colon S^{2k+1} \to M$, which is unique up to homotopy. Since the tangent bundle to M is stably framed (by hypothesis) and the tangent bundle to S^{2k+1} is stably framed (by the standard embedding $S^{2k+1} \to \mathbb{R}^{2k+2}$)[2], the normal bundle ν_f to f is stably framed. Recall (Definition 3.27) that the *destabilization obstruction* $\mathfrak{d}(f) \in \mathbb{Z}_2$ measures whether or not it is possible to reduce this stable framing of the normal bundle to a genuine framing.

8.51. Definition. For a self-transverse immersion $f\colon S^{2k+1} \to M$, define $\mathfrak{n}(f) \in \mathbb{Z}_2$ to be the total number (mod 2) of self-intersection points of the immersion f. Further define $\mu(f) \in \mathbb{Z}_2$ to be $\mathfrak{d}(f) + \mathfrak{n}(f)$.

We shall show in the next chapter (Proposition **??**) that the quantity $\mu(f)$ is a *homotopy invariant* of the immersion f. Thus we may consider μ as a $\mathbb{Z}_2$-valued function on the homology group $H_{2k+1}(M)$.

8.52. Proposition. *The function μ defined above is a quadratic refinement of the intersection form λ on $H_{2k+1}(M;\mathbb{Z})$.*

PROOF. It suffices to establish part (i) of Definition 8.22; in this case (skew-symmetric forms over $R = \mathbb{Z}$) part (ii) is trivial and part (iii) follows from part (i). That is, we must show that for homology classes x, y,

$$\mu(x+y) - \mu(x) - \mu(y) = \lambda(x,y).$$

Suppose that x and y are represented by transverse immersions f and g. Then $x + y$ is represented by the *connected sum* of f and g, which is an immersion $h\colon S^{2k+1} \# S^{2k+1} = S^{2k+1} \to M$ defined as follows (see figure 1). Run a path in M from a regular point (i.e. not a self-intersection point) $p \in f(S^{2k+1})$ to a regular point $q \in g(S^{2k+1})$. The path has a tubular neighborhood U diffeomorphic to $[0,1] \times \mathbb{R}^{4k+1}$, in such a way that $f(S^{2k+1})$ meets U in a copy of S^{2k} standardly embedded in $\{0\} \times \mathbb{R}^{2k+1} \times \{0\} \subset [0,1] \times \mathbb{R}^{4k+1}$, and similarly $g(S^{2k+1})$ meets U in a copy of S^{2k} standardly embedded in $\{1\} \times \mathbb{R}^{2k+1} \times \{0\} \subset [0,1] \times \mathbb{R}^{4k+1}$. We wish to join these two copies of S^{2k} by an immersion of a tube $[0,1] \times S^{2k}$ in U. There are two cases to consider:

(a) If the two copies of S^{2k} acquire opposite orientations from the immersed S^{2k+1}s of which they are a part, then we may simply connect them by the natural *embedding* of the product $[0,1] \times S^{2k}$ in $[0,1] \times \mathbb{R}^{2k+1} \times \{0\} \subset [0,1] \times \mathbb{R}^{4k+1}$.

[2] In cases $k = 0, 1, 3$ the tangent bundle to S^{2k+1} is actually trivial. Nevertheless it is very important to note that the framing provided by this trivialization is *not compatible* with the stable framing that we are using. Compare Example 3.29.

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FIGURE 1. Connected sum of immersions

(b) If on the other hand the two copies of S^{2k} acquire the same orientations, then we must connect them by an immersion of $[0,1] \times S^{2k}$ in $[0,1] \times \mathbb{R}^{4k+1}$ which reverses orientation. An explicit example of such a 'figure eight' immersion is described in Section 10.1; it has a single double point and its destabilization obstruction is equal to 1.

We now investigate the self-intersection points and destabilization obstructions of the connected sum h. The self-intersections of h comprise the self-intersections of f, the self-intersections of g, and the mutual intersections of f and g, together in case (b) above with the one extra self-intersection point introduced by the figure eight immersion. As for the destabilization obstruction, it is the sum of the destabilization obstructions of f and of g, together in case (b) above with an extra 1 coming from the figure eight immersion. Thus we have the following table

	$\mathfrak{n}(h)$	$\mathfrak{d}(h)$	$\mu(h)$
Case (a)	$\mathfrak{n}(f)+\mathfrak{n}(g)+\lambda(f,g)$	$\mathfrak{d}(f)+\mathfrak{d}(g)$	$\mu(f)+\mu(g)+\lambda(f,g)$
Case (b)	$\mathfrak{n}(f)+\mathfrak{n}(g)+\lambda(f,g)+1$	$\mathfrak{d}(f)+\mathfrak{d}(g)+1$	$\mu(f)+\mu(g)+\lambda(f,g)$

from which it can be seen that $\mu(x+y)-\mu(x)-\mu(y)=\lambda(x,y)$ in all cases. □

8.53. Definition. The *Arf invariant* of the stably framed manifold M is the Arf invariant of the quadratic refinement of its intersection form, described above.

We are going to carry out some calculations for M a surface embedded in $\mathbb{R}^3$. Let $F(M)$ be the principal bundle of oriented orthonormal 2-frames in M. The map

$$(\mathbf{x},\mathbf{y}) \mapsto (\mathbf{x},\mathbf{y},\mathbf{x}\times\mathbf{y})$$

(where $\times$ denotes the 'cross product' of vectors in $\mathbb{R}^3$) sends each 2-frame to an oriented 3-frame in $\mathbb{R}^3$, related to the standard 3-frame by an element of $SO(3)$. In this way be obtain a bundle map

$$F(M) \to SO(3) \times M.$$

Let $\widetilde{F}(M)$ denote the pull-back of $\mathrm{Spin}(3) \times M$ over this bundle map. It is a double cover of $F(M)$ called the *spin structure* induced by the embedding $M \to \mathbb{R}^3$.

We give M the stable framing induced by its embedding in $\mathbb{R}^3$.

8.54. Exercise. Let $f\colon S^1 \to M$ be an embedding. Show that $\mathfrak{d}(f)=0$ if and only if the unit tangent map

$$df\colon S^1 \to S(T(M)) \cong F(M)$$

does not lift to a map $S^1 \to \widetilde{F}(M)$. Deduce that if f is nullhomotopic, then $\mathfrak{d}(f) = 0$.

8.55. Exercise. Show that if $f\colon S^1 \to M$ is an embedding, and if there is an embedding $D^2 \to \mathbb{R}^3$ which extends f and which intersects M transversely and orthogonally along $\mathrm{Im}(f)$, then $\mathfrak{d}(f) = 0$. (Use Exercise 8.54.)

8.56. Example. Let $M = \mathbb{T}^2$ be the 2-torus. It acquires a stable framing $\mathfrak{f}_0$ from its embedding as a two-sided submanifold of $\mathbb{R}^3$. Let x and y be the standard homology generators. Then x, y, and $x + y$ may be represented by *embeddings* f, g, h of S^1 in M.

By Exercise 8.55, $\mathfrak{d}(f) = \mathfrak{d}(g) = 0$. It follows that $\mathfrak{d}(h) = 1$ (one can also check this directly). Thus the Arf invariant of the torus with framing $\mathfrak{f}_0$ is 0.

8.57. Example. By contrast let us now consider the torus $\mathbb{T}^2$ with the translation-invariant framing $\mathfrak{f}_1$ of its tangent bundle coming from the Lie group structure. With this framing we easily see that each of f, g, h have destabilization obstruction 1 (this essentially comes from the destabilization obstruction of $T(S^1)$ with its Lie-invariant framing, see Example 3.29). Thus the Arf invariant of the torus with framing $\mathfrak{f}_1$ is 1.

It can be shown that the Arf invariant is an invariant of framed cobordism. (We don't give a proof here, but this will follow from later results in

.) The two framings of the torus described above therefore give the two different elements of

$$\Omega_2^{fr} \cong \pi_2^s \cong \mathbb{Z}_2$$

(see Theorem 7.18), and the Arf invariant gives an isomorphism $\Omega_2^{fr} \to \mathbb{Z}_2$. This is how Pontrjagin computed π_2^s.

CHAPTER 9

Intersections and the fundamental group

In this chapter we shall refine the duality theory of Chapter 6. We want to use algebra to count intersections in manifolds M that are no longer simply connected. We also want to find the correct algebraic formulation of the notion of *self*-intersection of a middle-dimensional submanifold of such an M. It will turn out that the quadratic algebra of the previous chapter is exactly what is needed for that purpose.

9.1. Geometric versus algebraic intersections II

Theorem 6.31 shows that, in a simply connected manifold M, the algebraic intersection numbers of submanifolds[1] (defined by Poincaré duality) correspond after deformation to the natural geometric intersection numbers. In particular, if two submanifolds have algebraic intersection zero, they can be made disjoint by an isotopy. But, as Example 6.30 demonstrates, this statement is no longer true in a non-simply-connected M. To arrive at a suitable counterpart of Theorem 6.31, it is necessary to develop a theory of *equivariant intersection numbers*.

9.1. Definition. Let M be a compact connected manifold, with a preferred basepoint. A π_1*-trivial* submanifold N consists of a connected submanifold $N \subseteq M$ such that the image of $\pi_1 N \to \pi_1 M$ is the trivial group, together with a preferred homotopy class of paths from the basepoint of M to some fixed point of N.

We will usually apply this theory to simply connected manifolds N (namely spheres).

