Fractional Laplacians viscoacoustic wavefield modeling with $k$-space-based time-stepping error compensating scheme

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ABSTRACT

The spatial derivatives in decoupled fractional Laplacian (DFL) viscoacoustic and viscoelastic wave equations are the mixed-domain Laplacian operators. Using the approximation of the mixed-domain operators, the spatial derivatives can be calculated by using the Fourier pseudospectral (PS) method with barely spatial numerical dispersions, whereas the time derivative is often computed with the finite-difference (FD) method in second-order accuracy (referred to as the FD-PS scheme). The time-stepping errors caused by the FD discretization inevitably introduce the accumulative temporal dispersion during the wavefield extrapolation, especially for a long-time simulation. To eliminate the time-stepping errors, here, we adopted the $k$-space concept in the numerical discretization of the DFL viscoacoustic wave equation. Different from existing $k$-space methods, our $k$-space method for DFL viscoacoustic wave equation contains two correction terms, which were designed to compensate for the time-stepping errors in the dispersion-dominated operator and loss-dominated operator, respectively. Using theoretical analyses and numerical experiments, we determine that our $k$-space approach is superior to the traditional FD-PS scheme mainly in three aspects. First, our approach can effectively compensate for the time-stepping errors. Second, the stability condition is more relaxed, which makes the selection of sampling intervals more flexible. Finally, the $k$-space approach allows us to conduct high-accuracy wavefield extrapolation with larger time steps. These features make our scheme suitable for seismic modeling and imaging problems.

INTRODUCTION

Due to the intrinsic viscous character, seismic waves suffer amplitude loss, phase distortion, and frequency-content reduction during propagation in real underground media. These attenuating effects are widely observed in laboratory measurements (e.g., Wueneschel, 1965; Wang et al., 2007) and field surveys (e.g., McDonal et al., 1958). To accurately simulate wave propagation in the earth, the dispersion and dissipation caused by attenuation should be considered during wavefield extrapolation. Among various wave-modeling solvers (Aki and Richards, 1980; Day and Minster, 1984; Carcione et al., 1988; Blanch et al., 1995; Carcione et al., 2002; Zhu and Harris, 2014) incorporating the attenuation effects, the decoupled fractional Laplacian (DFL) wave equations (Zhu and Carcione, 2014; Zhu and Harris, 2014) have attracted much attention recently due to the following three favorable features. First, the DFL equation is able to characterize the frequency-independent $Q$ accurately. Second, the decoupled Laplacians can separate phase dispersion and amplitude attenuation effects, which would facilitate the attenuation-compensated reverse time migration (RTM) (Zhu et al., 2014; Li et al., 2016; Wang et al., 2017; Zhu and Sun, 2017) and full-waveform inversion (FWI) (Chen and Zhou, 2017; Xue et al., 2017). Third, different from the fractional time derivatives,
the fractional Laplacians are calculated using the fast Fourier transform (FFT), thus avoiding the requirement to store previous wavefields to improve the computational efficiency (Chen and Holm, 2004; Carcione, 2010; Treeby and Cox, 2010).

Nevertheless, two unresolved issues still exist in solving DFL wave equations: the mixed-domain problem and time-stepping errors. The first issue arises because the subsurface is heterogeneous so that the fractional Laplacian operators are spatial and wavenumber functions. Several strategies have been proposed to cope with this mixed-domain problem. The first category is to approximate the spatially varying fractional Laplacian by advanced numerical schemes, e.g., the low-rank approximation (Sun et al., 2015b; Chen et al., 2016a), finite-difference Hermitian approximation (Yao et al., 2017), phase shift plus interpolation (Ji et al., 2017), matrix transform (Chen et al., 2019a), and domain decomposition (Xue et al., 2018). The second one chooses a reformulation of the spatially varying Laplacians with weighted fixed-order Laplacians and finally derives a constant fractional-order DFL viscoacoustic wave equation (Chen et al., 2016a, 2019a, 2019b; Wang et al., 2018b; Xing and Zhu, 2018; Yang and Zhu, 2018), in which the mixed-domain problem is naturally avoided and the decoupling of attenuation and dispersion is inherited.

