Applications of boundary-preserving seismic tomography for delineating reservoir boundaries and zones of CO₂ saturation

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ABSTRACT

Delineating reservoir units is still a challenge for seismic approaches. Even high-resolution crosswell tomographic approaches that produce smooth velocity models to match traveltime data usually provide limited information about the boundaries of subsurface targets. A recent development of seismic traveltime tomography incorporated with a boundary-preserving regularization constraint promisingly helps to resolve ambiguities in reservoir boundaries, while allowing lateral variations. We applied this kind of boundary-preserving traveltime tomography to delineate boundaries of the reservoir and CO₂-saturated zones. We chose the minimum gradient support as the regularization to preserve boundaries of the geologic target by penalizing smaller model gradients and smoothing small model variations caused by noise. We evaluated several synthetic and real data applications. Synthetic examples demonstrated that the boundary-preserving algorithm produced improved recovery of the profile shape and velocity values of the blocky targets. Two real applications were for delineating the top and base of the reservoir in the King Mountain field, and for delineating a CO₂-saturated zone in the McElroy field. These inversion results suggested that the boundary-preserving inversion is able to provide better delineation of the top and base boundaries of the reservoir and boundaries of the CO₂-saturated zone than the conventional smooth-constrained inversion.

INTRODUCTION

Improving the delineation of boundaries of producing hydrocarbon reservoirs is of importance for the quantitative reservoir characterization and effective management of enhanced oil recovery processes (e.g., Harris et al., 1995). Ambiguous boundaries bring uncertainties for further geologic interpretation. The present study is motivated by a field problem from the Permian Basin of west Texas. The reservoir boundaries of the producible reservoir are ambiguous from previous tomographic studies, in particular uncertainties about the base of the saturated zone, knowledge of which critically influences plans for subsequent drilling. In addition, the lateral extent of the reservoir was below 3D surface seismic resolution and not clearly identified with crosswell velocity tomography (Langan et al., 1997). For further qualitative and quantitative assessment and development of the reservoir, sufficient information about its vertical and lateral extent, e.g., boundaries, is needed.

Crosswell seismic profiling (XSP) as a high-resolution seismic method is used for reservoir management to characterize and delineate reservoirs (e.g., Langan et al., 1997). An XSP includes velocity tomography and reflection imaging (e.g., migration). Traveltime tomography is used for velocity estimation, and reflection imaging is for delineating boundaries. In the conventional implementation, XSP-type seismic traveltime tomography inverts the first-arrival traveltime data to estimate P-wave velocity using a model constraint, such as first- or second-order Tikhonov regularization for flatness or smoothness constraints, respectively (Aster et al., 2005). With Tikhonov regularization, the solution will be at most a flattened and/or smoothed model of the earth’s subsurface. Nevertheless, the smooth model is not always the best model for geologic structures that are commonly characterized by sharp boundaries and comprise distinct geologic bodies, for example, the sharp petrophysical boundaries between the host rocks and the hydrocarbon reservoir zones (Michelena and Harris, 1991; Zhu and Harris, 2011). In this situation, sharp contrasts in the medium associated with lithologic boundaries are not easily identified from the tomography.

We seek a traveltime tomography algorithm that preserves boundaries within the framework of Tikhonov regularization. Several
boundary-preserving regularizations have been developed in image reconstruction to restore sharp edges and high-contrast images (Charbonnier et al., 1997). Some schemes have been applied to geophysical data, for example, the compactness body constraint (Last and Kubik, 1983; Ajo-Franklin et al., 2007), the $L_2$-norm (Claerbout and Muir, 1973; Bube and Langan, 1997; Farquharson, 2008; Zhang and Castagna, 2011; Li et al., 2012; Zhang et al., 2013), total variation regularization (e.g., Farquharson and Oldenburg, 1998), the minimum gradient constraint (Portniaguine and Zhdanov, 1999; Abubakar et al., 2008), and the level-set method (Lelièvre et al., 2012; Li and Leung, 2013; Zheglova et al., 2013; Li et al., 2014). Among them, the minimum gradient support (MGS) regularization proposed by Portniaguine and Zhdanov (1999) tends to produce more blocky models because it selects models where the spatial gradients of an anomaly, rather than the anomaly itself, are compact. They show that this approach successfully produces a desired sharp boundary model using 3D magnetic and gravity data. Later, Zhdanov et al. (2006) demonstrate its effectiveness in seismic traveltime tomography using synthetic examples. Youzwhisen and Sacchi (2006) apply this approach to reconstruct blocky acoustic-velocity models from synthetic seismic data sets. However, this approach has rarely been applied to field data sets. The goal of this paper is to investigate the feasibility of the boundary-preserving inversion of crosswell field data sets for delineating boundaries for a reservoir zone and a CO$_2$-saturated zone.