9.2. Remark. It amounts to the same thing to say that N is π_1-trivial if the inverse image of N under the universal covering map $\tilde{M} \to M$ is identified with a product $N \times \pi_1 M$. Equivalently, N is π_1-trivial if there is given a submanifold $\tilde{N}$ of $\tilde{M}$ such that the universal covering map is a homeomorphism $\tilde{N} \to N$.

Suppose now that N_1 and N_2 are two oriented π_1-trivial submanifolds of M, of complementary dimensions and intersecting transversely. Then to each intersection point $p \in N_1 \cap N_2$ we may associate an element $g_p \in \pi_1 M$, namely the homotopy class of the path that runs from the basepoint in M, via the preferred route to the preferred point of N_1, then by a path in N_1 to p, then back from p by a path in N_2 to the preferred point in N_2, and back by the preferred route to the basepoint in M. The assumption that N_1 and N_2 are π_1-trivial tells us that the choices that we made (of a path in N_1 from its basepoint to p, and of a path in N_2 from p to its basepoint) do not affect the homotopy class of g in $\pi_1 M$.

Suppose further that we now choose a local orientation for M at the basepoint (we can always do this, even though M need not be globally orientable). We can define a sign $\varepsilon(p) \in \{\pm 1\}$ for the intersection point p by comparing the orientation at p induced from the orientations of N_1 and N_2 with the orientation transported from the basepoint along the path for N_1.

9.3. Definition. In the situation of the previous paragraphs, let $\pi = \pi_1(M)$. Define the *equivariant intersection number* of N_1 and N_2, $\lambda(N_1, N_2)_\pi \in \mathbb{Z}[\pi]$ by

$$\lambda(N_1, N_2)_\pi = \sum_{p \in N_1 \cap N_2} \varepsilon(p) g_p.$$

We will omit the subscript π if it is clear from the context that we are dealling with *equivariant* intersections.

An alternative version of this definition can be given by considering the universal cover $\tilde{M}$ of M. As we remarked above, a π_1-trivial submanifold $N \subseteq M$ can equivalently be defined as one for which there is given a submanifold $\tilde{N}$ of $\tilde{M}$ that is mapped homeomorphically onto N by the universal covering map $\tilde{M} \to M$. Now suppose that we have two transversely intersecting π-trivial submanifolds N_1 and N_2 as above. The universal cover $\tilde{M}$ is oriented by the choice of orientation

[1] With suitable restrictions on the dimension, see the statement of the theorem for details.

at the basepoint of M. Then, for each $g \in \pi$, the submanifolds $\tilde{N}_1$ and $g^{-1}\tilde{N}_2$ have an ordinary intersection number $\lambda(\tilde{N}_1 : g^{-1}\tilde{N}_2)$, and it is not hard to verify the identity

$$\lambda(N_1, N_2)_\pi = \sum_{g \in G} \lambda(\tilde{N}_1 : g^{-1}\tilde{N}_2)g.$$

9.4. Remark. The ordinary intersection numbers are symmetric up to a sign which depends only on the dimension of N_1 and N_2; see Equation 3.14. The corresponding symmetry property of the equivariant intersection numbers is a bit more subtle, because our choice of sign $\varepsilon(p)$ for the intersection point p of N_1 and N_2 depended on transporting the orientation for M from the basepoint to p along a path *through* N_1. If we interchange N_1 and N_2 we may therefore change the transported orientation of M (and thus the sign of the intersection point) by a factor $w(g) = \pm 1$ according to whether the loop $g \in \pi_1(M)$ preserves or reverses the orientation of M. Interchanging N_1 and N_2 will also replace the associated loop $g \in \pi_1(M)$ by g^{-1}. Thus the equivariant counterpart of Equation 3.14 is

$$\lambda(N_2, N_1) = (-1)^{n_1 n_2} \lambda(N_1, N_2)^*.$$

Here $*$ is the *w-twisted involution* on $\mathbb{Z}[\pi]$ (Proposition 8.5) associated to the homomorphism $w\colon \pi_1 M \to \{\pm 1\}$. Recall that this homomorphism w defines the first Stiefel-Whitney class in $H^1(M; \mathbb{Z}_2)$ (Exercise 5.31).

9.5. Definition. If $x = \sum n_g g$ belongs to $\mathbb{Z}[\pi]$, we define $|x| = \sum |n_g|$.

We can now generalize Theorem 6.31 as follows. The statement is the same, except that simple-connectedness of M has been weakened to π_1-triviality of the submanifolds, and we use equivariant intersection numbers.

9.6. Theorem. *Let M be an n-dimensional manifold provided with a local orientation at its basepoint. Suppose that $N_1^{k_1}$ and $N_2^{k_2}$ are transversely intersecting oriented π_1-trivial submanifolds of M, $n = k_1 + k_2$, $k_1, k_2 \geqslant 3$. Then there exists an ambient isotopy of N_1 to a submanifold N_1' which intersects N_2 in precisely $|\lambda([N_1], [N_2])|$ points. In particular, if $\lambda([N_1], [N_2]) = 0$, then N_1 and N_2 can be made disjoint by an ambient isotopy.*

PROOF. The proof is the same as that of Theorem 6.31. Suppose that p and q are intersection points of N_1 and N_2 which contribute respectively $+g$ and $-g$ to the equivariant intersection number of N_1 and N_2. Let γ_1 and γ_2 be paths in N_1 and N_2 respectively from p to q. Then the path $\gamma_1^{-1}\gamma_2$ is homotopic in M to $g^{-1}g = 1$. Since the signs of the intersection points are opposite, Lemma 4.26 applies to show that they can be canceled by an ambient isotopy. $\square$

9.2. Homology with $\mathbb{Z}[\pi]$-module coefficients

Suppose that X is a finite simplicial complex, and recall that $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{Z})$ denotes the simplicial chain complex of X with integer coefficients. The universal cover $\tilde{X}$ of X is also a simplicial complex, on which $\pi = \pi_1(X)$ acts freely[2] with one π-orbit of simplices in $\tilde{X}$ corresponding to each individual simplex of X. It follows then that $\mathcal{C}(\tilde{X})$ may be thought of as a complex of finitely generated, free right $\mathbb{Z}[\pi]$-modules.

9.7. Exercise. Show that the total singular complex of $\tilde{X}$ also has the structure of a complex of $\mathbb{Z}[\pi]$-modules, and that it is chain equivalent to $\mathcal{C}(\tilde{X})$ as a complex of such modules. (In the next section we will see a systematic procedure for obtaining results of this sort from their non-equivariant counterparts.)

9.8. Definition. Let V be a left $\mathbb{Z}[\pi]$-module. The *homology of X with coefficients V*, written $H_*^{\pi}(X;V)$, is the homology of the complex

$$\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} V$$

of abelian groups.

In general $H_*^{\pi}(X;V)$ is an abelian group; if V is a $\mathbb{Z}[\pi]$-bimodule, then the homology is naturally a right $\mathbb{Z}[\pi]$-module.

9.9. Definition. Let W be a right $\mathbb{Z}[\pi]$-module. The *cohomology of X with coefficients W*, written $H^*_{\pi}(X;W)$, is the cohomology of the complex

$$\mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathcal{C}(\tilde{X}, W))$$

of abelian groups.

In general $H^*_{\pi}(X;W)$ is an abelian group; if W is a $\mathbb{Z}[\pi]$-bimodule, then the cohomology is naturally a left $\mathbb{Z}[\pi]$-module.

9.10. Remark. There is a pairing

$$H^k_{\pi}(X;W) \otimes_{\mathbb{Z}} H_k^{\pi}(X;V) \to V \otimes_{\mathbb{Z}[\pi]} W$$

between homology and cohomology.

9.11. Example. The group $\mathbb{Z}$ has a natural $\mathbb{Z}[\pi]$-bimodule structure in which every group element acts as the identity. The homology and cohomology groups $H_*^{\pi}(X;\mathbb{Z})$ and $H^*_{\pi}(X;\mathbb{Z})$ given by Definitions 9.8 and 9.9 are canonically isomorphic to the usual homology and cohomology groups of X with integer coefficients. Here is why: there is an obvious map of complexes of abelian groups

$$\mathcal{C}(\tilde{X}) \to \mathcal{C}(X)$$

which sends each cell of $\tilde{X}$ to its image in X. Since the image of a cell of $\tilde{X}$ under this map is the same as the image of each of its π-translates, the map passes to the quotient

$$\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z} \to \mathcal{C}(X).$$

This latter map is easily seen to be an isomorphism of chain complexes. A similar argument applies to cohomology.

[2]Our convention is that π acts on the *right*.

9.12. Example. Now consider homology and cohomology with coefficients in the bimodule $\mathbb{Z}[\pi]$. Since $\mathcal{C}(X) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi] = \mathcal{C}(X)$, the homology groups $H_*^\pi(X;\mathbb{Z}[\pi])$ with coefficients $\mathbb{Z}[\pi]$ are just the ordinary homology groups of $\tilde{X}$ with the natural right action of π.

The cohomology groups $H^*_\pi(X;\mathbb{Z}[\pi])$ with coefficients $\mathbb{Z}[\pi]$ are the *compactly supported* cohomology groups of $\tilde{X}$, with the natural left action of π. To see this, we note that a $\mathbb{Z}[\pi]$-module homomorphism from $\mathcal{C}(\tilde{X})$ to $\mathbb{Z}[\pi]$ is the same thing as a $\mathbb{Z}$-module homomorphism φ from $\mathcal{C}(\tilde{X})$ to $\mathbb{Z}$ with the additional constraint that for each cell σ, $\varphi(g\cdot\sigma)$ is nonzero for only finitely many $g \in \pi$. But since X has only finitely many simplices, this is exactly the same as a compactly supported cochain for $\tilde{X}$.