The second issue is associated with the second-order temporal accuracy in discretizing DFL wave equations. The time-stepping errors usually lead to unwanted numerical dispersion, in particular, which are accumulated to become unacceptable for long propagation times (Dablain, 1986). Various strategies have been proposed to suppress these temporal errors. The schemes include, for example, the Lax-Wendroff approach (Lax and Wendroff, 1964; Dablain, 1986; Chen, 2011), the rapid expansion method (Pestana and Stoffa, 2010; Tessmer, 2011), the one-step method (Zhang and Zhang, 2009; Sun et al., 2015a), the k-space method (Bojarski, 1982; Mast et al., 2001) or pseudoanalytical method (Etgen and Brandsberg-Dahl, 2009; Crawley et al., 2010), the Fourier finite-difference method (Song and Fomel, 2011; Song et al., 2013), various other optimization schemes (Liu and Sen, 2009, 2013; Liu, 2013; Chen et al., 2016c), and some filter-based postprocessing schemes to remove time dispersions from the synthetic data (Stork, 2013; Wang and Xu, 2015; Gao et al., 2016; Li et al., 2016a; Koene et al., 2018). Although there are many ways to suppress time dispersion, as mentioned above, the second-order-accuracy FD discretization has been widely used to solve the time derivative in DFL wave equations (Zhu and Carcione, 2013; Zhu and Harris, 2014; Li et al., 2016b; Wang et al., 2018a, 2018b; Zhao et al., 2018; Zhu and Bai, 2019), which limits the time step to a small value and, hence, decreases the computational efficiency. Chen et al. (2016a) propose an analytical three-step time marching scheme to suppress the temporal dispersion, which can accurately extrapolate the wavefield in a homogeneous medium, and this three-step scheme is further approximated to obtain the DFL wave equation in heterogeneous media. Although these approximations introduce a certain degree of errors, the temporal accuracy is still far superior to the traditional FD-PS scheme. In this study, we propose an alternative and simple k-space-based method to suppress the temporal dispersion in solving the DFL viscoacoustic wave equation.

A brief introduction of k-space methods is as follows: Fornberg and Whitham (1978) first attempt to use the k-space method to solve a nonlinear wave equation. Currently, the k-space method is widely used in bioengineering and geophysical problems to simulate the propagation of acoustic waves (Tabei et al., 2002; Tillett et al., 2009; Chen et al., 2016b), elastic waves (Firouzi et al., 2012; Chen et al., 2017; Gong et al., 2017), electromagnetic waves (Liu, 1994, 1995), and ultrasound waves (Treeby et al., 2011, 2012). For lossless media, the k-space correction term is a function of velocity and wavenumber and actually represents a space-wavenumber mixed-domain operator, when the velocity varies spatially. To treat this mixed-domain operator, Tabei et al. (2002) replace the spatially varying velocity with the maximum velocity for a given model. This maximum-velocity scheme may be acceptable in bioengineering problems because the velocities of biological tissues vary within a narrow range. However, this maximum-velocity scheme may be inapplicable to subsurface geologic media, where strong heterogeneity with large velocity contrasts is often the case. In geophysics, several reference velocities may be needed to achieve better accuracy (Etgen and Brandsberg-Dahl, 2009; Chu and Stoffa, 2011). In the same line, Fomel et al. (2013) develop an efficient low-rank approximation scheme to solve the mixed-domain operator for the heterogeneous case. The advantage of the low-rank approach is its ability to directly approximate the mixed-domain operator with a separable representation, which minimizes the number of FFTs per time step (Sun et al., 2015b).

The proposed method is essentially a generalization of the k-space concept to solve the DFL wave equations. However, differing from the existing k-space methods, our scheme includes two correction terms to compensate for the time-stepping errors in dispersion-dominated and loss-dominated operators, respectively. In the numerical implementation, we combine the correction terms and the original spatially varying Laplacians and solve the combined mixed-domain operators using the low-rank approximation (Fomel et al., 2013; Sun et al., 2015a; Chen et al., 2019b). Although the computation complexity of the combined mixed-domain operators is increased, the approximation accuracy is not degraded, and the computation cost barely increases. The proposed k-space method provides better temporal accuracy and a more relaxed stability condition than the traditional FD-PS method. This allows us to perform high-accuracy wavefield extrapolation with larger time steps, thus improving the overall computation efficiency.

The rest of this paper is organized as follows. We first review the DFL viscoacoustic wave equation (Zhu and Harris, 2014) and its traditional FD-PS simulation scheme. Then, we provide a detailed description of the proposed time-stepping error compensation approach, followed by a description of the stability analysis and numerical implementation. We verify the accuracy and computational efficiency using three simulation examples. Finally, we conduct discussion before drawing some conclusions.