The paper is organized as follows: First, we briefly introduce the seismic traveltime inversion algorithm with the boundary-preserving minimum-gradient support regularization. Then, the algorithm is tested on two synthetic cases to evaluate the quality of the reconstruction of velocity boundaries with this type of regularization. Finally, we present two real applications of this algorithm in west Texas fields: The first case is delineating reservoir boundaries in the King Mountain field, and the second is delineating a CO$_2$-saturated zone in the McElroy field.

**METHODODOLOGY**

In this section, we briefly introduce the framework of seismic inversion and smooth and boundary-preserving regularizations.

**Framework of seismic inversion**

The task of geophysical inversion is to find an optimized model of the subsurface that satisfies the measurements. Because of incoherent noise in the measurements, the inverse procedure usually is an ill-posed problem that must be regularized by additional constraints to reduce the nonuniqueness and stabilize the solution. The conventional way of solving this ill-posed inverse problem is the Tikhonov approach (Tikhonov and Arsenin, 1977). The mathematical formulation of the Tikhonov regularized method is

$$\Phi(m) = \Phi_d(m) + \lambda \phi_m(m), \quad (1)$$

$$\phi_d(m) = \frac{1}{2} \|Gm - d^{obs}\|^2, \quad (2)$$

where the first term $\phi_d(m)$ is the data misfit that minimizes the difference between the predicted data and the observed data, and the second term $\phi_m(m)$ represents regularization that enforces a priori knowledge of the shape and structure of the model. We will discuss more details of $\phi_m(m)$ in the following section. The regularization parameter $\lambda$ is a positive value that determines the goodness of fit to the measured data and the relative fit to the priori model. The forward modeling $Gm$ provides predicted data, given the vector of real model parameters $m$ of size $N_m = N_xN_yN_z$, and $d^{obs}$ is the observed data vector of size $N_d$.

**Boundary-preserving regularization**

To choose the sharp boundary of geologic targets, we can choose the boundary-preserving regularization — MGS (Charbonnier et al., 1997; Portniaguine and Zhdanov, 1999). This MGS regularization attempts to produce blocky models by preserving large spatial gradients of a model and smoothing small spatial gradients. The functional is written as

$$\phi_m^{mgs}(m) = \sum_i \phi_i, \quad (3)$$

where

$$\phi_i = \frac{|\nabla m_i|^2}{|\nabla m_i|^2 + \beta^2}, \quad (4)$$

subscript i is the pixel index, and $\beta$ is a scaling factor to influence what solutions we obtain. When $\beta > \max(\nabla m)$, equation 4 reduces to the first-order operator regularization; when $\beta < \min(\nabla m)$, equation 4 becomes the zero-order operator regularization (Aster et al., 2005). To sharpen the model, we can pick up $\beta$ in the range of the model gradient; i.e., $\min(\nabla m) < \beta < \max(\nabla m)$. We then adjust $\beta$ by trial and error to produce an acceptable model.

We use a constrained Gauss-Newton scheme to minimize the objective function (equation A-1) that combines the data misfit with the MGS model misfit. The inverted model parameters are forced to lie within their physical bounds using a nonlinear transformation procedure (Abubakar et al., 2008). We further enforce a reduction in the objective function using a line-search method after each iteration (Pidlisecky et al., 2007). The computational details of solving the objective function with the boundary-preserving regularization are documented in Appendices A and B.

The forward traveltime calculation in the inversion algorithm is computed by solving the eikonal equation using the finite-difference scheme (Hole and Zelt, 1995). Raypaths are obtained by following the steepest gradient of the traveltime map from a receiver back to the source (Vidale, 1990). In the synthetic and field examples below, ray coverage is well defined because of dense shots and receivers along the model.