9.13. Example. As an explicit example, let us consider $X = S^1$ with its usual cell structure[3] with one 0-cell and one 1-cell. We take $\pi = \pi_1(S^1) = \mathbb{Z}$, so $\mathbb{Z}[\pi] = \mathbb{Z}[t,t^{-1}]$. The complex $\mathcal{C}(\tilde{X})$ is then

$$0 \longleftarrow \mathbb{Z}[t,t^{-1}] \xleftarrow{1-t} \mathbb{Z}[t,t^{-1}] \longleftarrow 0$$

One readily computes that its homology is $\mathbb{Z}$ in dimension 0 and trivial in dimension 1, in agreement with the ordinary homology of the universal cover $\tilde{X} = \mathbb{R}$.

Similarly the dual complex $\operatorname{Hom}(\mathcal{C}(\tilde{X}, \mathbb{Z}[\pi])$ is

$$0 \longrightarrow \mathbb{Z}[t,t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t,t^{-1}] \longrightarrow 0$$

whose cohomology is trivial in dimension 0 and $\mathbb{Z}$ in dimension 1, in agreement with the compactly supported cohomology of $\mathbb{R}$. Notice that Poincaré duality still seems to hold! We will investigate this in the next section.

With reference to this example, one might be puzzled by the following question: homology (with $\mathbb{Z}[\pi]$ coefficients) is naturally a right $\mathbb{Z}[\pi]$-module, cohomology is naturally a left module. How then can there be a natural Poincaré duality isomorphism between them? The answer is that $\mathbb{Z}[\pi]$ is provided with some extra structure — an involution — which relates left and right actions. The involution allows us to understand the symmetry properties of Poincaré duality , which are critical in setting up the surgery obstruction groups.

9.14. Example. Let $w\colon \pi_1(X) \to \{\pm 1\}$ be a homomorphism. Recall from Exercise 8.7 that we defined $\mathbb{Z}^w$ to be the $\mathbb{Z}[\pi]$-bimodule structure on $\mathbb{Z}$ for which a group element g acts on the right by 1 and on the left by $w(g)$. The homology $H_*(X;\mathbb{Z}^w)$ is then what is usually called the homology with *local coefficients* in the local coefficient system defined by w.

Suppose in particular that X is a combinatorial homology n-manifold. Then for each vertex $\hat{\sigma}$ of the barycentric subdivision of X we have

$$H_k(X, X \ominus \hat{\sigma}; \mathbb{Z}^w) \cong \begin{cases} \mathbb{Z} & \text{when } k = n \\ 0 & \text{otherwise} \end{cases}$$

since the closed star of $\hat{\sigma}$ is simply connected. We make the following definition, analogous to 6.20.

[3] We described homology and cohomology for *simplicial* complexes above, because we shall use simplicial methods in the next section; but there is a completely analogous version of the theory for *cell* complexes.

9.15. Definition. An *orientation* of X for the orientation character w is a class $[X] \in H_n^\pi(X; \mathbb{Z}^w)$ which restricts to a generator of $H_n^\pi(X, X \ominus \hat{\sigma}; \mathbb{Z}^w)$ for each vertex $\hat{\sigma}$ of the barycentric subdivision.

When $w = 1$ this reduces to Definition 6.20. But now we have

9.16. Proposition. *Every (connected, compact) manifold X is orientable for the orientation character defined by the first Stiefel-Whitney class.*

PROOF. Any manifold X has an *orientation cover* $\bar{X}$, a $\mathbb{Z}_2$ cover whose fiber over $p \in X$ consists of the possible orientations of X at p. The orientation cover $\bar{X}$ is the $\mathbb{Z}_2$-cover associated to the Stiefel-Whitney class $w = w_1 \colon \pi_1 X \to \mathbb{Z}_2$, and it is (tautologically) orientable. Moreover, an orientation for $\bar{X}$ can be described by a chain which is *equivariant* for the natural $\mathbb{Z}_2$ action on $\bar{X}$. However, one can see as in Example 9.11 that the complex of such equivariant chains can be naturally identified with $\mathcal{C}(\tilde{X}) \otimes \mathbb{Z}^w$. $\square$

9.3. Geometric modules and assembly

Fix a group π. In this section, we are going to consider finite simplicial complexes with fundamental group π. In fact for each such complex K we are going to define a functor, called the *assembly functor*, from the category of $(\mathbb{Z}, K)$-modules (as defined in Chapter 6) to the category of $\mathbb{Z}[\pi]$-modules.

Let $\tilde{K}$ be the universal cover of K. The covering map $q\colon \tilde{K} \to K$ takes each simplex of $\tilde{K}$ to a simplex of K. The inverse image under q of a simplex σ of K is a free π-orbit of simplices of $\tilde{K}$. Thus there exist isomorphisms of π-spaces

$$q^{-1}(\sigma) \cong \sigma \times \pi;$$

but, and this is very important, there is no *canonical choice* of such an isomorphism, because there is no canonical choice of 'base simplex' among the various simplices making up the orbit[4].

Recall that a $(\mathbb{Z}, K)$-module M is just a list $\{M_\sigma\}$ of $\mathbb{Z}$-modules parameterized by the simplices of K.

9.17. Definition. The *assembly* of the $(\mathbb{Z}, K)$-module M is the $\mathbb{Z}[\pi]$-module $A(M)$ described as follows. As a $\mathbb{Z}$-module, $A(M)$ is a direct sum over all the simplices of $\tilde{K}$; the summand corresponding to a simplex $\tilde{\sigma}$ of $\tilde{K}$ is a copy of $M_{q(\tilde{\sigma})}$. The π-action is given by the π-action on the simplices of $\tilde{M}$.

9.18. Remark. By choosing a 'base simplex' in each π-orbit, we can find module-isomorphisms $A(M) \cong M \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]$. However, these isomorphisms are not canonical, for the reasons explained above; in particular, they are not necessarily consistent with the process of assembly of $(\mathbb{Z}, K)$-module morphisms, which we shall describe next.

Recall in fact that a $(\mathbb{Z}, K)$-module morphism $\varphi\colon M \to N$ is a list $\{\varphi_{\sigma,\tau}\}$ of $\mathbb{Z}$-module morphisms $M_\sigma \to N_\tau$, such that $\varphi_{\sigma,\tau}$ is zero unless $\sigma \leqslant \tau$ (that is, unless σ is a face of τ).

9.19. Definition. The *assembly* of the $(\mathbb{Z}, K)$-module morphism $\varphi\colon M \to N$ is a direct sum over pairs of simplices of $\tilde{K}$; if $\tilde{\sigma}$ is a face of $\tilde{\tau}$ then the summand is $\varphi_{q(\tilde{\sigma}),q(\tilde{\tau})}$, and otherwise the summand is zero.

9.20. Exercise. Check that this process is well-defined and gives a $\mathbb{Z}[\pi]$-module map $A(M) \to A(N)$.

9.21. Exercise. Show that the assembly of the chain complex $\mathcal{C}_*(K')$ of $(\mathbb{Z}, K)$-modules (defined in Example 6.7) is the chain complex $\mathcal{C}_*(\widetilde{K'})$, considered as a complex of $\mathbb{Z}[\pi]$-modules as in Section 9.2.

9.22. Exercise. Show that the assembly of the cochain complex $\mathcal{C}^*(K)$ (Example 6.4) is the *locally finite* cochain complex of $\tilde{K}$.

An important example for us will be the assembly of a *diagonal approximation* (see Chapter 5).

REVISED TO HERE

Our intention is to set up Poincaré duality for manifolds in the context of twisted homology. Having obtained the notion of orientation, the next task is to define a suitable

[4]The issue is the same as that arising when we distinguish between a *vector space* and the associated *affine space*. Any choice of base-point gives an isomorphism between an affine space and its associated vector space; but there is no canonical choice of such a base-point.

cap-product. So, let X be a finite complex (or just a compact Hausdorff space, if we use singular theory), with fundamental group π, and let w be an orientation character for π. Let V be a right $\mathbb{Z}[\pi]$-module. For every $a \in H_r^\pi(X; \mathbb{Z}^w)$ we want to define a *cap-product*

$$\frown a : H_\pi^s(X; V^o) \to H_{r-s}^\pi(X; V)$$

which is a homomorphism of abelian groups. To do this, we begin with an Eilenberg-Zilber diagonal approximation

$$\mathcal{C}(\tilde{X}) \to \mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}} \mathcal{C}(\tilde{X}).$$

One can manufacture such a diagonal approximation (see Appendix D) which is *equivariant* with respect to the π-action on the tensor product by $(x \otimes y)g = (xg) \otimes (yg)$. Now tensor on the right by the module $\mathbb{Z}^w$. This gives a chain map

$$\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w \to (\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}} \mathcal{C}(\tilde{X})) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w.$$

According to Proposition 8.8, the complex on the right of this display is naturally isomorphic to $\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathcal{C}(\tilde{X})^o$. Tensoring over $\mathbb{Z}$ with the complex $\mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathcal{C}(X), V^o)$ which computes the cohomology, this gives us a diagram

$$\begin{array}{c}
\mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathcal{C}(X), V^o) \otimes_{\mathbb{Z}} (\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w) \\
\Big\downarrow \\
\mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathcal{C}(X), V^o) \otimes_{\mathbb{Z}} (\mathcal{C}(X) \otimes_{\mathbb{Z}[\pi]} \mathcal{C}(X)^o) \\
\Big\downarrow \text{evaluation} \\
V^o \otimes_{\mathbb{Z}[\pi]} \mathcal{C}(X)^o
\end{array}$$

in which the arrows are maps of complexes. Passing to homology this gives the desired product

$$H_r^\pi(X; \mathbb{Z}^w) \otimes H_\pi^s(X; V^o) \to H_{r-s}^\pi(X; V) \tag{9.23}$$

9.24. Definition. The product defined in Equation 9.23 above is called the *cap product* between H_π^* and H_*^π.

Assuming for simplicity that $w = +1$ we can obtain a more geometric picture of the cap product as follows. There is defined an *infinite transfer* map T from the ordinary homology $H_r(X; \mathbb{Z})$ to the locally finite homology $H_n^{lf}(\tilde{X}; \mathbb{Z})$ of the universal cover. The usual (locally finite) cap product with $T(a)$ defines a map

$$H_c^s(X; \mathbb{Z}) \to H_{r-s}(X; \mathbb{Z}).$$

The cap-product above is just this map.

