Solving fractional Laplacian viscoacoustic wave equation using Hermite Distributed Approximating Functional Method
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SUMMARY

Accurate seismic modeling in realistic media severes the basis of seismic inversion and imaging. Recently viscoacoustic seismic modeling incorporating attenuation effects was done by solving a fractional Laplacian viscoacoustic wave equation. In this equation, attenuation, being spatially heterogeneous, is represented partially by the spatially varying power of the fractional Laplacian operator previously approximated by the global Fourier method. In this paper, we present a local spectral approach, based on Hermite distributed approximating functional (HDAF) method, to implement the fractional Laplacian in the viscoacoustic wave equation. The proposed approach combines local methods’ simplicity and global methods’ accuracy. Several numerical examples are presented to demonstrate the feasibility and accuracy of using the HDAF method for the frequency-independent Q fractional Laplacian wave equation.

INTRODUCTION

Attenuation and associated dispersion should be taken into account to accurately characterize wave propagation in real media. Many methods have been proposed to model acoustic attenuation effects during wave propagation (Stekl and Pratt, 1998; Carcione, 2007; Zhu et al., 2013). One of them is using constant-Q wave propagation in the time domain (Kjartansson, 1979; Carcione et al., 2002). It is accurate in producing desirable constant-Q behavior at all frequencies. However, it requires using a fractional time derivative to simulate the power-law stress-strain relation (Carcione et al., 2002; Carcione, 2010). One drawback of the fractional time derivative approach is that its implementation requires storing all the previous wavefields. Even when the fractional operators are truncated after certain time steps, the memory requirements are still too high (Podlubny, 1998; Carcione, 2008). To overcome the memory expense, Chen and Holm (2004) proposed to use fractional Laplacian operators to model anomalous attenuation. It successfully avoids additional memory required by the fractional time derivatives. Based on this idea, various types of wave equations using fractional Laplacians are derived to model wave propagation with attenuation and velocity dispersion (Carcione, 2010; Treeby and Cox, 2010). For example, Zhu and Harris (2014) developed a decoupled wave equation that accounts separately for amplitude attenuation and phase dispersion effects.

The fractional Laplacian was previously implemented mainly using the generalized Fourier transform approach (Zhu and Harris, 2014; Zhu and Carcione, 2014), by using an averaged fractional power of the Laplacian operator. This simple average of the power variable of the Laplacian operator undoubtedly introduces numerical errors of simulations, especially when attenuation model is strongly heterogeneous. To approximate the spatially varying power variable, Chen et al. (2014) and Sun et al. (2014) applied a low-rank approximation scheme to implement the spatially varying fractional Laplacians. The low rank approach can directly approximate the mixed-domain wave extrapolation operator with a separate representation and enable one to evaluate the variable fractional power of the Laplacian operator. However, these methods intensively rely on Fourier transform.

In this paper, we propose a local approach, based on the Hermite distributed approximating functional (HDAF), to implement the fractional Laplacians of Zhu and Harris (2014). The HDAF was originally introduced by Hoffman et al. (1991) and Hoffman and Kouri (1992) as a computational tool for treating a variety of problems in physics and chemistry (Wei et al., 1998a,b; Zhang et al., 1998; Kouri et al., 1999; Zhang et al., 1999; Pindza and Maré, 2014; Lesage et al., 2015). In the HDAF approach, an approximation to the Delta function is constructed by using even regularized Hermite polynomials. The study of the HDAF indicates that it can deliver spectral accuracy when used to solve partial differential equations. In addition, the discretization of HDAF method has some features similar in spirit to most finite-difference schemes (Zhang et al., 1998). Derivatives of a continuous function can be easily obtained by the convolution between the HDAF kernel and the function. Hence, the HDAF is a local spectral method. It exhibits global methods accuracy for differentiation and local methods flexibility in handling complex geometries and boundary conditions (Wei et al., 1998a).