### THEORY

#### DFL viscoacoustic wave equation

Derived from the frequency-independent Q model (Kjartansson, 1979), the DFL viscoacoustic wave equation (Zhu and Harris, 2014) is expressed as

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \eta (-\nabla^2)^{\gamma+1} p + \tau \frac{\partial}{\partial t} (-\nabla^2)^{\gamma+1/2} p, \quad (1)$$

where
Visco-modeling with error compensation

\[
\frac{\partial^2 p}{\partial t^2} = \int_{-\infty}^{\infty} \tilde{p}(\omega)(-i\omega)^2 e^{-i\omega t} d\omega. \tag{9}
\]

The ratio of the integrand in equations 8 and 9 is expressed as

\[
\phi(\omega) = \frac{e^{-i\omega R} + e^{i\omega R} - 2}{-\omega^2 \Delta t^2} = \text{sinc}^2(\omega \Delta t / 2), \tag{10}
\]

with \(\phi(\omega)\) being the time-stepping error correction term. When we select an infinitesimal time step \((\Delta t \to 0)\), the value of \(\phi(\omega)\) approaches 1; therefore, the FD discretization of the time derivative (equation 8) infinitely approaches the exact solution (equation 9).

Besides adopting a small time step, compensating the time-stepping errors via the correction term of \(\phi(\omega)\) to suppress the temporal dispersion during wavefield extrapolation is a more reasonable choice. In the numerical implementation, incorporating the time-stepping error correction term into the wavenumber domain is rather straightforward (Treeby et al., 2017). By transforming \(\phi(\omega)\) to \(\phi(k)\) via the dispersion relation, the time-stepping errors can be compensated effectively by multiplying with \(\phi(k)\) in the wavenumber domain.

Because, in the DFL viscoacoustic wave equation, the dispersion-dominated and the loss-dominated operators are decoupled; therefore, these two operators require two different correction terms. To do that, we propose the following scheme:

\[
p^{r+\Delta t} + p^{r-\Delta t} - 2p^{r} = \eta c^2 F^{-}\{k^{2r+2} \tilde{p}^{r}\}
\]

\[
+ \tau c^2 F^{-}\{k^{2r+1} (\tilde{p}^{r} - \tilde{p}^{r-\Delta t})\}, \tag{11}
\]

where \(\tilde{p}\) denotes the pressure in the wavenumber domain, \(F^{-}\) denotes the inverse Fourier transform, and \(\Delta t\) is the temporal sampling interval. Note that, for brevity, we use \(k\) to represent the norm of wavenumber vector \(\mathbf{k}\).

Compensation for the time-stepping errors

To demonstrate the time-stepping errors, we first define the following Fourier transforms:

\[
\begin{cases}
p^{r} = \int_{-\infty}^{\infty} \tilde{p}(\omega) e^{-i\omega t} d\omega, \\
p^{r+\Delta t} = \int_{-\infty}^{\infty} \tilde{p}(\omega) e^{-i\omega t} e^{i\omega R} d\omega, \\
p^{r-\Delta t} = \int_{-\infty}^{\infty} \tilde{p}(\omega) e^{-i\omega t} e^{-i\omega R} d\omega,
\end{cases} \tag{7}
\]

where \(\tilde{p}(\omega) = 1/2\pi \int_{-\infty}^{\infty} p^{r} e^{i\omega t} d\omega\). Substituting equation 7 into equation 5 yields

\[
\frac{\partial^2 p^{r+\Delta t} + p^{r-\Delta t} - 2p^{r}}{\Delta t^2} = \int_{-\infty}^{\infty} \tilde{p}(\omega)\left(\frac{e^{-i\omega R} + e^{i\omega R} - 2}{\Delta t^2}\right) e^{-i\omega t} d\omega. \tag{8}
\]

According to the differential property of the Fourier transform, we have the following exact expression:

\[
\eta = -c_0^{-2} \omega_0^{-2} \cos(\pi \gamma), \quad \tau = -c_0^{-2} \omega_0^{-2} \sin(\pi \gamma),
\]

\[
c = c_0 \cos(\pi \gamma / 2), \quad \gamma = \arctan(1/Q)/\pi,
\]

where \(p\) represents the pressure in the time domain, \(\nabla^2\) represents the Laplacian, and \(c_0\) is the reference velocity at the reference angular frequency \(\omega_0\). Here, we set \(\omega_0 = 20\pi f_d\), where \(f_d\) denotes the dominant frequency of the source. According to Zhu and Harris (2014), equation 1 can be divided into the dispersion-dominated wave equation

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \eta (\nabla^2)^{r+1} p, \tag{3}
\]

and the loss-dominated wave equation

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \nabla^2 p + \tau \frac{\partial}{\partial t} (\nabla^2)^{r+1/2} p. \tag{4}
\]

The second-order time derivative in equation 1 is commonly solved by the FD operator with second-order accuracy:

\[
\frac{\partial^2 p^{r+\Delta t} + p^{r-\Delta t} - 2p^{r}}{\Delta t^2} = \eta c^2 F^{-}\{k^{2r+2} \tilde{p}^{r}\}
\]

\[
+ \tau c^2 F^{-}\{k^{2r+1} (\tilde{p}^{r} - \tilde{p}^{r-\Delta t})\}, \tag{6}
\]

where \(\tilde{p}\) denotes the pressure in the wavenumber domain, \(F^{-}\) denotes the inverse Fourier transform, and \(\Delta t\) is the temporal sampling interval. Note that, for brevity, we use \(k\) to represent the norm of wavenumber vector \(\mathbf{k}\).