9.25. Remark. In the situation of the cap-product, above, suppose that V is not merely a right module but a *bimodule*. Then $H_\pi^s(X; V^o)$ is a right $\mathbb{Z}[\pi]$-module and $H_{r-s}^\pi(X; V)$ is a left $\mathbb{Z}[\pi]$-module. The cap-product with a is now a module map from the opposite of cohomology to homology.

When $V = \mathbb{Z}[\pi]$ itself, which will be the most important case, we can apply the result of Exercise 8.6 that $V \cong V^o$ as bimodules and express the cap-product with a as a map of left $\mathbb{Z}[\pi]$-modules

$$H_\pi^s(X; \mathbb{Z}[\pi])^o \to H_{r-s}^\pi(X; \mathbb{Z}[\pi]).$$

When calculating with this expression of the cup product it is important not to forget the extra involution that has been introduced by the isomorphism $\mathbb{Z}[\pi] \cong \mathbb{Z}[\pi]^o$.

Let M^n be a compact manifold, oriented with orientation character w. Then the cap-product with the fundamental class defines $\mathbb{Z}[\pi]$-module morphisms

$$D\colon H^r(M;\mathbb{Z}[\pi])^o \to H^w_{n-r}(M;\mathbb{Z}[\pi]). \tag{9.26}$$

Following word-for-word the proofs in the non-equivariant case, we find

9.27. Theorem (Universal Poincaré duality)**.** *The equivariant duality maps D for a compact oriented manifold, defined in Equation 9.26 above, above are isomorphisms.* □

Now we make the connection to intersection theory. Let N_1 and N_2 be transversely intersecting oriented π-trivial submanifolds of M, of complementary dimensions. Recall that a π-trivial submanifold N^k has a preferred lift $\tilde{N}$ to a submanifold of the universal cover of M. The fundamental class of $\tilde{N}$ then maps to a class $[N] \in H^w_k(M;\mathbb{Z}[\pi])$. We use the orientation character coming from the first Stiefel-Whitney class.

9.28. Theorem. *The equivariant intersection number of transversely intersecting submanifolds as above is related to equivariant Poincaré duality by*

$$\lambda(N_1, N_2)_\pi = D^{-1}[N_1][N_2].$$

PROOF. The equivariant intersection number of N_1 and N_2 is a sum, over $g \in \pi$, of the ordinary intersection numbers of $\tilde{N}_1$ and $g^{-1}\tilde{N}_2$. Now apply ordinary intersection theory in the universal cover. □

Together with corollary **??**, this theorem provides the essential link between quadratic algebra over $\mathbb{Z}[\pi]$ and geometric intersections. Our application will be to discover when homology classes can be represented by disjoint embedded spheres, so that we can do surgery on them.

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9.6. The quadratic intersection form

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The algebraic surgery sequence

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CHAPTER A

The h-cobordism theorem

A.1. Definition. An *h-cobordism* is a cobordism[1] $(W; \partial_- W, \partial_+ W)$ such that the inclusions $\partial_\pm W \to W$ are homotopy equivalences.

The notion of h-cobordism was introduced by Thom (circa 1957) into the study of exotic spheres, as a substitute for the relation of diffeomorphism, which seemed at that time to be inaccessible to algebraic study. If that were still the case, surgery theory would still work, but it would produce a classification of manifolds up to h-cobordism only[2]. However, around 1961 Smale proved the *h-cobordism theorem*, which states that any simply-connected h-cobordism has a product structure, and consequently that h-cobordant simply-connected manifolds are diffeomorphic. Thus the h-cobordism and the diffeomorphism classifications of simply-connected manifolds are the same. For general manifolds there is a difference between the classifications, but it is measured by an algebraic K-theory invariant (Whitehead torsion), and this invariant vanishes in several interesting cases (for example when the fundamental group is free abelian.)

Let us recall now the notion of *cellular homology* for CW-complexes. Suppose that X is such a complex, and recall the notion X^k for its k-skeleton. Choose (arbitrarily) an orientation for each cell. Using this orientation, we may identify the relative homology group $H_k(X^k, X^{k-1}; \mathbb{Z})$ with the free abelian group $C_k(X)$ generated by the k-cells of X.

We can define a boundary map $\partial\colon C_k(X) \to C_{k-1}(X)$ by composing pieces of the exact sequences for the pairs (X^k, X^{k-1}) and (X^{k-1}, X^{k-2}): it is the composite

$$H_k(X^k, X^{k-1}) \to H_{k-1}(X^{k-1}) \to H_{k-1}(X^{k-1}, X^{k-2}).$$

Notice that $\partial^2 = 0$ (because we compose two successive maps in the same exact sequence). Thus we have a complex.

A.2. Proposition. *The homology of the complex $(C_*(X), \partial)$ is canonically isomorphic to the ordinary homology of X. There is a similar result for cohomology using the dual complex $C^*(X) = \mathrm{Hom}(C_*(X), \mathbb{Z})$.*

PROOF. Note that $H_r(X^k, X^{k-1}) = 0$ if $r \neq k$, and consider the spectral sequence associated to the filtration of X by skeleta. $\square$

The advantage of cellular homology is the very geometric nature of the boundary map. Indeed, the map $\partial\colon C_k \to C_{k-1}$ is an integer matrix whose (ij) matrix entry is the degree of the map $S^{k-1} \to X^{k-1} \to S^{k-1}$, where the first map is the attaching map for the i'th

[1] We will frequently use this notation for a cobordism. It means that W has a boundary which is a disjoint union of two pieces $\partial_- W$ and $\partial_+ W$, which are oriented respectively with the induced orientation from W and the opposite of that orientation.

[2] Something like this still *is* the case for the exotic homology manifolds of Bryant, Ferry, Mio and Weinberger.

k-cell and the second is the map which collapses the complement of the j'th $(k-1)$-cell to a point. More geometrically still, this matrix entry is the generic number of inverse images, under the attaching map f_i for the boundary sphere of the i'th k-cell, of a point in the interior of the j'th $(k-1)$-cell.

A.3. Example. Suppose X is a finite ordered simplicial complex. Then it has a CW structure with simplices for cells. The complex $C_*(C)$ associated to this CW structure is the usual simplicial chain complex of X.

Now let us apply this to a manifold M^n with the CW structure determined by a handle decomposition. We choose (arbitrarily) an orientation for each handle, by which we mean an orientation for its attaching disc. Moreover, we will assume that the handle decomposition is *ordered* in the following sense: there is a sequence $\emptyset = M_{-1}, M_0, \ldots, M_n = M$ of codimension zero submanifolds with boundary such that M_k is obtained by attaching k-handles to the boundary of M_{k-1} (so that M_k corresponds to the k-skeleton in the CW structure). It can be shown that it is always possible to attach handles in increasing order in this way[3]. Now the attaching sphere α of a k-handle is a submanifold of ∂M_{k-1}, and the belt sphere β of a $(k-1)$-handle is also a submanifold of ∂M_{k-1}; and their dimensions add up to $\dim \partial M_{k-1} = n-1$, so that generically they intersect in a finite set of points. Furthermore, α is oriented (by the boundary of the orientation chosen for the attaching disc of the k-handle), and the *normal bundle* to β is oriented (because its fibre is just the attaching disc of the $(k-1)$-handle). In these circumstances we can attach a sign to each intersection point, so we can define an intersection number $[\alpha : \beta] \in \mathbb{Z}$.

A.4. Proposition. *Let M be a manifold with an ordered handle decomposition, as above. Then the homology of M is computed by the following complex: the chain group C_k is the free abelian group on the k-handles, and the boundary map $C_k \to C_{k-1}$ is the matrix $\mathfrak{M}_k$ whose (ij)'th entry is equal to the intersection number $[\alpha_i : \beta_j]$ of the attaching sphere of the i'th k-handle with the belt sphere of the j'th $(k-1)$-handle.*

PROOF. Let $H = D^{n-k+1} \times D^{k-1}$ be the j'th $(k-1)$-handle, with $\partial_1 H = \beta_j \times D^{k-1}$ contained in ∂M_{k-1} and the remainder of the boundary contained in ∂M_{k-2}. According to our description above of the matrix entries of the boundary map, the matrix entry we want is just the generic number of preimages of some point p in the interior of D^{k-1} under the map $\alpha_i \to D^{k-1}$ which shrinks D^{n-k+1} to a point. But plainly this is just the intersection number of α_i with the belt sphere $S^{n-k+2} \times \{p\}$. $\square$

This result is due to Smale. The intersection numbers can also be described in terms of Morse theory as the numbers of flow lines of the gradient flow between a critical point of index k and one of index $k-1$. In this form, the result was rediscovered by Witten in the early eighties, with a highly original proof based on ideas of quantum-mechanical tunnelling. Witten's work set off an explosion of interest in the relationship between analysis and Morse theory.