VISCOACOUSTIC WAVE EQUATION

Based on the frequency-independent Q model, in which the attenuation coefficient is linear with frequency (Kjartansson, 1979), Zhu and Harris (2014) derived a viscoacoustic wave equation with decoupled fractional Laplacians:

\[ \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \eta (-\nabla^2)^{\gamma} p + \tau \frac{\partial}{\partial t} (-\nabla^2)^{\gamma+1/2} p \]

where \( P(x,t) \) is the pressure wavefield, and the fractional power \( \gamma = \arctan(1/Q)/\pi \). For any positive value of Q, we have \( 0 < \gamma < 0.5 \). We also have

\[ c^2 = c_0^2 \cos^2 (\pi \gamma/2), \]

\[ \eta = -c_0^2 \omega_0^2 \cdot 2 \cos (\pi \gamma), \]

\[ \tau = -c_0^2 \omega_0^{-1} \cdot 2 \sin (\pi \gamma), \]

with \( c_0 \) is the velocity defined at the reference frequency \( \omega_0 \). Note that the phase velocity \( c_0 \) and \( \gamma \) are spatially varying variables. The merit of this viscoacoustic wave equation is that the
velocity dispersion and amplitude loss phenomena of seismic attenuation can be separately modeled, which enables seismic imaging to compensate for attenuation with less effort (Zhu et al., 2014).

As we can see, when $\gamma$ is zero, viscoacoustic equation (1) reduces to the classical acoustic wave equation. The extra computation compared to acoustic is to calculate the fractional Laplacian operators. The fractional Laplacian operators are solved by Fourier based approach without difficulties when $\gamma$ is independent of position, i.e., medium is homogeneous. This is not always the case for subsurface geologic media. When heterogeneity in the subsurface is taken into account, the fractional Laplacian operator with spatially varying $\gamma$ becomes a mixed wavenumber-space domain problem when applying Fourier transform. Although the spatially varying $\gamma$ was approximated by an averaged value in smoothly heterogeneous media (Zhu and Harris, 2014), this is inaccurate in heterogeneous media. Recent advances in approximating the mixed-domain operators by low-rank approximation are possible to make better approximation to fractional Laplacian operators (e.g., (Chen et al., 2014; Sun et al., 2014, 2015). This approach is accurate to approximate such a mixed-domain fractional Laplacian operator, but it intensively uses Fourier transform. In this paper, we introduce a local method, HDAF, to compute the fractional Laplacian operators.

APPROXIMATING A FUNCTION AND ITS DERIVATIVES USING HERMITE DISTRIBUTED APPROXIMATING FUNCTIONAL

HDAF can be viewed as an approximate mapping of a certain set of continuous $L^2$ functions to themselves, accurate to a given tolerance (Zhang et al., 1997). The HDAF is able to provide an analytical representation of a function and its derivatives in terms of a discrete set of values of the function only. This is central to its success in various computational applications (Zhang et al., 1997; Wei et al., 1997).

It is known that the Dirac delta function has the following properties

$$u(x) = \int \delta(x-x')u(x')dx', \quad (5)$$

and

$$u^{(\alpha)}(x) = \int \delta^{(\alpha)}(x-x')u(x')dx', \quad (6)$$

where $\alpha \in \mathbb{R}$ and $u^{(\alpha)}(x)$ is $\alpha$th derivative of function $u(x)$.

However, equations 5 and 6 are of little numerical interest for practical application due to the occurrence of the delta distribution. In the HDAF approach, an approximation to the delta function is defined as (Hoffman and Kouri, 1995)

$$\delta_M(x-x', \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|x-x'|^2}{2\sigma^2}} M^{\alpha/2} \sum_{n=0}^{M/2} (-1)^n \frac{1}{n!} H_{2n}(\frac{x-x'}{\sqrt{2}\sigma}), \quad (7)$$

where the right hand side is a two-parameter delta sequence; $\sigma$, $M$ are the delta sequence (HDAF) parameters, and the $\rho^{th}$ Hermite polynomial is defined by

$$H_{\rho}(x) = (-1)^{\rho} e^x \frac{d^\rho}{dx^\rho} e^{-x}, \forall x \in \mathbb{R}. \quad (8)$$

The HDAF is constructed by using even Hermite polynomials (since the delta distribution is even in its argument) and is dominated by its Gaussian envelope, $\exp(-(x-x')^2/2\sigma^2)$. For any fixed $\sigma$, the HDAF becomes exactly identical to the delta function in the limit of infinite $M$. Additionally, for a fixed $M$, the HDAF becomes identical to the delta function when $\alpha$ goes to zero (Hoffman and Kouri, 1995; Zhang et al., 1997).

Unlike the delta function, the HDAF expression (equation 7) can be discretized by quadratures. The continuous approximation to a function $u(x)$ generated by the HDAF delta sequence is

$$u(x) \approx u_D(x) = \int \delta_M(x-x', \sigma)u(x')dx'. \quad (9)$$

Given a discrete set of functional values on a grid, the HDAF approximation to the function at any point $x_j$ can be obtained by

$$u(x_j) \approx u_D(x_j) = \Delta \sum_i W^i \delta_M(x_j-x_i, \sigma)u(x_i), \quad (10)$$

where $\Delta$ is the uniform grid spacing, and $W$ is the number of stencil points.