The characteristics of temporal dispersion

According to Levander (1988), spatial dispersion usually leads to a phase delay, whereas temporal dispersion usually results in a phase advance. Here, we present an intuitive illustration of the characteristics of the temporal dispersion by wavefield simulations. The model is homogeneous with a reference velocity of \(c_0 = 2000\) m/s, \(Q = 100\), and the dominant frequency of the source is 12.5 Hz.
Figure 1a shows four subsnapshots at $t = 3$ s computed by applying the FD-PS method to equation 6 with different temporal sampling intervals of 1, 2, 3, and 4 ms. It is evident that the advanced numerical dispersion gradually becomes severe with larger time steps. We also test a longer time ($t = 18$ s) simulation using the FD-PS method with a time step of 4 ms and show the snapshot in Figure 1b, from which we see that the wavefront is blurred significantly. This severe temporal dispersion would likely decrease the resolution of migrated images and eventually affect the reliability of seismic interpretation. Correspondingly, snapshots in Figure 1c and 1d generated by the proposed $k$-space approach (equation 11) show a stable simulation and exhibit no visible temporal dispersion.

**Numerical implementation**

Assuming that the medium is homogeneous, it is straightforward to extrapolate wavefields using equation 6 by the traditional FD-PS method or using the proposed $k$-space formulation 11. To model wave propagation in heterogeneous media, we must deal with mixed-domain fractional Laplacian operators in equation 6.

In equation 6, the mixed-domain operators of spatial variable-order fractional Laplacians can be expressed as

$$
L_{\text{FD-PS1}}(x, k) = k^{2\gamma(x)+2},
$$

$$
L_{\text{FD-PS2}}(x, k) = k^{2\gamma(x)+1}.
$$

In equation 11, apart from the fractional Laplacians, there are the additional mixed-domain correction terms $\phi_1(k)$ and $\phi_2(k)$. We combine these two parts of the mixed-domain operators, respectively, into one as

$$
\begin{bmatrix}
L_{k,1}(x, k) = \phi_1(x, k)k^{2\gamma(x)+2}, \\
L_{k,2}(x, k) = \phi_2(x, k)k^{2\gamma(x)+1}
\end{bmatrix}
$$

To approximate the mixed-domain operators in equations 13 and 14, we use the low-rank decomposition method (Fomel et al., 2013); a detailed description of low-rank approximation can also be found in Sun et al. (2016) or Guo et al. (2016).

Next, we test the low-rank decomposition accuracy by using a 2D model with a total grid number of $N = 128 \times 128$. A linearly increasing velocity model is shown in Figure 2, and the $Q$ model is built from the velocity model by the empirical function of $Q = 3.516(c_0/1000)^{3/2}$ (Li, 1993). We set the spatial grid spacing to 10 m and the time step to 1 ms. Figure 3a and 3c displays the exact matrices of $L_{\text{FD-PS2}}$ in equation 13 and $L_{k,2}$ in equation 14, respectively. We approximate these two exact matrices using the low-rank method with the rank of $m = n = 2$ and illustrate the differences between the exact matrices and approximation matrices in Figures 3b and 3d. We implement these low-rank approximations using the MATLAB 2016a platform on a Linux workstation (Intel Xeon CPU v4, 3.00 GHz, 256 GB RAM). The calculation times of approximating $L_{\text{FD-PS2}}$ and $L_{k,2}$ are 7.3 and 9.4 s, respectively. Compared with $L_{\text{FD-PS2}}$, the complexity of $L_{k,2}$ increases, and the computation time of $L_{k,2}$ is slightly increased. However, as shown in Figure 3, the approximate accuracy of $L_{k,2}$ is not degraded.