A handle has a certain symmetry which we have not exploited so far. In fact, suppose given an ordered handle decomposition and corresponding filtration of M, as above. Let $\bar{M}_k = M \setminus M_{n-k+1}$. Then $\bar{M}_{k+1}$ is obtained from $\bar{M}_k$ by attaching k-handles; these are "the same" handles as the $(n-k)$-handles in the original decomposition, except that the rôles of the attaching and belt discs have been interchanged. We call this the dual

[3]This is a simple application of transversality. Suppose, for example, that we attach a 2-handle after a 3-handle. We can make the attaching sphere of the 2-handle transverse to the belt sphere of the 3-handle, and, counting dimensions, this means that they don't meet at all. So we can slide this attaching sphere right off the 3-handle, which means that we might as well have attached the 2-handle first.

handle decomposition of M (thinking of the Morse function as the "height" above some plane, what we have done is to turn M upside-down). Suppose now that M itself is oriented. Then an orientation of the belt disc of a handle determines an orientation of its attaching disc, by the requirement that the unique intersection point should have positive sign. Hence, an orientation for a handle decomposition determines an orientation for the dual handle decomposition. Plainly the matrices of intersection numbers for the dual handle decomposition are (up to some signs depending only on the dimension) just the transposes of the corresponding matrices for the original handle decomposition; or, to put it more canonically, there is an isomorphism of chain complexes

$$\bar{C}_*(M) \cong \operatorname{Hom}(C_{n-*}(M), \mathbb{Z}) = C^{n-*}(M).$$

Since both handle decompositions compute the (co)homology of M, we find another proof of Poincaré duality from this: the homology of the oriented manifold M is isomorphic to the cohomology in the complementary dimension.

REMARK: The *Morse inequalities* are another well-known consequence of A.4, whose proof does not in fact require the explicit description of the boundary map. Let M^n be a manifold equipped with a Morse function f. Let b_i be the i'th Betti number of M, and let c_i be the number of critical points of f having index i. Then the Morse inequalities are

$$\begin{aligned}
b_0 &\leqslant c_0 \\
b_1 - b_0 &\leqslant c_1 - c_0 \\
b_2 - b_1 + b_0 &\leqslant c_2 - c_1 + c_0 \\
&\cdots \\
b_n - b_{n-1} + \cdots \pm b_0 &= c_n - c_{n-1} + \cdots \pm c_0
\end{aligned}$$

To see this one simply notes that by A.4 the c_i are the dimensions of the chain spaces in a complex that computes the rational homology. Elementary linear algebra now completes the proof.

21.1. Handle Calculus

The results above show that complete homological information about the structure of a manifold or a cobordism is contained in the matrices $\mathfrak{M}_k$ of intersection numbers between the attaching spheres of k-handles and the belt spheres of $(k-1)$-handles. *Handle calculus* shows that certain algebraic operations on these matrices can be 'performed geometrically', by changing the handle decomposition of the given manifold. In particular, if the matrices can be made trivial by suitable algebraic operations, the handle structure can be made trivial, and so the original cobordism will have a simple structure. One should not here the analogy with the Whitney lemma, which also shows that under certain circumstances algebra (intersection numbers, in this case) faithfully reflects geometry.

The following result, called the Cancellation Lemma, is due to Smale.

A.5. Proposition. *Suppose that the manifold W' is obtained from W^n by adding successively a q-handle and a $(q+1)$-handle to the boundary, and suppose that the attaching sphere of the $(q+1)$-handle intersects the belt sphere of the q-handle transversely in a single point. Then W' is diffeomorphic to W.*

The application to intersection matrices is

A.6. Corollary. *Suppose that the simply-connected manifold W^n has a handle presentation in which one of the intersection matrices $\mathfrak{M}_k$, $4 \leqslant k \leqslant n-3$, has i'th row and j'th column intersecting in an entry ± 1 and having zeroes elsewhere. Then one can remove the corresponding k-handle and $(k-1)$-handle from the presentation without affecting the intersection numbers between other handles.*

To prove the corollary one uses the Whitney trick to ensure that all the algebraic intersection numbers are actually equal (in absolute value) to the number of geometric intersection points, and then uses Smale's lemma to cancel the two handles; because all the other intersection numbers are assumed to be zero, this operation can be carried out away from all the other handles so it doesn't affect them.

PROOF. (OF SMALE'S LEMMA) Let's begin by thinking what the region in ∂W must look like along which the two handles are attached. First, the q-handle is attached along $S^{q-1} \times D^{n-q}$. Then the q-handle is attached. Because of the transversality assumption, we may assume that its attaching sphere S^q is made up by joining two discs D^q, one of which is a disc $D^q \times \{p\}$, $p \in S^{n-q-1}$, in the other boundary of the q-handle, and the other is a disc D^q contained in ∂W and spanning $S^{q-1} \times \{p\}$. The whole attaching region in ∂W therefore looks like $S^{q-1} \times D^{n-q} \cup_{S^{q-1} \times D^{n-q-1}} D^q \times D^{n-q-1}$, and this is in fact naturally identified (as a manifold with corners) with $D^q \times D^{n-q-1}$. See the first figure.

Now we see after a little thought that the two attached handles together must make up the cell $D^1 \times D^q \times D^{n-q-1}$, attached along its face $\{0\} \times D^q \times D^{n-q-1}$. See the second figure. In this figure, the front face of the cube is the attaching region in ∂W. The light grey is the q-handle; it runs 'round the back' (invisibly in the figure) to form a neighbourhood of the two sides and the back of the cube. The dark grey is the $(q+1)$-handle.

Thus the effect of attaching two handles to W is simply to attach a disc along one face in its boundary. But plainly this produces a manifold diffeomorphic to W. (We leave corner-smoothing questions to the reader.) □

We will now use Smale's lemma to start getting rid of handles. Our first victims are the 0-handles.

A.7. Lemma. *Let W be a connected cobordism with $\partial_- W$ nonempty. Then W has a presentation without 0-handles. If $\partial_- W$ is empty, it has a presentation with just one 0-handle.*

PROOF. The second statement follows from the first (just punch a hole). To prove the first statement, consider a presentation with the minimal number of 0-handles. Since W is connected, W_1 is connected; so, if h^0 is a 0-handle, there is a 1-handle k^1 connecting it to somewhere else. But then its attaching sphere is an S^0 with one point on h^0 and one point somewhere else; so by Smale h^0 and k^1 can be cancelled. □

The 1-handles are more recalcitrant. We have to pay a price in 3-handles for their elimination.

A.8. Lemma. *Let W^n be a connected cobordism, with $\partial_- W \to W$ a 1-connected map and $n \geqslant 5$. Suppose that W is equipped with a presentation without 0-handles. Then there is a new presentation of W without 0-handles or 1-handles, and having the same handles as before in dimensions $\geqslant 4$.*

PROOF. (Outline) First we eliminate the 0-handles using the previous lemma. Then W_0 is connected. Consider now a presentation with the minimal number of 1-handles; for simplicity suppose that there is just one 1-handle h^1 attached to $\partial_+ W_0$. Let Γ_1 be a simple closed curve in $\partial_+ W_1$ which intersects the belt sphere of h^1 transversely in one point and returns through $\partial_+ W_0$. We may assume (by transversality) that Γ_1 is disjoint from the attaching spheres of all 2-handles, so it in fact lies in $\partial_+ W_2$. Now Γ_1 must be nullhomotopic in W (since it clearly doesn't come from $\partial_- W$); since W is obtained by adding handles of indices 3 and up to $\partial_+ W_2$, this means that Γ_1 is already nullhomotopic in $\partial_+ W_2$.

Now we attach to W_1 a cancelling 2-handle and 3-handle, $k^2 \cup l^3$, attached somewhere out of the way of everything else. This doesn't change W_1. The attaching sphere of k^2 is another null-homotopic simple closed curve Γ_2 in $\partial_+ W_2$, by the same argument as before.

Now we need an 'unknotting theorem': Two null-homotopic simple closed curves in a k-manifold, $k \geqslant 4$, are in fact isotopic there. At least for $k \geqslant 5$ this can easily be proved by using Whitney's embedding theorem to embed a cylindrical homotopy between the two curves, and then manufacturing an isotopy by 'pushing' along the cylinder. Thus Γ_1 and Γ_2 are isotopic, and we can use the isotopy to move our trivial handle pair so that k^2 actually is attached along Γ_1.

But now $h^1 \cup k^2$ is trivial, by the cancellation lemma. Thus

$$W_2 = W_0 \cup h^1 \cup k^2 \cup l^3 \cup \text{2-handles} = W_0 \cup l^3 \cup \text{2-handles}$$

and we have eliminated the 1-handle and acquired instead an extra 3-handle. □

This process is called *handle trading*. The two lemmas above are the first of a series of handle-trading lemmas, which culminate in the following

A.9. Proposition. *Let W^n be a cobordism with $(W, \partial_- W)$ k-connected, $k \leqslant n-4$. Given any presentation of W, it can be modified to produce a new presentation without handles of dimension $\leqslant k$, and with the number of handles of dimensions $\geqslant k+3$ unchanged.*

Now remember that any cobordism can be given a dual presentation, in which the k-handles are the $(n-k)$-handles of the original presentation. This means that we can trade down as well as up (if things are connected enough). After sufficient trading we arrive at the following situation.