The approximation of the derivatives of a continuous function can also be generated using the HDAF

$$u^{(\alpha)}_D(x) \approx u^{(\alpha)}_D(x) = \int \delta^{(\alpha)}_M(x-x', \sigma)u(x')dx'. \quad (11)$$

where $\delta^{(\alpha)}_M(x-x', \sigma)$ is the $\alpha$th derivative of $\delta_M(x-x', \sigma)$, and is given by the two parameter delta sequence

$$\delta^{(\alpha)}_M(x-x', \sigma) = (-1)^{\alpha} \frac{\sigma^{\alpha}}{\sqrt{2\pi}\sigma^\alpha} M^{\alpha/2} \sum_{n=0}^{M/2} (-1)^n \frac{1}{n!} \frac{d^{2n+\alpha}}{dx^{2n+\alpha}} e^{-\frac{|x-x'|^2}{2\sigma^2}}. \quad (12)$$

When $\alpha$ is integer, the expression of HDAF is explicitly given by (Hoffman and Kouri, 1995)

$$\delta_M^{(\alpha)}(x-x', \sigma) = -\frac{(-1)^{\alpha}}{\sqrt{2\pi}\sigma^{\alpha+1}} \frac{1}{\sigma^\alpha} \frac{1}{\sqrt{2\sigma}} \sum_{n=0}^{M/2} (-1)^n \frac{1}{n!} H_{2n+\alpha}(\frac{x-x'}{\sqrt{2}\sigma}). \quad (13)$$

For non-integer $\alpha$, there are several known numerical algorithms to calculate the fractional order derivatives of HDAF, such as the Grunwald-Letnikov method (Petráš, 2008; Kumar and Rawat, 2012), finite difference quadrature approach (Huang and Oberman, 2014) and Fourier transform method (Carcione, 2010; Tseng et al., 2000). We employ the Fourier transform method:

$$\delta_M(x, \sigma) \xrightarrow{\text{FFT}} \delta_M^k(k, \sigma) \xrightarrow{(ik)^\alpha} \delta_M^{(\alpha)}(k, \sigma) \xrightarrow{\text{FFT}} \delta_M^{(\alpha)}(x, \sigma). \quad (14)$$
When uniformly discretized by Gauss quadrature, equation 11 gives
\[
 u^{(a)}(x_j) \approx u_p^{(a)}(x_j) = \frac{1}{\Delta} \sum_{i=1}^{W} \delta^a_{M}(x_j - x_i, \sigma) u(x_i). \tag{15}
\]

Equation 15, together with the expression of the fractional order HDAF, provides an algebraic way to approximate fractional derivatives of functions. With a proper choice of the parameters (\(\sigma, M,\) and \(W\)), HDAF provides arbitrarily high accuracy for estimating the derivatives of functions. Compared with other differentiation methods, the HDAF is extremely simple. The kernel of the HDAF fractional order differentiator is required to be calculated only once. And arbitrary order derivatives of a function can be obtained by the convolution between the fractional order derivatives of the HDAF and the function.

High dimensional HDAFs can be obtained using tensor products (Zhang et al., 1999). For example, the two dimensional HDAF is given as
\[
 \delta_{M}(x,x',z,z',\sigma) = \delta_{M}(x-x',\sigma_1) \delta_{M}(z-z',\sigma_2). \tag{16}
\]
In general, the parameters \(M\), and \(\sigma\) can be different in \(x\) and \(z\) directions. Then, one can write the HDAF approximation for a two-dimensional function \(u(x,z)\) as
\[
 u(x,z) = \int dx' dz' \delta_{M}(x,x',z,z',\sigma) u(x',z). \tag{17}
\]

**NUMERICAL EXAMPLES**

**Different Q media**

First we investigate the accuracy of the solution of the constant-Q wave equation using the proposed HDAF method for different Q values. The velocity model is homogeneous with a reference velocity \(c_0 = 2000\) m/s, defined at high frequency 1500 Hz. Simulations are performed with grid points of \(201 \times 201\), \(\Delta h = 10.0\) m spacing in the \(x\) and \(z\) directions, and a second order finite difference scheme for time discretization with a time step of \(0.25 \times 10^{-3}\) s. The source is chosen as a Ricker wavelet with the central frequency of 25 Hz located at the center of the model in simulations. Figure 1 shows snapshots taken at 400 ms with \(Q = (a) 100,\) (b) 30, (c) 10, and (d) 5. In this case, the HDAF parameters are chosen as: \(\sigma = 1.65\Delta h\), \(M = 24\), and \(W = 51\). We can see decreased amplitudes and delayed phase with decreasing Q values. Figure 2 compares the seismograms recorded 500 m away from the source using pseudospectral and HDAF methods corresponding to the Q values in Figure 1. The corresponding rms error values are \(2.6 \times 10^{-3}\), \(2.6 \times 10^{-3}\), \(6.5 \times 10^{-3}\), \(1.4 \times 10^{-3}\), respectively. We found that the smaller Q value results in the large error. The rms errors can be reduced by increasing the stencil length \(W\).