**SYNTHETIC EXAMPLES**

First, we demonstrate the performance of this method on two synthetic crosswell seismic data sets of increasing complexity and realism. The purpose of these model studies is to explore the potential of this approach for delineating the geometry of geologic anomalies and even the internal structure inside targets.

The starting models for the two cases of inversions are homogeneous with mean values of the true models. The reference models
are identical to the starting models when a priori information is not available. In the synthetic examples, the coefficients for the regularization matrix in the x- and z-directions are \( a_x = 1.0, a_z = 1.0 \), and \( a_{\text{lap}} = 1.0 \) (see equation 16 in Pidlisecky et al., 2007). We solve equation B-1 to find \( \lambda \). The scale factor \( c \) is selected visually to make solutions similar to the true model. For synthetic tests, we choose \( c = 1.0 \). The accuracy is further assessed by computing the data error, which is defined as

\[
E = \sqrt{\sum_{j=1}^{N_d} (d_{\text{obs}}^j - d_{\text{calc}}^j)^2},
\]

where \( N_d \) is the number of data unknowns and \( d_{\text{obs}} \) and \( d_{\text{calc}} \) are the observed and calculated traveltime values.

In the first example, we consider a velocity model with two triangle anomalies, one with high velocity and one with low velocity, shown in Figure 1a. The model dimension is 300 m in the horizontal direction, and it is 600 m in the vertical direction with uniformly spaced sampling of 10 m. We set up 56 sources and receivers in two wells to collect the data. The well spacing is 240 m. Synthetic data are computed by solving the eikonal equation using the finite-difference scheme (Hole and Zelt, 1995). The inversion zone is the same as the forward computation size, and it is discretized in 31 \( \times \) 61 grids.

First, we ran our inversion algorithm using the \( L_2 \)-norm smooth-constrained regularization (Pidlisecky et al., 2007) with \( \lambda = 2.75 \times 10^{-5} \). Convergence was reached at 10 iterations. At the end of the iteration, the data error was reduced from 5.6\% to 0.39\%. The inversion result is shown in Figure 1b. The image is quite smeared. Although it retrieves the locations of two anomalies, their boundaries are smoothed. Also, the inverted velocity values of the two anomalies deviate from the true values. In addition, there are side lobes in the inverted model (Figure 1b) directly above and below the true location of the anomalies.

Second, we ran the inversion using the boundary-preserving constraint (equation 4). We kept the same \( \lambda \) as the smooth-constrained inversion. A \( \beta \) value of 0.1 was used for the boundary-preserving inversion. At the end of the 10 iterations, the data error reduces from 5.6\% to 0.33\%, about the same as with the normal regularization. The resulting velocity model is shown in Figure 1c. The two anomalies are recovered quite well, and the side lobes above the true locations seen in Figure 1b are mostly eliminated, except the remaining lower zone. The boundaries of anomalies also are clearly much better preserved, and the values of the anomalies are surprisingly closer to the true values.

In the second example, we built a more realistic model based on the velocity model from the King Mountain field study in west Texas (Langan et al., 1997). The synthetic velocity model is shown in Figure 2a. The velocity of the target area is lower than the surrounding rocks, and the transitional feature is imagined for a reservoir with varying fluid saturation inside. Another low-velocity anomaly is located in the bottom-left corner.

The model dimension is 400 m in the horizontal direction and 500 m in the vertical direction. There are 43 sources and receivers in the two wells to collect data, respectively. The well spacing is 350 m. Figure 2a also shows the source-receiver geometry setup. Synthetic data were generated by solving the viscoacoustic wave equation (Zhu and Harris, 2014). Figure 2b shows a common source gather at the source depth 190 m, where the red line indicates the picked first-arrival times. Similarly, we picked all first-arrival times for 43 common source gathers. Then, we corrupted the picked traveltime data with Gaussian random noise (1\% of the mean data value).