A.10. Proposition. *Let W^n be an h-cobordism, $n \geqslant 5$. Then W has a presentation with only k-handles and $(k+1)$-handles; here k can be any integer between 2 and $n-3$.*

We now proceed towards the proof of the h-cobordism theorem. Let W^n be an h-cobordism where $n \geqslant 7$. (One can get the dimension down to 6 using the improved versions of the Whitney trick. We won't worry about this.) Then it has a presentation with 3-handles and 4-handles only. The homology is determined by a single matrix $\mathfrak{M}_4$ of intersection numbers; by the Whitney trick, we can assume that these intersection numbers are equal (in absolute value) to the actual geometric number of intersection points between the belt spheres of the 3-handles and the attaching spheres of the 4-handles.

Now the homology computed by the cellular chain complex in this case is the *relative* homology $H_*(W, \partial_- W)$, and consequently it is zero (because of the assumption that W is an h-cobordism). Thus $\mathfrak{M}_4$ is an *invertible* matrix of integers; in particular, it is square.

Now recall that the *elementary row operations* on a matrix of integers are the following:

(i) To interchange two rows;
(ii) To multiply a row by ± 1;
(iii) To add n times one row to another row, $n \in \mathbb{Z}$.

The *elementary column operations* are defined similarly.

A.11. Proposition. (SMITH NORMAL FORM THEOREM) *Any integer matrix can be reduced by elementary row and column operations to a diagonal matrix in which each diagonal entry divides the next one. In particular, any invertible integer matrix can be reduced by elementary row and column operations to the identity matrix.*

This theorem is usually proved as part of the classification of finitely generated abelian groups. Its relevance here comes from the last of our geometric tool theorems:

A.12. Proposition. *Let W be a cobordism equipped with a presentation having only k-handles and $(k+1)$-handles. Then any elementary row or column operation on $\mathfrak{M}_{k+1}$ can be implemented geometrically by a suitable change of the presentation.*

We won't prove this here.

A.13. Theorem. (h-COBORDISM THEOREM) *A simply-connected h-cobordism in dimension 7 or above is a product.*

As we mentioned, the dimension 7 can be improved to 6; Donaldson gave counterexamples in dimension 5.

PROOF. By the results above, we can assume that there is a presentation having only 3-handles and 4-handles, and such that $\mathfrak{M}_4$ is the identity matrix. By the Whitney trick, we may adjust by isotopies so that the attaching sphere of each 4-handle meets the belt sphere of just one 3-handle transversely in a single point, and meets no other belt spheres. Now by the cancellation lemma we may get rid of all the handles. Thus W has a presentation with no handles at all, which makes it a product. $\square$

A.14. Remark. Suppose that the inclusions $\partial_\pm W \to W$ are 1-connected and that $H_*(W, \partial_- W; \mathbb{Z}\pi) = 0$. Then W is an h-cobordism. This follows from the Hurewicz theorem and Poincaré duality.

21.2. Some consequences of the h-cobordism theorem

Here we follow Milnor's book in listing some results that can be easily deduced from the h-cobordism theorem.

A.15. Theorem. (DISC CHARACTERIZATION THEOREM) *Let W^n be a compact simply-connected smooth manifold, $n \geqslant 6$, with simply-connected boundary and having the integral homology of a point. Then W^n is diffeomorphic to the n-disc.*

PROOF. Let D_0 be an n-disc in the interior of W. Then, by the remark above, $D \setminus D_0^\circ$ is a simply-connected h-cobordism; hence it is a product. Thus D is a disc with a cylinder attached to the boundary, i.e. a disc. $\square$

A.16. Theorem. (GENERALIZED POINCARÉ CONJECTURE IN DIMENSION $\geqslant 6$) *If a smooth manifold M^n, $n \geqslant 6$, has the homotopy type of an n-sphere, then M is homeomorphic to S^n (by a homeomorphism which is smooth except at one point).*

PROOF. Remove the interior of a disc D_0 from M. The complement then satisfies the conditions of A.15, so it is another disc. Thus M is a *twisted sphere* — it consists of two n-discs glued together by some diffeomorphism of their boundary $(n-1)$-spheres.

To prove that any twisted sphere is homeomorphic to an ordinary sphere, it clearly suffices to prove that any homeomorphism $S^{n-1} \to S^{n-1}$ can be extended to a homeomorphism $D^n \to D^n$. This is done by regarding D^n as the cone on S^{n-1} and extending the homeomorphism radially[4]. If the original homeomorphism is a diffeomorphism, the extended homeomorphism is smooth except at the cone point. $\square$

The argument does *not* show that M is smoothly a standard sphere, and in fact this is false, as we shall see. However, it does allow one to identify the following three objects:

(a) Θ^n, the set of smooth structures on the standard n-sphere (i.e. $\Theta^n = \mathcal{S}^{DIFF}(S^n)$;
(b) Γ^n, the group of diffeomorphisms of S^{n-1} modulo those that extend to D^n;
(c) A^n, the group of units in the monoid of all smooth n-manifolds under connected sum.

A.17. Theorem. (DIFFERENTIABLE SCHOENFLIES THEOREM IN DIMENSIONS $\geqslant 6$) *Let Σ be a smoothly embedded $(n-1)$)-sphere in S^n, $n \geqslant 6$. Then there is an ambient isotopy of S^n that carries Σ onto the standard $S^{n-1} \subset S^n$.*

PROOF. By the Jordan-Brouwer separation theorem the complement of Σ has two components having Σ as their common boundary. Consider the closure C of one component. It is a simply-connected manifold with boundary Σ and having the homology of a point; hence, by A.15, it is a disc. Now we appeal to the *disc theorem*[5] of Palais and Cerf: Any two orientation-preserving embeddings of a closed n-disc in a connected n-manifold are ambient isotopic. From this we obtain an isotopy that carries C to the lower hemisphere and hence Σ to the equator. $\square$

In the last two results the dimension 6 can be reduced to 5 by additional arguments.

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[4]This is called the *Alexander trick*.

[5]We have in fact already made extensive use of this theorem and its generalized version, the uniqueness of tubular neighbourhoods. They are needed to show that the operations of connected sum, surgery, and so on, are well-defined.

21.3. The Whitehead group

A.18. Definition. Let R be a ring with unit. The group $GL(R)$ is the inductive limit of the matrix groups $GL_n(R)$; in other words, $GL(R)$ is the group of infinite invertible matrices over R which differ in only finitely many places from the infinite identity matrix.

A matrix in $GL(R)$ is called *elementary* if it differs from the identity only in one place, not on the diagonal.

A.19. Lemma. (WHITEHEAD LEMMA) *The subgroup $E(R)$ of $GL(R)$ generated by the elementary matrices is precisely the commutator subgroup of $GL(R)$.*

PROOF. The proof of the Whitehead lemma begins with the computation that any elementary matrix is the commutator of two other elementary matrices. Therefore

$$E(R) = [E(R), E(R)] \subseteq [GL(R), GL(R)]$$

and we need only prove the reverse inclusion. Now it is elementary (pardon the pun) to see that any upper or lower triangular matrix is the product of elementary matrices. The identity

$$\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

shows that the left hand side belongs to $E(R)$ for any $x \in GL_n(R)$, and multiplying the left hand sides for $x = a$ and for $x = -1$ shows that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

also belongs to $E(R)$. Now finally the identity

$$\begin{pmatrix} a^{-1}b^{-1}ab & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1}b^{-1} & 0 \\ 0 & ba \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

shows that $a^{-1}b^{-1}ab \in E(R)$, as required. $\square$

Consequently, $GL(R)/E(R)$ is an abelian group.

A.20. Definition. $K_1(R) = GL(R)/E(R)$.

It follows from the calculations above that the addition in $K_1(R)$ can be defined either by group product or by direct sum; these are equivalent.

A.21. Remark. A pleasant description of algebraic K-theory is that it studies 'the deep structure of linear algebra over R'. To make some sense of this statement, suppose that we have an invertible matrix over R. Then we may ask: Why is it invertible? It might be invertible for the trivial reason that it is a product of elementary matrices: this is trivial in that it does not depend at all on the structure of the ring R. Or it might be invertible for some other reason, dependent on the structure of R. Now K_1R is just the 'group of non-trivial reasons for invertibility' in this sense.

For any commutative ring R one can define the determinant $\det\colon M_n(R) \to R$; it enjoys the usual multiplicative properties. Since the determinant of an elementary matrix is 1, we get an induced homomorphism

$$\det\colon K_1R \to R^{\times}, \tag{$*$}$$

where $R^{\times}$ denotes the group of units in R.

If R is a field, we know from undergraduate linear algebra that any invertible matrix of determinant 1 is a product of elementary matrices. Thus the determinant homomorphism $(*)$ is an isomorphism in this case. In view of the discussion above this can perhaps be phrased as follows.

A.22. Proposition. *Linear algebra over a field is shallow.*

In the last chapter we discussed the Smith normal form theorem for matrices over $\mathbb{Z}$. This theorem works (with the same proof) for matrices over any Euclidean domain, so that an invertible matrix over such a domain is a product of elementary matrices and a diagonal matrix with units down the diagonal. This proves that linear algebra over a Euclidean domain is shallow too, or, more formally

A.23. Proposition. *For any Euclidean domain R,* $\det\colon K_1R \to R^\times$ *is an isomorphism.*

In particular, $K_1\mathbb{Z} = \{\pm 1\}$.

Beware that this theorem does not remain true for a principal ideal domain R — despite the fact that the structure theory for finitely generated modules, usually proved as a consequence of the Smith theorem, still holds in this case! As an exercise, think about the proof of the structure theory in matrix language, and try to see why it does not prove $K_1R = 0$.