**Two layer model**

This example is aimed to test the HDAF method for viscoacoustic wave modeling in the presence of sharp contrast in velocity and Q. We use an isotropic two-layer model with \(c_0 = 1800\) m/s in the top layer and \(c_0 = 3600\) m/s in the bottom layer. The interface is at a depth of 1040 m. The reference frequency is the same as above. The model is discretized on a \(200 \times 200\) grid, with \(8\) m spacing in both directions. The time step is 1.0 ms. A Ricker wavelet with 25 Hz center frequency is located at the position \((800\) m, \(800\) m) of the model. The snapshots of wavefield at 330 ms are shown in Figure 3. Figure 3a shows the wavefield with homogeneous attenuation, where \(Q = 30\). Figure 3b shows the wavefield using HDAF method for the same velocity but with \(Q = 30\) for the top layer and \(Q = 100\) in the bottom layer. In Figure 3c, velocity and Q remain the same as those in Figure 3b; however, the fractional power of Laplacian \(\gamma\) is the averaged value, which corresponds to the original implementation in Zhu and Harris (2014). Figure 3d is the reference wavefield computed by the pseudospectral method for spatial varying Laplacian operator. It is implemented by calculating fractional Laplacian operator twice in wavenumber domain for two layer parameters and transforming them back to space domain.

To better compare the results, a middle trace at \(x = 800\) m is extracted from the wavefield snapshots. Figure 4a shows the two traces from Figure 3a and 3b, along with their difference. In the top layer, the velocity and Q values are the same for both models. Hence, the first propagating waveforms are identical. The effect of different attenuations can be observed from the other two waveforms. The transmitted arrival exhibits less attenuation for \(Q = 100\). Figure 4b shows two traces from Figures 3b and 3c. Note the difference between the two traces attributed to error of using an average \(\gamma\) when implementing fractional Laplacian operators. Figure 4c shows two traces from Figure 3b and 3d. We find that the results using HDAF and pseudospectral methods agree very well.

**CONCLUSION**

We propose the HDAF method to compute numerical approximations for the fractional Laplacian in the viscoacoustic wave equations. An advantage of the HDAF is that it transforms the Laplacian operator into a matrix vector multiplication that involves banded matrix representations, similar to local methods (i.e., finite-difference, finite-element) while preserving exponential accuracy of global methods such as spectral methods. Therefore, the HDAF can obtain the same level of accuracy as spectral methods, but also have sufficient flexibility to handle complicated geometries. Numerical results demonstrate that the proposed method can model wave propagation in attenuating media with high accuracy.

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Figure 1: Four snapshot parts corresponding to four Q values: a) Q=100, b) Q=30, c) Q=10 and d) Q=5. A point source located at the center of the model. Snapshots are recorded at 0.2 s in homogeneous attenuating media.

Figure 2: Comparison between two seismograms calculated by the pseudospectral method (solid) and HDAF method (circle) corresponding to four Q values: a) Q=100, b) Q=30, c) Q=10 and d) Q=5 at point 500m away from the source.

Figure 3: Snapshots at 330 ms for two layer model: (a) wavefield for homogeneous $Q = 30$; (b) wavefield for $Q = 30$ in the top layer and $Q = 100$ in the bottom layer; (c) wavefield using a constant averaged fractional power $\gamma$ in the same model as panel (b); (d) reference wavefield calculated with pseudospectral method in the same model as panel (b).

Figure 4: Trace at $x = 800$ m extracted from wavefield snapshots and their difference: (a) variable $\gamma$ (solid) using HDAF, homogeneous $Q = 30$ (dashed), and their difference (dashed-dot); (b) variable $\gamma$ (solid) using HDAF, average $\gamma$ (dashed), their difference (dashed-dot); (c) variable $\gamma$ (solid) using HDAF, variable $\gamma$ (solid) using pseudospectral method (dashed), their difference (dashed-dot).
REFERENCES