For inversion, the synthetic traveltime data were limited to a 45° aperture. The inversion grid size was identical to the forward computation size, that is, discretized with a 41 \( \times \) 51 grid. Again, we ran the \( L_2 \)-norm smooth-constrained inversion. An optimal value of \( \lambda = 0.48 \times 10^{-5} \) was found to give the best model. This value corresponds to a data misfit of \( \phi_d = 0.0035 \) at the tenth iteration.
The second row of Figure 3 shows the smooth-constrained inversion results. The recovered model is acceptable, but the boundaries and internal structure of the reservoir are only smoothly resolved. From the gradients of the resulting model in Figure 3e and 3f, we can observe that the boundaries of the target are smeared out and the velocity contrasts are too low, which creates errors for interpretation.

In contrast, the boundary-preserved inversion results in the last row of Figure 3 sharpen the geometry of the anomalies. We can observe the top and the base of the reservoir clearly in Figure 3h. The small target below the reservoir is delineated as well. Surprisingly, the dipping internal structures appear, but they are hidden in the results of the smooth-constrained inversion. These boundaries are also clearly identified in gradient maps, which are, respectively, the vertical and horizontal gradients (Figure 3h and 3i). For the boundary-preserved inversion, we choose a value of $\lambda = 0.77 \times 10^{-4}$ and a $\beta$ value of 0.2, resulting in a data misfit of $\phi_d = 0.0034$ at the 10th iteration. The final data misfit of this approach is slightly better than that of the $L_2$-norm regularized inversion.

REAL DATA APPLICATIONS

The boundary-preserving inversion approach has been applied to two field data sets to investigate the capability of this approach in the delineation of the reservoir bodies and CO$_2$-saturated zones. The details of the test configuration and results are discussed below.

Case I: Delineating reservoir boundaries

In the Permian Basin of west Texas, the stratigraphy and structure in and near a complex limestone reservoir are not well resolved from previous surface seismic studies (Langan et al., 1997). Although further qualitative information about the reservoir boundaries was provided by the crosswell tomographic approach (Langan et al., 1997), the reservoir boundaries, in particular the bottom of reservoir, are still ambiguous to some degree in the previous tomograms. In this section, we apply our boundary-preserving inversion approach to this data set for delineating the reservoir boundaries.

The crosswell data were collected for 201 sources (piezoelectric cylindrical bender source) spaced at approximately 1.5 m (5 ft) depth intervals and 203 receivers, also at 1.5 m spacing. The input source signal was a 250–1250 Hz linear upswep. Thus, we have approximately 40,000 traces of raw data. The data quality is good enough to pick the first arrivals, although tube waves are obvious at a depth of approximately 2730 m (8950 ft). Primary picking was carried out with common shot gatherers, an example of which is shown in Figure 4a for a source at a depth of approximately 2600 m (8600 ft). The peak of the first arrivals is usually picked indicated by the red line in Figure 4a. The traveltime picks were subsequently checked for consistency in four domains: common shot, common receiver, common offset, and common middepth. Figure 4b shows the final picked traveltime map in the source and receiver location display. The traveltime variation in the diagonal (i.e., common middepth gather) reveals the velocity varying in the vertical depth. Several anomalies of smaller traveltimes are related to high-velocity zones.

The 2D inverse domain size is $21 \times 206$, for the total number of 4326 unknowns. The grid spacings in the horizontal and vertical directions are 9.5 and 1.5 m (32 and 5 ft), respectively. We will run $L_2$-norm smooth-constrained inversion and our boundary-preserving constrained inversion. The starting velocity models for two inversions have constant velocity, which is the average velocity value obtained from the zero-offset data. The coefficients for the regularization matrix in the $x$- and $z$-directions are $a_x = 1.0$, $a_z = 0.5$, and $a_{\text{off}} = 1.0$. The parameters $\lambda$ for $L_2$-norm smooth and boundary-preserving inversions are obtained by solving equation B-1, which gives 0.0017 and 0.0011 at the first iteration. The cooling coefficient is $c = 0.8$. A $\beta$ value of 0.01 is experimentally chosen for the boundary-preserving inversion. The inversion converged within 20 iterations. The parameters $\lambda$ decrease to approximately $1.24 \times 10^{-5}$ and $1.36 \times 10^{-6}$, respectively.