Our interest is mainly in the functor K_1, but for completeness we will also define K_0. Let R be a ring, and consider the collection $P(R)$ of isomorphism classes of finitely generated projective modules over R. It is a monoid under direct sum.

A.24. Definition. The group $K_0(R)$ is the Grothendieck group of the monoid $P(R)$.

To be precise, this means that $K_0(R)$ is an abelian group characterized by the following universal property: there is a homomorphism (of monoids) $P(R) \to K_0(R)$, and any other homomorphism from $P(R)$ to an abelian group factors through this one.

A.25. Proposition. *If R is a principal ideal domain, then $K_0(R) = \mathbb{Z}$.*

PROOF. Notice first that since R is an integral domain (no zero-divisors) it has no non-trivial projections. Thus if $R \to P$ is a surjection, with P projective, we must have $P = R$ or $P = 0$. The structure theory for modules over a PID gives that any finitely-generated R-module M is the direct sum of finitely many modules M_i, each of which is a homomorphic image of R; if M is projective so is each M_i, so M is isomorphic to R^k for some k. We have proved that any finitely generated projective module over R is free, and it is clearly determined up to isomorphism by its rank k (tensor with the field of fractions to see this is well-defined); the map $[M] \mapsto \operatorname{rank} M$ gives the desired isomorphism. $\square$

Now we specialize to the case of group rings. Suppose that $R = \mathbb{Z}\pi$. Notice that for $g \in \pi$ the elements $\pm g$ are units of R, that is members of $GL_1(R)$, hence they define elements of $K_1(R)$.

A.26. Definition. The *Whitehead group* $\mathrm{Wh}(\pi)$ is the quotient of $K_1(\mathbb{Z}\pi)$ by the subgroup $\{\pm g : g \in \pi\}$.

Naturally one would like to know about the Whitehead groups $\mathrm{Wh}(\pi)$. Perhaps they are all trivial? Unfortunately this is rather far from being the case in general.

A.27. Example. Let $\pi = \mathbb{Z}/5$, generated say by t. The element $u = 1 - t - t^4$ is then a unit in $\mathbb{Z}\pi$, by direct computation; its inverse is $1 - t^2 - t^3$. Hence u defines a class $[u] \in K_1(\mathbb{Z}\pi)$. Now let $\theta = e^{2\pi i/5} \in \mathbb{C}$; then $t \mapsto \theta$ defines a homomorphism $\mathbb{Z}\pi \to \mathbb{C}$, so we get a homomorphism

$$K_1(\mathbb{Z}\pi) \to K_1(\mathbb{C}) \to \mathbb{R}^+$$

where the second arrow is induced by the determinant. Under the above homomorphism, the subgroup $\pm\pi \in K_1(\mathbb{Z}\pi)$ clearly maps to 1, so we obtain a homomorphism $\varphi\colon \mathrm{Wh}(\pi) \to \mathbb{R}^+$. But plainly $\varphi(u) = 1 - 2\cos\frac{2\pi}{5} \neq 1$, so that $[u]$ generates an infinite cyclic summand in $\mathrm{Wh}(\pi)$.

For torsion-free groups we are in better shape. In fact there is no counterexample to the following conjecture.

A.28. Conjecture. *Let π be a finitely generated torsion-free group; then* $\mathrm{Wh}(\pi) = 0$.

A key positive result here is

A.29. Proposition. (BASS-HELLER-SWAN THEOREM) *Let π be a finitely generated free abelian group. Then* $\mathrm{Wh}(\pi) = 0$.

This result was proved in the sixties; the computation $\mathrm{Wh}(\mathbb{Z}) = 0$ (a special case) was done by Higman in the forties.

We won't go through the proof, but here are some of the ideas. Inductively it is sufficient to give a computation of the K-theory of the group ring $R[\mathbb{Z}]$ in terms of the K-theory of R. Now $R[\mathbb{Z}]$ can profitably be written as the ring of 'Laurent polynomials' $R[t, t^{-1}]$, and we see that it is some kind of algebraic analogue of the "R-valued functions on a circle." (In terms of C^*-algebras, this corresponds to the identification of the group C^*-algebra $C^*_r\mathbb{Z}$ with $C(S^1)$, via Fourier analysis.) In classical (topological) K-theory the effect of taking a product with a circle is

$$K^1(X \times S^1) = K^1(X) \oplus K^0(X)$$

and this suggests that something analogous should be true here. In fact Bass, Heller and Swan proved a formula

$$K^1 R[t, t^{-1}] = K_1 R \oplus K_0 R \oplus \mathrm{Nil}\, R \oplus \mathrm{Nil}\, R$$

where the groups $\mathrm{Nil}\, R$ vanish if R is a left regular ring (in particular, if R itself is a Laurent polynomial ring over $\mathbb{Z}$ in finitely many variables). Let's consider for simplicity the case $R = \mathbb{Z}$, so the Bass-Heller-Swan formula is computing $K_1(\mathbb{Z}[\mathbb{Z}])$. We know $K_1\mathbb{Z} = 0$, $K_0\mathbb{Z} = \mathbb{Z}$. The BHS map from $K_0 R$ to $K_1 R[t, t^{-1}]$ proceeds by associating to a projective module P over R the endomorphism t of the projective $R[t, t^{-1}]$-module $P \otimes_R R[t, t^{-1}]$. Here, then, we find that $K_1\mathbb{Z}[t, t^{-1}]$ is $\mathbb{Z}$ generated by t. But t is a group element, so $\mathrm{Wh}(\mathbb{Z}) = 0$, as asserted.

The Bass-Heller-Swan theorem leads to an inductive definition of 'lower K-theory' groups K_i, $i < 0$, such that the formula

$$K_i R[t, t^{-1}] = K_i R \oplus K_{i-1} R \oplus \mathrm{Nils}$$

remains true.

21.4. Whitehead torsion

Imagine that you are teaching a first course in algebraic topology, and you want to motivate the concept of homotopy equivalence. So, you explain, the disc is homotopy equivalent to a point; an annulus is homotopy equivalent to a sphere and so on.

Each of these homotopy equivalences is of a comparatively elementary sort. To be precise:

A.30. Definition. Let X be a space, $f\colon D^{n-1} \to X$ a map. Let $Y = X \cup_f D^n$, where the identification is made along one hemisphere of $\partial D^n = S^{n-1}$. Then we say that Y is obtained from X by an *elementary expansion*, or that X is obtained from Y by an *elementary collapse*.

Plainly X and Y are homotopy equivalent. In terms of CW-theory, Y may be thought of as obtained from X by attaching first an $(n-1)$-cell (along $f_{|S^{n-2}}$) and then an n-cell (along a map half of which is f and the other half of which is the identity map to the previously attached $(n-1)$-cell). The two cells are analogous to the two cancelling handles in Smale's cancellation lemma.

A.31. Definition. A homotopy equivalence is *simple* if it can be obtained by a succession of elementary expansions and collapses.

One reason for interest in this notion from the point of view of manifold theory is the following. Let P be a polyhedron and let K be a complex. K is said to be *full* if every simplex of P which has all its vertices in K is a simplex of K. In this case the *characteristic function* $\lambda\colon P \to [0,1]$, defined to be the unique simplexwise affine function which is 1 on the vertices of K and 0 on the other vertices, has $\lambda^{-1}\{1\} = K$, and we may make the

A.32. Definition. A *regular neighbourhood* of K in P is $\lambda^{-1}([0,t])$ for some $0 < t < 1$.

This is the PL substitute for the notion of tubular neighbourhood. Up to isotopy, the choice of t does not matter.

A.33. Theorem. *Let X and Y be finite complexes in a PL manifold (Euclidean space, for example). Suppose that X and Y are related by an elementary expansion (or collapse). Then X and Y have PL homeomorphic regular neighbourhoods.*

I won't prove this, but hope the figure makes it plausible. For a proof see Rourke and Sanderson thm 3.26.

Two PL manifolds M and N are said to be *stably PL homeomorphic* if $M \times \mathbb{R}^n$ is PL homeomorphic to $N \times \mathbb{R}^n$ for some large n.

A.34. Corollary. *Let M and N be two stably parallelizable compact PL manifolds. Then M and N are stably PL homeomorphic if and only if they are simple homotopy equivalent.*

In the 1940's, Whitehead asked: Are there homotopy equivalences which are not simple? To answer this question he introduced the idea of the *torsion* of a homotopy equivalence. Let $i\colon X \to Y$ be a homotopy equivalence of CW complexes. By a classical construction (mapping cylinder) we may and shall assume that X is a subcomplex of Y and that i is the inclusion. Then (Y, X) is a relative CW-complex, and Y is obtained from X by attaching successively 0-cells, 1-cells, 2-cells and so on. We may then consider the relative cellular chain complex $C_*(Y, X)$; this is defined as in the last chapter, except that I now want to consider coefficients in $\mathbb{Z}\pi$. We need to choose a π-trivialization and an orientation of each cell; when we have done this, the cellular chain complex is a complex of finitely generated, based, free $\mathbb{Z}\pi$-modules, and it has zero homology (i.e. it is acyclic) because i is a homotopy equivalence.

We will define an invariant of such complexes, which one could think of as a 'deep Euler characteristic'. Let R be any ring and consider finite acyclic chain complexes C of fg based free R-modules.

A.35. Definition. If such a chain complex contains only two non-vanishing groups and one non-trivial differential $d_n\colon C_n \to C_{n-1}$, let $\mathfrak{M}_n$ be the matrix of d_n relative to the given bases, and define $\tau(C) \in K_1(R)$ to be $(-1)^n$ times the class of $\mathfrak{M}_n$.