**Stability analysis**

In this section, we analyze the stability of the FD-PS and $k$-space approaches for the 2D case using the eigenvalue method (Gazdag, 1981). Equations 6 and 11 are transformed to the wavenumber domain; then, they are rewritten in the matrix form as

$$
\begin{bmatrix}
p^{t+\Delta t} \\
p^t
\end{bmatrix}
= \begin{bmatrix}
2 + a \Delta t^2 + b \Delta t & -1 - b \Delta t \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
p^{t+\Delta t} \\
p^t
\end{bmatrix},
$$

where $a = c_0^2 \gamma \Delta t^{2\gamma+2}$ and $b = c_0^2 \gamma \Delta t^{2\gamma+1}$ for the FD-PS method and $a = \phi_1(k)c_0^2 \gamma \Delta t^{2\gamma+2}$ and $b = \phi_2(k)c_0^2 \gamma \Delta t^{2\gamma+1}$ for the $k$-space method. For stable simulation, the eigenvalues of the matrix on the right-hand side of equation 15 must be less than or equal to 1.0 in magnitude (Gazdag, 1981). Solving the eigenvalues $\lambda$ of the matrix and assuming $\lambda \leq 1$ gives (Zhu and Harris, 2014)

$$
\Delta t \leq \frac{-b + \sqrt{-2a}}{a}.
$$

Substituting the corresponding values of $a$ and $b$ into equation 16, we obtain the stability condition corresponding to the FD-PS method as

$$
\Delta t \leq \frac{\tan(\pi \gamma)}{c_0 k} + \frac{\sqrt{2} a \gamma}{(c_0 k)^{2\gamma+1} \sqrt{\cos(\pi \gamma)}}
$$

and the $k$-space method as
\[
\Delta t \leq -\frac{\phi_2(k) \tan(\pi \gamma)}{\phi_1(k)} + \frac{\sqrt{2} \omega_0^c}{c_0 k (c_0 k)^{\gamma+1} \sqrt{\cos(\pi \gamma) \phi_1(k)}}. \quad (18)
\]

Next, we explain the stability condition of these two methods with the following assumptions: The reference velocity is fixed at \(c_0 = 2000 \text{ m/s}\), \(Q\) increases gradually from 10 to 150, and the wavenumber is selected as the Nyquist spatial wavenumber of \(k = \pi / \Delta x\). There are two terms on the right side of equation (17), and we define the ratio of the two terms as \(\zeta = \left(\frac{\sqrt{2} \omega_0^c}{(c_0 k)^{\gamma+1} \sqrt{\cos(\pi \gamma)}\phi_1(k)}\right) / \left(\tan(\pi \gamma) / c_0 k\right)\). As displayed in Figure 4a, the value of \(\zeta\) increases with \(Q\); even the minimum is larger than 10, which means that the second term \(\sqrt{2} \omega_0^c / (c_0 k)^{\gamma+1} \sqrt{\cos(\pi \gamma)}\) is dominant in equation (17). Similarly, in equation (18), \(\phi_1(k)\) closes to \(\phi_2(k)\) (as shown in Figure 4b), the second term \(\sqrt{2} \omega_0^c / (c_0 k)^{\gamma+1} \sqrt{\cos(\pi \gamma)}\phi_1(k)\) is also the dominant operator. Comparing the second terms in equations (17) and (18), because \(\phi_1(k)\) is less than 1, in theory, the \(k\)-space method enjoys a more relaxed stability condition than the FD-PS method.
Although we have formulated the stability condition of the \( k \)-space method as shown in equation 18, it is difficult to give an explicit expression because \( \phi_1(k) \) and \( \phi_2(k) \) are functions of \( \Delta t \). Thus, we compare the stability conditions of the FD-PS method and the proposed \( k \)-space approaches by solving equations 6 and 11 using the low-rank approximation in a wide range of subsurface media parameters. We take \( \Delta x = \Delta z = h \) and denote the Courant number as \( \alpha = c \Delta t / h \). We increase \( Q \) from 10 to 150, and the maximum Courant numbers, which can be adopted to ensure stable wavefield propagation for 10 s, are displayed in Figure 5, where the line with the solid circles represents the FD-PS method and the line with the hollow circles corresponds to the proposed \( k \)-space approach. Obviously, the proposed \( k \)-space method can simulate the wave propagation with a larger Courant number. It can be seen that with a larger \( Q \), a larger Courant number can be adopted, which indicates that the attenuation has a limited effect on the stability conditions.