Figure 5a displays the inversion results by the $L_2$-norm smoothness constraint and Figure 5b with the boundary-preserving constraint. Overall, a smoothness-constrained inversion only roughly produces the boundaries of the reservoir, and it does not exhibit clear boundaries at depths of approximately 2650 (~8700 ft) and 2700 m (~8900 ft) (Figure 5a). In contrast, the boundary-preserved results shown in Figure 5b give improved delineation of the reservoir volume (unit B). Besides the top and bottom of the reservoir, the lateral extent of the reservoir between approximately 2650 and 2700 m (8700 and 8900 ft) is better constrained. Several other striking features are identified in Figure 5b. These include the

![Figure 3. (a-c) True effective velocity model, (d-f) inversion results by smooth-constrained regularization, and (g-i) boundary-preserving regularization. The corresponding gradients in the vertical and horizontal directions are shown in the middle and right panels for each row.](image-url)

Figure 3. (a-c) True effective velocity model, (d-f) inversion results by smooth-constrained regularization, and (g-i) boundary-preserving regularization. The corresponding gradients in the vertical and horizontal directions are shown in the middle and right panels for each row.
boundaries at approximately 2600 (8500 ft) and 2650 m (8700 ft), corresponding to the shale layer (unit A). Notably, the high-velocity zone (unit C) in the bottom-right part at approximately 2730 m (~9100 ft) is more pronounced. The lower velocity layer (unit D) exhibits a significant lateral variation below a depth of 2730 m (9100 ft). The boundaries of unit D show a good correlation with the well logs.

These observations are also verified in the corresponding gradient distribution shown in Figure 6. Figure 6b further shows some small features in the left side that have a good correlation to \( V_p \) well logs. In addition to these visual geometric features, the velocity values in the reservoir unit are more or less close to that by \( L_2 \)-norm smoothness inversion.

The two inversions converge to the same approximate data misfits as shown in Figure 7. The rms data misfit value of the boundary-preserving result reduces to \( 8.1 \times 10^{-5} \) compared with the counterpart value of \( 7.7 \times 10^{-5} \) by conventional smoothness. The boundary-preserving result has a slightly larger rms error than that of the \( L_2 \)-norm regularized inversion. Remarkably, the boundary-preserving regularization inversion converges in fewer iterations than does the \( L_2 \)-norm regularized inversion.

**Case II: Delineating the CO\(_2\)-saturated zone**

Our boundary-preserving inversion approach is subsequently applied to a field time-lapse data set for delineating a CO\(_2\)-saturated zone. A time-lapse monitoring project was conducted in the McElroy field in west Texas using crosswell acquisition geometry. The baseline data set was acquired in 1993, and a monitor data set was acquired in 1995. The project was executed as a pilot study to monitor changes (e.g., seismic velocity) in the reservoir in response to CO\(_2\) injection into the reservoir. We use data collected between wells A and C. The well spacing is approximately 190 m (~620 ft). Harris et al. (1995) describe the data collection procedure and relevant survey parameters (also see Lazarotos and Marion, 1997; Arogumati and Harris, 2012). The first-arrival traveltimes for baseline (1993) and repeat surveys (1995) were picked and subtracted to obtain the travelt ime difference. Using this travelt ime difference data, we then perform differential tomography to estimate velocity changes. The formulation of this differential tomography approach will be

\[
\Delta d = G \Delta m, \tag{6}
\]

where \( G \) is the raypath matrix, \( \Delta d \) is the travelt ime data difference between two surveys, and \( \Delta m \) is the model change. The raypaths will be traced in the baseline model for the inversion be-
The velocity decreases after injection in McElroy field. Two crosswells were placed up-dip to 860 m (2850 ft) from the injection well. The reservoir zone is connected to the injection layer. The velocity change area in the overburden layer is approximately 860 (2850 ft) to 900 m (2930 ft). The boundary-preserving constrained inversion produces sharper boundary of the velocity change zone (see Figure 8b) than the result based on the smooth-constrained schemes (Figure 8a).