We would like to reduce the general case to the case of complexes of length two. In analysis, dealing say with elliptic complexes, one does this by taking adjoints and forming $d + d^*\colon C_{\text{even}} \to C_{\text{odd}}$. This technique is not available for complexes of general R-modules. However, the basic function of the inner product is to provide 'complements' to submodules, and this can be done by arguments using projectivity.

A.36. Lemma. *Let (C, d) be a finite*[6] *acyclic chain complex of projective R-modules; then it is chain contractible, and each boundary submodule $B_k = Z_k = \ker d_k$ is a direct summand in C_k.*

PROOF. By induction, using the definition of 'projective'. □

Thus our complex has all the B_k finitely generated projective R-modules. By adding on trivial complexes of length 2 (geometrically this corresponds to introducing cancelling pairs of cells or handles) we can assume that all the B_k are actually *free* modules. We choose (arbitrarily) bases for them and complements X_k for them in C_k. Let $\mathfrak{N}_k$ be the matrix of

$$C_k = X_k \oplus B_k \to B_{k-1} \oplus B_k.$$

This is an invertible matrix and it is well defined up to the choice of projection onto B_k, an upper triangular term and hence a product of elementary matrices. Thus its class in $K_1 R$ is well defined. We now define the *torsion* in general to be the alternating sum

$$\tau(C) = \sum (-1)^k [\mathfrak{N}_k] \in K_1 R.$$

It is easy to see that this agrees with the previous definition for complexes of length two.

For a homotopy equivalence $f\colon X \to Y$, want to define the torsion as the torsion of the mapping cylinder. We have to choose bases for the free modules appearing in the cellular chain complex. These bases are not canonical[7], but the ambiguities appearing therein only change the torsion by the action of the group $\pm\pi$ on K_1. Thus the torsion $\tau(f)$ is well-defined in $\mathrm{Wh}(\pi)$. Using cellular approximation, one can define $\tau(f)$ even for a non-cellular homotopy equivalence.

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[6] Actually, we only require it to be bounded below.

[7] The *ordering* of the bases is not canonical either, but this does not matter. The reason for this is that it is possible to write the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as a product of elementary matrices. Since we have explicitly factored out the action of -1, it follows that any transposition of rows or columns acts trivially on the Whitehead group.

A.37. Remark. There are various useful formulae for the torsions of certain kinds of composite maps. For instance, say that $f\colon X \to Y$ and $g\colon Y \to Z$ are homotopy equivalences. Then one has
$$\tau(g \circ f) = \tau(f) + \tau(g).$$
Suppose that $f\colon X \to Y$ and consider $f \times 1\colon X \times Z \to Y \times Z$. Then
$$\tau(f \times 1) = \tau(f)\chi(Z)$$
where χ denotes the Euler characteristic.

A.38. Theorem. (WHITEHEAD) *A homotopy equivalence f between finite complexes is simple if and only if $\tau(f) = 0$.*

PROOF. (SKETCH) This is like the proof of the h-cobordism theorem. The easy way around, we verify by hand that the torsion of an elementary expansion or collapse is zero. Then the additivity of torsion tells us that any simple homotopy equivalence has torsion zero. The other way round, suppose we have a homotopy equivalence with zero torsion. Then we can do 'cell-trading' to arrange that the relative cellular complex only has cells in two consecutive dimensions. Since the torsion is zero, the boundary matrix of the relative cellular complex is a product of elementray matrices. One shows that these elementary matrices represent moves that can be effected geometrically, by expansions and collapses. □

Clearly the torsion of a cellular homeomorphism is zero. For a long time it was unknown whether the Whitehead torsion of a (non-cellular) homeomorphism must be zero. This was resolved by Chapman (1972) using infinite-dimensional topology.

A.39. Definition. The *Hilbert cube Q* is the product $\prod_{i=1}^{\infty}[0,1]$ of countably many copies of the unit interval.

A.40. Theorem. (CHAPMAN) *Two finite complexes X and Y are simple homotopy equivalent if and only if $X \times Q$ and $Y \times Q$ are homeomorphic. In fact, $f\colon X \times Y$ is a simple homotopy equivalence if and only if $f \times 1\colon X \times Q \to Y \times Q$ is homotopic to a homeomorphism.*

A.41. Corollary. (TOPOLOGICAL INVARIANCE OF WHITEHEAD TORSION) *If f is a homeomorphism, then it is a simple homotopy equivalence, i.e., $\tau(f) = 0$.*

There are now other proofs of this using controlled topology. See later, perhaps.

21.5. The s-cobordism theorem

Let us now consider the problem of determining whether a *non*-simply-connected h-cobordism is a product. The argument given in the last chapter for the h-cobordism theorem breaks down only at the very last stage, where we needed to apply the Whitney trick in order to see that the algebraic intersection numbers of the attaching and belt spheres were actually attained geometrically. We know that in the non-simply-connected case this need not be true if we use ordinary intersection numbers, but it becomes true once again if we use the $\mathbb{Z}\pi$ intersection numbers. What changes will we have to make to do this?

Clearly the attaching discs of handles are π-trivial submanifolds, since in fact they are contractible. Choosing (arbitrarily) π-trivializations for the attaching discs, the $\mathbb{Z}\pi$ intersection numbers can be organized into matrices giving us a complex of $\mathbb{Z}\pi$-modules. Just as before, one can show that this complex computes the homology with coefficients in $\mathbb{Z}\pi$.

The handle trading lemmas still apply, so that a high-dimensional h-cobordism can be reduced to one with 3-handles and 4-handles only. The intersection matrix $\mathfrak{M}_4$ is now a matrix over $\mathbb{Z}\pi$, and the row operations that we can perform geometrically are:

(i) To interchange two rows;
(ii) To multiply a row by $\pm g$, $g \in \pi$ (this corresponds to a change of choice of π-trivialization and/or orientation);
(iii) To add a times one row to another row, $a \in \mathbb{Z}\pi$.

We need to know, then, whether an invertible matrix over $\mathbb{Z}\pi$ can always be reduced to the identity by employing these row operations (and the corresponding column operations). The answer is given by Whitehead torsion.

Let W be an h-cobordism, and let it be given a presentation with 3-handles and 4-handles only. The matrix $\mathfrak{M}_4$ is then an invertible matrix over the group ring $\mathbb{Z}\pi$, so it defines an element of $K_1(\mathbb{Z}\pi)$.

A.42. Definition. The image of this class $[\mathfrak{M}_4]$ in $\mathrm{Wh}(\pi)$ is called the *torsion* $\tau(W)$ of the h-cobordism.

Plainly, this is just the Whitehead torsion of the inclusion $\partial_- W \to W$ (which is a homotopy equivalence).

A.43. Theorem. (s-COBORDISM THEOREM) *A high-dimensional s-cobordism is a product if and only if its torsion vanishes.*

PROOF. In one direction, $M \times I$ has a handle-decomposition without handles, hence with torsion zero. This means that the torsion is an obstruction to the existence of a product structure. In the other direction, suppose that $\tau(W)$ vanishes. Then by means of elementary row and column operations $\mathfrak{M}_4$ can be brought to diagonal form where the diagonal entries are $\pm g$, $g \in \pi$. By the non simply connected Whitney trick, we can make further isotopies to arrange that the handles intersect in pairs with geometric intersection number 1; then we can cancel them by Smale's lemma as before. □

There is something a bit asymmetrical here: an h-cobordism is trivial if the torsion of the inclusion $\partial_- W \to W$ vanishes. What about the inclusion $\partial_+ W \to W$? The answer is that there is a duality formula, just like Poincaré duality for homology.

A.44. Proposition. (MILNOR DUALITY) *If W^n is an h-cobordism, then $\tau(W) = (-1)^{n-1}\tau(\bar{W})^*$, where $\bar{W}$ denotes the dual h-cobordism, and $*$ is the involution on the Whitehead group* *induced by the involution on $\mathbb{Z}\pi$ coming from the first Stiefel-Whitney class; in other words,* $\tau(\partial_- W \to W) = (-1)^{n-1}\tau(\partial_+ W \to W)$.

This is pretty clear once we have rolled up to have handles in two consecutive dimensions only. One has a similar formula for a homotopy equivalence f of *closed* manifolds, namely $\tau(f) = (-1)^{n-1}\tau(f)^*$; that is, $\tau(f)$ lives in a certain involution-invariant subgroup of the Whitehead group.

Here is a geometric corollary (a 'rolling-up' argument). For it we need the following observation. Let W be an h-cobordism and Z a closed manifold. Then $W \times Z$ is also an h-cobordism. It can be shown that

$$\tau(W \times Z) = \tau(W)\chi(Z)$$

where χ denotes the Euler characteristic. This is just a corollary of the product formula for Whitehead torsion.

A.45. Proposition. *Let M and N be compact high-dimensional manifolds. Suppose that $M \times \mathbb{R}$ and $N \times \mathbb{R}$ are diffeomorphic; then they are diffeomorphic by a diffeomorphism which is equivariant for the natural $\mathbb{Z}$-actions by translation.*

PROOF. By gluing M onto one end of $M \times \mathbb{R} \cong N \times \mathbb{R}$ and N onto the other (in the natural way) we obtain an h-cobordism between M and N. The torsion of this h-cobordism vanishes on crossing with S^1, so by the s-cobordism theorem $M \times S^1$ and $N \times S^1$ are diffeomorphic, by a diffeomorphism which on the homotopy level is just the product of a homotopy equivalence $M \to N$ with the identity on S^1. Passing to infinite cyclic covers we get the asserted periodic diffeomorphism between $M \times \mathbb{R}$ and $N \times \mathbb{R}$. $\square$

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