**NUMERICAL EXAMPLES**

**A homogeneous model**

The first example is to validate the accuracy of the proposed \( k \)-space approach by comparing the traces for a long propagation distance of 5 km in a homogeneous medium. Simulations are performed using a 2D model with a reference velocity of \( c_0 = 2000 \text{ m/s} \) defined at the dominant 12.5 Hz source. The model contains \( 300 \times 300 \) cells with a uniform grid spacing of 20 m. The traces obtained by the traditional FD-PS method (the blue lines) using equation 6 and the proposed \( k \)-space approach (the red lines) using equation 11 are shown in Figure 6 and are compared with the analytical solution that is computed by convolving the source wavelet with the Green’s function (Carcione, 2007). In Figure 6, from left to right, the time step increases from 1 to 4 ms at an interval of 1 ms; from the top to the bottom, the quality factor \( Q \) of each row is 5, 20, and 50, respectively.

We use the following formula to quantitatively evaluate the accuracy of the traditional FD-PS method and the proposed \( k \)-space scheme:

\[
\text{ERR} = \left[ \sum_{i=1}^{nt} \frac{(x_i - xo_i)^2}{\sum_{i=1}^{nt} x_{oi}^2} \right] \times 100\%, \quad (19)
\]

where \( x_i \) represents the numerical solution of the reference, \( xo_i \) represents the trace calculated by the FD-PS method or our \( k \)-space scheme, \( i \) stands for the index of the sample point, and \( nt \) stands for the number of temporal sampling points. In Figure 6, the relative error between the reference trace and the trace generated by the FD-PS method is indicated as ERR1, and the relative error between the reference and the trace calculated by the \( k \)-space method is indicated as ERR2. The FD-PS method can maintain a relatively high time accuracy when the time steps are small (1 and 2 ms). However, as the time step gradually increases, the disturbance of the advanced phase gradually appears, which indicates that the temporal dispersion becomes notable; for example, ERR1 increases to 29.07% for the case of \( \Delta t = 4 \text{ ms} \) and \( Q = 50 \). Nevertheless, almost no obvious waveform distortions appear in the traces obtained by the proposed \( k \)-space approach even for the large time step of \( \Delta t = 4 \text{ ms} \), which indicates its ability to compensate for the time-stepping errors.

**The Marmousi model**

Second, we present the results of two simulations in the highly heterogeneous Marmousi model with a more realistic level of structural complexity. The velocity model is displayed in Figure 7, and the \( Q \) model is built from the velocity model with the function of \( Q = 3.516(c_0/1000)^{2.2} \) (Li, 1993). According to the stability condition shown in Figure 5, for this Marmousi model, the allowed maximum time steps of the FD-PS and \( k \)-space methods are approximately 2.0 and 3.0 ms, respectively. The models are discretized into \( 760 \times 256 \) grid points with uniform vertical and horizontal grid spacing of 20 m. A Ricker-wavelet excitation source with a dominant frequency of 20 Hz is located at (7600 m, 200 m). Figure 8 shows wavefield snapshots at 1.5 s: Figure 8a shows a reference snapshot, Figure 8b and 8c represents the snapshots generated using the traditional FD-PS method with a time step of 2 and 1 ms, respectively, and Figure 8d displays the snapshot calculated using the proposed \( k \)-space approach with a time step of 2 ms. For a clear comparison, we enlarge two parts of the wavefront in the
rectangular boxes. All of the snapshots are displayed in the same amplitude range. In Figure 8b, some obvious disturbances can be observed in the rectangular boxes, which mean that the FD-PS method suffers serious temporal dispersion for the time step of 2 ms. When the time step of the FD-PS method is reduced to 1 ms, the waveform distortion, as shown in Figure 8c, decreases significantly. A good agreement exists between Figure 8a, 8c, and 8d, which indicates that the k-space approach is capable of performing accurate wavefield extrapolation with twice the time step of the FD-PS method. Figure 9 displays the seismic traces within 1.0–3.0 s recorded at (5000 m, 300 m), where the relative errors are calculated using equation 19. Similar to wave snapshot observations, obvious numerical dispersion can be observed from the trace obtained by the FD-PS method with $\Delta t = 2$ ms (trace a), and when the time step is reduced to $\Delta t = 1$ ms, the relative error of the trace obtained by the PS method decreases notably (trace b). Trace c, generated by the k-space approach with $\Delta t = 2$ ms, is even more accurate than that calculated by the FD-PS method with $\Delta t = 1$ ms. When a time step of 1 ms is adopted, the k-space approach can simulate almost the same trace as the reference.