These anomalies may be caused by inversion-related artifacts or traveltime errors. The first reason is that the inversion-related velocity changes in the overburden are very small (<1% from Figure 9a). The second reason is that their distribution seems to extend along the raypath from the bottom to the top corner (Figure 8a). This may imply that these velocity anomalies are associated with time-lapse data acquisition errors. Conversely, the boundary-preserving constrained inversion that preserved large gradients and smoothed small gradients can reduce anomalous artifacts in the overburden layer in Figure 8b. In addition, the reservoir zone is connected in Figure 8b, although there exists the lateral variation. These results are compared in detail with available sonic logs in Figure 9, where the velocity difference is displayed as a percentage change from the baseline and expanded around the reservoir zone. The top and bottom boundaries of the CO$_2$ injection layer correlate well with the well log (black arrows in Figure 9). Interestingly, the heterogeneities in these velocity changes may reveal the spatial distribution of CO$_2$ and the pore pressure zone.

**DISCUSSION**

As we observe from our velocity tomogram results, smooth-constrained inversion presents a relatively smoothed velocity model. It is inaccurate to use such maps to interpret the boundaries of geologic structures that are commonly characterized by sharp boundaries associated with lithologic boundaries. In this study, boundary-preserving regularized inversion successfully sharpens these geologic boundaries. Comparisons between the performances of boundary-preserving inversion and smooth-constrained inversion using synthetic and field data show the superiority of the former when applied to data from geologic structures with clear boundaries.

It is also clear from synthetic examples (Figures 1 and 3) that the boundary-preserving inversion provides a slightly better velocity estimate than that by smooth-constrained inversion,
largely because the anomaly is focused into an appropriate geometry. In real data applications, we further verify our observations by comparing them with other known information. For instance, we can observe that the maximum P-wave velocity reduction is approximately 300 m/s (up to 1000 ft/s) from Figure 8b. The large reductions are attributable to the CO₂ saturation and pore pressure increase in the McElroy field (Harris et al., 1995; Wang et al., 1998). The P-wave velocity decreases by an average of 5% (Figure 9b), which agrees with observations from previous studies (Lazaratos and Marion, 1997; Wang et al., 1998). In contrast, smooth-constrained inversion that overaverages velocity in the inversion domain might underestimate the velocity difference as a result of CO₂ injection (by an average of 3% in Figure 9a).

Another advantage of boundary-preserving constrained inversion is preserving large gradients and meanwhile smoothing small gradients. For example, Figures 1 and 3 show clearly sharp boundaries of geologic zones but less artifacts elsewhere. Similar observations can also be found in real data applications in Figures 5 and 8.

Although in this paper, field applications include delineating velocity anomalies associated with reservoir and CO₂-saturated zones, there are many other applications; for example, delineating the fault zone from the velocity tomogram by boundary-preserving tomography is promising. We should be aware that the identification of boundaries in real applications might be subjective and may require additional information for verification, e.g., from well logs or production data. However, the boundary-preserving inversion method can be considered as an additional and complementary tool for obtaining the most consistently and reasonably geologic results for a given data set.

CONCLUSIONS

We have presented applications of the seismic boundary-preserving inversion algorithm that can incorporate a minimum-gradient support regularization. Synthetic results of the boundary-preserving constrained inversion are compared against smooth-constrained inversion, showing that boundary-preserving inversion is capable of providing velocity models with sharper contrasts. Also, it provides a better estimation of velocity values. The applications of two field data sets suggest that boundary-preserving inversion sharpens delineation of boundaries of the reservoir body and the CO₂-saturated zone, which are further verified by the well logs. Interestingly, the coherent variation in the velocity change zone can be observed in the boundary-preserving inversion results but not in the smooth-constrained inversion results. The velocity variation might reveal the spatial distribution of CO₂ and pore pressure changes. Based on these tests, therefore, we conclude that this boundary-preserving inversion is a valuable addition for reservoir characterization (e.g., delineation) as well as providing useful information for quantitative CO₂ monitoring.

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APPENDIX A

GAUSS-NEWTON MINIMIZATION

From equations 1–3, the linearized objective function is

$$\min \Phi(m) = \frac{1}{2} ||Gm - d^{\text{obs}}||^2 + \lambda \sum_i q_i. \quad (A-1)$$

To minimize equation A-1, we use a favorable Gauss-Newton strategy that should only require a few iterations to converge if the starting model is chosen appropriately (Nocedal and Wright, 1999). Newton-based methods also allow us to choose a different regularization parameter $\lambda$ at each inversion iteration. We can thus optimize regularization parameter $\lambda$ using the cooling approach (Haber et al., 2000).
The gradient operator of the objective function (equation A-1) is

\[ J = G^T(Gm - d^{mb}) + \lambda L(m)m, \]  

(A-2)

and the Hessian operator with dropping the second-order terms is

\[ H \approx G^TG + \lambda L(m), \]  

(A-3)

where

\[ L(m)m = D \text{diag}(\phi^c)Dm, \]  

(A-4)

and \( D \) is the first-order finite-difference approximate matrix, whose size is \( N_xN_z \times (N_x-1)(N_z-1) \) for the 2D problem. The elements \( b_{ii} \) of the diagonal matrix \( \text{diag}(\phi^c) \) are

\[ b_{ii} = \frac{\beta^2}{(\nabla m_i^c)^2 + \beta^2}, \]  

(A-5)

where \( i = 1, 2, \ldots, (N_x-1)(N_z-1) \). This auxiliary variable \( b \) marks the location of discontinuities, where \( b \) is equal to a large constant number \((1/\beta^2)\) when the model gradient is much less than \( \beta \). The regularization works like \( L_2 \)-norm smooth regularization and tends to produce the smooth model. Conversely, \( b \) has little effect when the model gradient is large \((\beta < \min(\nabla m))\); i.e., solutions do not penalize the large gradients: Thus, the boundary is preserved.

The second regularization term \( \frac{\beta_m^2}{\beta_m^2} \) is usually nonquadratic, but it is quadratic and convergent in \( m \) when the auxiliary variable \( b \) is fixed, as shown by Charbonnier et al. (1997).

At the iteration of the Gauss-Newton minimization process, the objective function reaches its minimum when the search direction \( \delta m \) satisfies

\[ H\delta m = -J. \]  

(A-6)

This system is solved using the conjugate gradient least-squares technique. The model is updated via

\[ m^n = m^{n-1} + \alpha \delta m, \]  

(A-7)

where \( m^n \) is the updated model at the \( n \)th iteration and \( \alpha \) is the step length to be determined by a line search algorithm (Pidlisecky et al., 2007).

To avoid unreasonable velocity values during inversion, we implement bounds on the physical values for velocities \( m_{\text{min}} \leq m^n \leq m_{\text{max}} \) using nonlinear transformation (Abubakar et al., 2008). This allows us to efficiently include a priori knowledge about the lower and upper velocity bounds. The stop criteria of inversion are the following: The total number of iterations exceeds a fixed maximum iteration, the data misfit function value is within a fixed tolerance factor, and the updated model values at the \( n \)th iteration are less than a tolerance factor. Either condition will stop the inversion procedure.

**APPENDIX B**

**CHOICE OF REGULARIZATION COEFFICIENT \( \lambda \)**

In equation A-1, we can see that the regularization coefficient \( \lambda \) aims to adjust the absolute values of Hessian matrix \( G^TG \) and regularization matrix \( L(m) \), which may differ by many orders of magnitude. To balance this difference, it is natural to choose the coefficient (Hansen, 2010)

\[ \lambda = \frac{\|G^TG\|}{\|L\|_p}. \]  

(B-1)

This holds for any matrix norm \((p = 1, 2, \infty)\). For \( p = 2 \), \( \|L\|_p = s_{\text{max}}(L) \), where \( s_{\text{max}} \) is the largest singular value of \( L \). Therefore, using the equation with the \( L_2 \)-norm equalizes the largest singular values of the Hessian and regularization matrix.

To minimize computational effort of computing the \( L_2 \)-norm of two matrices, we only compute them at the first iteration and obtain \( \lambda \). Then, we can conduct a cooling approach (Haber et al., 2000); i.e., we let the \( \lambda \) gradually decrease during the inversion. We scale the \( \lambda = \lambda c \) if the convergence rate \( \Delta = |\phi_c(\lambda m^n) - \phi_c(\lambda m^{n-1})|/(|\phi_c(\lambda m^n)|) \) is less than 1\%, where \( 0 < c \leq 1 \). As a result, we allow smaller singular values to gradually gain influence on the solution, which are likely associated to small-scale features in the model (Hansen, 1998).

**REFERENCES**


Hansen, P. C., 2010, Discrete inverse problems: Insight and algorithms: SIAM.