**The 3D overthrust model**

The last example demonstrates the generalization and feasibility of our scheme in the 3D overthrust velocity model shown in Figure 10. The P-wave velocity value ranges from 2100 to 6000 m/s. Again, the $Q$ model is also built from the velocity model with an empirical function of $Q = 3.516(c_0/1000)^{2.2}$ (Li, 1993). This model contains $600 \times 300 \times 187$ grid points with uniform grid spacing of 15 m. A Ricker-wavelet source with a dominant frequency of 20 Hz is located at (4500 m, 2250 m, 30 m). In this numerical example, we adopt the maximum time steps allowed by the stability conditions and compare the accuracies of the FD-PS and $k$-space methods. The maximum allowed time steps of the FD-PS and k-space methods are 0.75 and 1.2 ms, respectively. Figure 11a–11c displays the reference common-shot gather and the common-shot gathers calculated by the FD-PS and the proposed k-space methods, respectively. Figure 11d and 11e displays the differences of the shot gathers obtained by the FD-PS method and the k-space approach, respectively, to the reference shot gather.

![Figure 6](https://via.placeholder.com/150)

Figure 6. A comparison between the analytical solution and simulated traces obtained by the FD-PS method and the $k$-space approach. The relative errors between the analytical solution and the traces by the FD-PS method (ERR1); the analytical solution and the traces by the $k$-space approach (ERR2) are defined in equation 19.

![Figure 7](https://via.placeholder.com/150)

Figure 7. The Marmousi P-wave velocity model.
record. In Figure 11, the top of the data cube shows a horizontal slice at 1.2 s, the right side shows the profile at $x = 4.5$ km, and the front side shows the profile at $y = 2.25$ km. Compared with Figure 11e, larger errors are observed in Figure 11d, in particular, reflections that are important for further waveform imaging/inversion tend to be erroneous, which means that the FD-PS method suffers from severe numerical dispersion. For a detailed comparison of late arrivals, we show the seismograms between 1.4 and 2.5 s at (3000 m, 3000 m, 30 m) in Figure 12, where the black line represents the reference trace, the blue line represents the trace obtained by the FD-PS method with $\Delta t = 0.75$ ms, and the red line represents the trace calculated by the $k$-space approach with $\Delta t = 1.2$ ms. Figure 12 shows that the trace corresponding to the FD-PS method exhibits a visible misfit off the reference trace, whereas the proposed $k$-space approach not only allows a larger time step (almost double) but also generates the trace that matches well with the reference.

DISCUSSION

Accuracy at high frequencies

As shown in Figure 6, when the time step is fixed and $Q$ is gradually increased, the relative error also gradually increases. It is not surprising that the attenuation has a certain suppression effect on the temporal dispersion. To explain this phenomenon, we compute a Fourier transform of the traces in Figure 6 corresponding to different values of $Q$ (the dominant frequency of the wavelet is 12.5 Hz) and display their frequency spectra in Figure 13. It is shown that with a lower $Q$, the higher frequency content is “naturally” attenuated, and thus less of the signal is susceptible to dispersion remains. The quality factor $Q$ determines the retained frequency components, which is the reason for the effect of $Q$ on temporal dispersion.

Figure 8. (a) The reference snapshot and the snapshots generated using the FD-PS method with a time step of (b) 2 ms and (c) 1 ms; (d) the snapshot generated using the $k$-space method with a time step of 2 ms. The snapshots are generated at 1.5 s.

Figure 9. The comparison of numerical solutions between the reference (black line) and simulated traces (red line) obtained by the FD-PS method with a time step of (a) 2 ms and (b) 1 ms, and by the $k$-space method with a time step of (c) 2 ms and (d) 1 ms.

Figure 10. The P-wave velocity (in km/s) of the 3D overthrust model.
The high-frequency components are more susceptible to server numerical dispersion, but they are critical for achieving a high-resolution seismic image (Zhang and Yao, 2013). To validate numerical accuracy in the presence of high-frequency components, we fix the time step of 4 ms with $Q = 50$. We use the FD-PS method to generate snapshots with dominant frequencies of 20 (Figure 14a) and 50 Hz (Figure 14c) and use the $k$-space approach to obtain the snapshots with dominant frequencies of 20 (Figure 14b) and 50 Hz (Figure 14d). Comparing the lower-left quarters of Figure 14a and 14c, it is evident that the FD-PS method presents obvious temporal dispersion with the increase in dominant frequency. On the other hand, the wavefields shown in the lower-left quarters of Figure 14b and 14d calculated using the $k$-space approach exhibit nonvisible numerical dispersion, which indicates the ability of the $k$-space approach to suppress the time dispersion for large time steps and high frequencies.

**Computation cost**

We discuss the computation cost with the assumption that all mixed-domain operators are handled by the low-rank approximation method. As mentioned above, besides the mixed-domain operator of the Laplacian, our $k$-space approach needs to handle the mixed-domain operator of the correction term. In this case, the low-rank approximation time of the combined mixed-domain operators in our $k$-space increases slightly. The detailed computation time comparison corresponding to Figures 9 and 12 are listed in Tables 1 and 2. The 2D simulation is conducted using MATLAB 2016a on a Linux workstation (Intel Xeon CPU v4, 3.00 GHz, 256 GB RAM),

![Figure 11](image1.png)

**Figure 11.** The common-shot gather of (a) the reference solution and those calculated by (b) the FD-PS method with $\Delta t = 0.75$ ms and (c) our $k$-space method with $\Delta t = 1.2$ ms. (d) The difference between (a and b), and (e) the difference between (a and c).

![Figure 12](image2.png)

**Figure 12.** The reference (the black line) and the simulated traces obtained by the FD-PS method with $\Delta t = 0.75$ ms (the blue line) and the $k$-space method with $\Delta t = 1.2$ ms (the red line).
and the 3D simulation is implemented using CUDA programming on a GTX 1080 GPU. In Table 1, the trace computed by the FD-PS method with a 1 ms time step is comparable to the trace obtained by the \( k \)-space approach with 2 ms time step. The computation time indicates that for the wavefield simulation in the 2D Marmousi model, our \( k \)-space approach enjoys two times higher computational efficiency than the traditional FD-PS method to achieve comparable accuracy. As shown in Table 2, the trace computed using the \( k \)-space approach with a time step of 1.2 ms coincides with the reference trace well. To achieve comparable accuracy, the FD-PS method needs to adopt a time step of 0.5 ms, which means that the \( k \)-space approach achieves a speed-up ratio of 2.4 for the 3D simulation.

### Compensation residual-error analysis

The proposed scheme compensates the time-stepping errors well, but it still has some residual errors. The compensation residual error is likely caused by the following two reasons. First, as mentioned above, the time-stepping errors in the loss-dominated operator are compensated by an approximated correction term of \( \phi_2(k) \), which is similar to the compensation operator used in lossless acoustic media.

#### Table 1. Calculation time comparison for the 2D simulations in Figure 9.

<table>
<thead>
<tr>
<th>Numerical method</th>
<th>Time step (ms)</th>
<th>Relative error (%)</th>
<th>Calculation time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD-PS</td>
<td>2</td>
<td>24.93</td>
<td>1076</td>
</tr>
<tr>
<td>FD-PS</td>
<td>1</td>
<td>1.68</td>
<td>2159</td>
</tr>
<tr>
<td>( k )-space</td>
<td>2</td>
<td>1.60</td>
<td>1083</td>
</tr>
</tbody>
</table>

#### Table 2. Calculation time comparison for the 3D simulations in Figure 12.

<table>
<thead>
<tr>
<th>Numerical method</th>
<th>Time step (ms)</th>
<th>Relative error (%)</th>
<th>Calculation time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD-PS</td>
<td>0.75</td>
<td>17.40</td>
<td>1253</td>
</tr>
<tr>
<td>FD-PS</td>
<td>0.5</td>
<td>1.39</td>
<td>1870</td>
</tr>
<tr>
<td>( k )-space</td>
<td>1.2</td>
<td>1.41</td>
<td>781</td>
</tr>
</tbody>
</table>

Figure 13. The amplitude spectra of the traces shown in Figure 6 corresponding to different \( Q \) values.

Figure 14. Snapshots generated using the FD-PS method when the dominant frequency of the source is (a) 20 Hz and (c) 50 Hz, and snapshots generated using the \( k \)-space method when the dominant frequency of the source is (b) 20 Hz and (d) 50 Hz.
Two numerical examples carried out on heterogeneous media further demonstrate that the proposed $k$-space approach can increase the computational efficiency by approximately 2.0 (2.4) times for 2D (3D) modeling. Therefore, the proposed scheme for efficient viscoacoustic modeling has a high potential for developing practical RTM and FWI methods.

CONCLUSION

We have developed a $k$-space-based time-stepping error compensation scheme for solving the DFL viscoacoustic wave equation. The advantages of our scheme over the traditional FD-PS method are the introduction of a compensation term that did not require rewriting the original wave modeling code; also, the relaxed stability condition allows a large time step during simulations. The proposed scheme almost does not introduce additional computational complexity but allows us to extrapolate wavefields with a larger time step; thus, it is more economical in terms of computational cost. The advantages of our scheme over the traditional FD-PS method for solving the DFL viscoacoustic wave equation.

REFERENCES


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DATA AND MATERIALS AVAILABILITY

Data associated with this research are available and can be obtained by contacting the corresponding author.

Figure 15. The comparison between the analytical solution and simulated traces obtained by the complex $k$-space approach. The relative error (ERR) is defined in equation 19.


